Jaroslav Pelant

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RELATIONS BETWEEN GENERALIZED SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

JAROSLAV PELANT, Praha

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This paper is concerned with existence theorems for ordinary differential equations with discontinuous right-hand sides in a space of finite dimension for various definitions of generalized solutions. We substitute the Viktorovskij definition [2] by an equivalent definition in terms of differential inclusions and then we establish the relations between the new definition and the Filippov definition [1].

This paper will be followed by another one dealing with a modification of the Viktorovskij definition and with an equivalent definition in terms of differential inclusion, which will be shown to coincide with the Filippov definition. Consequently, we shall obtain an existence theorem for the modified Viktorovskij solution.

I. AUXILIARY LEMMAS AND DEFINITIONS

Let us introduce the following notation. Let (E_n, S, μ) be a space with a Lebesgue measure μ, where E_n is an n-dimensional real linear normed space with the norm $\| \ \|$, S is a σ-algebra of Lebesgue measurable subsets. Let the closed convex hull of the subset $E \subset E_n$ be denoted by $\overline{\text{conv}}\, E$. The base formed by n linearly independent vectors $e_1, \ldots, e_n$ will be denoted by $(e_1, \ldots, e_n)$. $U(x, \delta)$ will denote an open δ-neighbourhood of the point x in the space E_n considered.

Definition 1. A function f defined in a measurable set $E \subset E_n$, $f(E) \subset E_r$ will be called *weakly asymptotically continuous* at the point x_0 with respect to E if it satisfies the condition

$$\forall(\varepsilon > 0)\, \forall(\delta > 0)\, \exists(\delta_0 \in (0, \delta\rangle)\, \exists(N)\, \forall(x \in E)\,.$$
$$.\, (\|x - x_0\| < \delta_0,\ x \notin N \Rightarrow \|f(x) - f(x_0)\| < \varepsilon)$$

where $\mu(N) < \mu(U(x_0, \delta_0) \cap E)$ for $\mu(U(x_0, \delta_0) \cap E) > 0$ and if $\mu(U(x_0, \delta_0) \cap E) = 0$ then N is arbitrary.

Definition 2. A point x will be called a point of *metrical density* of the measurable set $E \subset E_n$, if $\mu(U(x, \delta) \cap E) > 0$ for arbitrary $\delta > 0$.

Lemma 1. *Let an arbitrary measurable set $E \subset E_n$ be given. If E' is the set of all points of metrical density of the set E, then the set $E - E'$ is of measure zero.*

Proof. We shall use Vitali's covering of the set $E - E'$ with the cubes H chosen small enough to satisfy the condition $\mu(H \cap E) = 0$ for each H. Following Vitali's theorem, an at most countable disjoint system of cubes H_i can be chosen so that $\mu((E - E') - \bigcup_i H_i) = 0$ holds. This implies $\mu(E - E') = 0$.

Lemma 2. *For every simple measurable function f defined on $E \subset E_n$, $f(E) \subset E_r$, the set of all the points of E at which the function f is not w.a. cont. with respect to E is of measure zero.*

Proof. For $\mu(E) = 0$ the assertion is trivial. Suppose therefore $\mu(E) > 0$. Let f be an arbitrary measurable simple function defined on E by the formula $f(x) = e_i$ for $x \in B_i$, $i = 1, \ldots, m$ where $B_i \subset E$ are measurable disjoint sets which satisfy $\bigcup_{i=1}^{m} B_i = E$, and e_i are points in E_r. The set E' is the set of all points of metrical density of the set E. From now on it is sufficient to consider the sets $E'' = E' \cap E$, $B_i'' = B_i \cap E''$ instead of E, B_i, respectively, since $\mu(E - E') = 0$.

Let us choose an arbitrary $x_0 \in E''$ and suppose $x_0 \in B_i''$ for a certain fixed i. The following cases may occur:

1) $\mu(U(x_0, \delta_0) \cap (E'' - B_i'')) = 0$ holds for a certain $\delta_0 > 0$. Then $f(x)$ is w. a. cont. at the point x_0.

2) $\mu(U(x_0, \delta) \cap (E'' - B_i'')) > 0$ holds for every $\delta > 0$.

a) $\mu(U(x_0, \delta) \cap (E'' - B_i'')) < \mu(U(x_0, \delta) \cap E'')$ for every $\delta > 0$, then $f(x)$ is w. a. cont. at x_0.

b) There exists $\delta_1 > 0$ such that $\mu(U(x_0, \delta_1) \cap (E'' - B_i'')) = \mu(U(x_0, \delta_1) \cap E'')$. This implies $\mu(U(x_0, \delta_1) \cap B_i'') = 0$ and, therefore, $f(x)$ is not w. a. cont. at x_0.

It will be shown that the set of all the points in B_i'' at which the function f is not w. a. cont. is of measure zero. To every point $x \in B_i''$ with that property there exists $\delta_x > 0$ such that $\mu(U(x, \delta_x) \cap B_i'') = 0$. Now it suffices to use Lemma 1 with B_i'' written instead of E. As the number of the disjoint sets B_i'' is finite and $E'' = \bigcup_{i=1}^{m} B_i''$, the measure of the set of all the points in E'' at which the function f is not w. a. cont. with respect to E'' is zero.

Lemma 3. *For every function f defined on $E \subset E_n$, $f(E) \subset E_r$, which is the uniform limit of a sequence of simple measurable functions, the measure of the set of all the points in E at which f is not w. a. cont. with respect to E, is zero.*

Proof. We shall prove the non-trivial case i.e. $\mu(E) > 0$. Suppose $f(x) = \lim_{m \to \infty} f_m(x)$ uniformly on E where f_m are simple measurable functions. We shall omit the set D

of measure zero where the functions f_m are not w. a. cont. We shall show that the function f is w. a. cont. on $E - D$ with respect to $E - D$ and, therefore, also with respect to E because $\mu(D) = 0$. Suppose $x_0 \in E - D$ is an arbitrary point of metrical density of the set $E - D$. Let us prove the inequality

$$\|f(x) - f(x_0)\| \leqq \|f(x) - f_m(x)\| + \|f_m(x) - f_m(x_0)\| + \|f_m(x_0) - f(x_0)\| < \varepsilon .$$

For an arbitrary $\varepsilon > 0$ we find m such that $\|f(x) - f_m(x)\| < \frac{1}{3}\varepsilon$ on $E - D$. Now, for the fixed function f_m and for every $\delta > 0$ there exist $\delta_0 \in (0, \delta\rangle$ and a set N satisfying $\mu(N) < \mu(U(x_0, \delta_0) \cap E)$ such that the implication $(\|x - x_0\| < \delta_0, x \notin N \Rightarrow \|f_m(x) - f_m(x_0)\| < \frac{1}{3}\varepsilon) \Rightarrow (\|x - x_0\| < \delta_0, x \notin N \Rightarrow \|f(x) - f(x_0)\| < \varepsilon)$ holds.

If $x_0 \in E - D$ but x_0 is not a point of metrical density, then the weakly asymptotical continuity is obvious.

Lemma 4. *For every function defined and measurable on $E \subset E_n$, $f(E) \subset E_r$ the measure of the set D_E of all the points of E at which f is not w. a. cont. with respect to E, is zero.*

Proof. It suffices to suppose $\mu(E) > 0$ and that all points of the set E are its points of metrical density. Given a measurable subset $A \subset E$ consisting exclusively of its points of metrical density, then $D_A \supset D_E \cap A$. Let $\mu(E) < +\infty$. Following Egoroff Theorem, to an arbitrary $\varepsilon > 0$ there exists $E_\varepsilon \subset E$ such that $\mu(E_\varepsilon) > \mu(E) - \varepsilon$, and there exist simple measurable functions on E_ε uniformly converging to f. Now we shall use the results of Lemmas 2 and 3. Let the set of all the points of metrical density of E_ε be denoted by E'_ε. Let $E'_\varepsilon \cap E$ be denoted by E''_ε. This set satisfies again $\mu(E''_\varepsilon) > \mu(E) - \varepsilon$, and moreover, $D_{E_\varepsilon''} \supset D_E \cap E''_\varepsilon$ where $\mu(D_{E_\varepsilon''}) = 0$. We may write $D_E = (D_E \cap E''_\varepsilon) \cup (D_E \cap (E - E''_\varepsilon))$. Then $\mu(D_E) = \mu(D_E \cap (E - E''_\varepsilon)) < \varepsilon$ where ε is an arbitrary positive number. Hence $\mu(D_E) = 0$, q. e. d. In the case of $\mu(E) = +\infty$ it is possible to use a countable covering of the set E by sets of finite measure.

Lemma 5. *To every function f defined and measurable on $E \subset E_n$, $f(E) \subset E_r$, there exists a set $N_0 \subset E$ such that $\mu(N_0) = 0$, $\bigcap_{N,\mu(N)=0} \overline{f(E - N)} = \overline{f(E - N_0)}$, $\bigcap_{N,\mu(N)=0} \overline{\operatorname{conv}} f(E - N) = \overline{\operatorname{conv}} f(E - N_0)$.*

Proof. Let N_0 contain all the points of E at which the function f is not w. a. cont. with respect to E as well as all the points that are not points of metrical density of the set E. Lemmas 1 and 4 imply $\mu(N_0) = 0$.

Lemma 6. *For any measurable function f defined and bounded on an open set $E \subset E_n$, $f(E) \subset E_r$*

$$\bigcap_{\delta>0} \bigcap_{N,\mu(N)=0} \overline{\operatorname{conv}} f(U(x, \delta) - N) \neq \emptyset \quad \textit{holds for every} \quad x \in E .$$

Proof. Use Lemma 5 for every fixed $\delta > 0$, then Cantor's theorem on intersection of compact sets.

II. DEFINITION OF GENERALIZED SOLUTIONS

Considering an ordinary differential equation $\dot{x} = f(t, x)$, we suppose the right-band side $f(t, x)$ to be a function defined almost everywhere on an open connected set $G \subset E_{n+1}$, and to map this set into E_n.

Remark 1. Definition 3 was introduced by A. F. FILIPPOV (cf. [1]), Definition 4 is due to E. E. VIKTOROVSKIJ [2].

Definition 3. A function $x(t)$ defined on an interval $T = \langle t_1, t_2 \rangle$ where $(t, x(t)) \in G$ for every $t \in T$, is an *F-solution* of the equation $\dot{x} = f(t, x)$ if it is absolutely continuous on T and if there exists a subset $T_1 \subset T$, $\mu(T_1) = \mu(T)$ such that $\dot{x}(t) \in \bigcap_{\delta > 0} \bigcap_{N, \mu(N) = 0} \overline{\operatorname{conv}} f(t, U(x(t), \delta) - N)$ for every $t \in T_1$.

Remark 2. The intersection of the sets in Definition 3 will be written briefly as $K^F(f, t, x(t))$.

Remark 3. When passing from one base $(e_1, \ldots, e_n)$ where the system in Definition 3 has the form $\dot{x}_i = f(t, x_1, \ldots, x_n)$, $i = 1, 2, \ldots, n$ with a solution $x(t) = (x_1(t), \ldots, x_n(t))$, to another base $(b_1, \ldots, b_n)$ the system transforms into the form $\dot{y}_i = g_i(t, y_1, \ldots, y_n)$ and the solution assumes the form $y(t) = (y_1(t), \ldots, y_n(t))$ where $y(t) = C\,x(t)$, $g(t, y) = C f(t, C^{-1} y)$ and C is a regular matrix of the corresponding transformation. Vectors $x(t)$, $y(t)$, g, f are taken as column vectors. When passing from one base to another, the set T_1 in Definition 3 remains unchanged. This is directly concluded from the properties of linear mapping represented by a regular matrix C. Hence, Definition 3 does not depend on the choice of the base.

Definition 4. A function $x(t)$ defined on an interval $T = \langle t_1, t_2 \rangle$ where $(t, x(t)) \in G$ for every $t \in T$, is a *V-solution* of the equation $\dot{x} = f(t, x)$ with respect to a given base B, where the equation is represented by $\dot{x}_i = f_i(t, x_1, \ldots, x_n)$, $i = 1, 2, \ldots, n$, if it is absolutely continuous on T and if to any $\varepsilon > 0$ and to an arbitrary set $N \subset G$, $\mu(N) = 0$ there exist functions $\psi^i(t)$ defined on T, with their ranges in E_n and with the following properties:

For $i = 1, 2, \ldots, n$,

$$(t, \psi^i(t)) \in G \quad \text{for every} \quad t \in T, \tag{1}$$

$$f_i(t, \psi^i(t)) \quad \text{are integrable on} \quad T, \tag{2}$$

$$\|x(t) - \psi^i(t)\| < \varepsilon \quad \text{on} \quad T, \tag{3}$$

(4) $$\left|x_i(t) - \left(x_i(t_1) + \int_{t_1}^{t} f_i(\tau, \psi^i(\tau))\,\mathrm{d}\tau\right)\right| < \varepsilon \quad \text{on} \quad T,$$

and

(5) $$(t, \psi^i(t)) \notin N \quad \text{almost everywhere on} \quad T.$$

Definition 5. A function $x(t)$ defined on an interval $T = \langle t_1, t_2 \rangle$ where $(t, x(t)) \in G$ for every $t \in T$ is an *MF-solution* of the equation $\dot{x} = f(t, x)$ with respect to a given base B where the equation is represented by $\dot{x}_i = f_i(t, x_1, \ldots, x_n)$, $i = 1, \ldots, n$, if it is absolutely continuous on T and if there exists a subset $T_1 \subset T$ such that $\mu(T_1) = \mu(T)$ and for every $t \in T_1$ it is $\dot{x}(t) \in K_B^{MF}(f, t, x(t))$ where

$$K_B^{MF}(f, t, x(t)) = \prod_{i=1}^{n} K^F(f_i, t, x(t))$$

and

$$K^F(f_i, t, x(t)) = \bigcap_{\delta > 0} \bigcap_{N, \mu(N) = 0} \overline{\operatorname{conv}} f_i(t, U(x(t), \delta) - N)$$

for $i = 1, \ldots, n$.

Remark 4. If the right-hand side of the equation $\dot{x} = f(t, x)$ is defined on G, measurable on G and continuous in x for arbitrary (but fixed) t, then $K_B^{MF}(f, t, x) = K^F(f, t, x) = f(t, x)$ and, therefore, every solution in the sense of Definition 3 and Definition 5 is a solution in the sense of Carathéodory.

Definition 6. A function $x(t)$ defined on an interval $T = \langle t_1, t_2 \rangle$ where $(t, x(t)) \in G$ for every $t \in T$ is an *MV-solution* of the equation $\dot{x} = f(t, x)$ if it is absolutely continuous on T, and if to any $\varepsilon > 0$ and to an arbitrary set $N \subset G$, $\mu(N) = 0$ there exists a function $\psi(t) \in E_n$ defined on T, with the following properties:

(6) $$(t, \psi(t)) \in G \quad \text{for every} \quad t \in T,$$

(7) $$f(t, \psi(t)) \quad \text{is integrable on} \quad T,$$

(8) $$\|x(t) - \psi(t)\| < \varepsilon \quad \text{on} \quad T,$$

(9) $$\left\|x(t) - \left(x(t_1) + \int_{t_1}^{t} f(\tau, \psi(\tau))\,\mathrm{d}\tau\right)\right\| < \varepsilon \quad \text{on} \quad T,$$

and

(10) $$(t, \psi(t)) \notin N \quad \text{a. e. on} \quad T.$$

III. RELATIONS BETWEEN GENERALIZED SOLUTIONS

Remark 5. Everywhere in this chapter we suppose that the right-hand side $f(t, x)$ of the system $\dot{x} = f(t, x)$ is defined a. e. on an open connected set $G \subset E_{n+1}$, and that it maps this set into E_n. Let the function $f(t, x)$ be measurable on G. Assume that to every compact set $K \subset G$ there exists a locally integrable function $m(t)$ defined a. e. on the projection of the set K to the axis t, satisfying $\|f(t, x)\| \leqq m(t)$ a. e. on K. It follows from this remark and Lemma 6 that the sets $K^F(f, t, x(t))$ are non-empty almost everywhere on T provided $x(t)$ is a continuous function defined on a closed interval T, $(t, x(t)) \in G$ for every $t \in T$.

Definition 7. We shall say that an absolutely continuous function $x(t)$ defined on $T = \langle t_1, t_2 \rangle$ fulfils condition CF if $\exists(T_1 \subset T : \mu(T_1) = \mu(T))\, \forall(i)\, \forall(B_j)\, \forall(t \in T_1)\,.$ $.\, \{\alpha \vee \beta\}$, where $B_j = C_j B$ and B is a given orthonormal base and C_j are all the regular matrices of the type (n, n) with rational elements. Hence $\{B_j\}$ is a countable system of orthonormal bases in E_n. An index i is the index of the coordinate in a given base. The conditions α and β read as follows:

α) for an arbitrary open interval $I \subset G(t)$ with $x(t) \in I$ and for any $\varepsilon > 0$ it holds $\mu(M^i_{\varepsilon,I,t}) > 0$, where $M^i_{\varepsilon,I,t} = \{x \in I : |\dot{x}_i(t) - f_i(t, x)| < \varepsilon\}$.

β) for an arbitrary open interval $I \subset G(t)$ with $x(t) \in I$, it holds $\mu(N^i_{1,I,t}) > 0$ as well as $\mu(N^i_{2,I,t}) > 0$ where $N^i_{1,I,t} = \{x \in I : f_i(t, x) > \dot{x}_i(t)\}$, $N^i_{2,I,t} = \{x \in I :$ $: f_i(t, x) < \dot{x}_i(t)\}$ and $G(t)$ is the projection of the set G into E_n with fixed t.

Theorem 1. $(F \Leftrightarrow CF)$ *An absolutely continuous function $x(t)$ defined on $T =$ $= \langle t_1, t_2 \rangle$ is an F-solution of the system $\dot{x} = f(t, x)$ from Remark 5 in the sense of Definition 3 if and only if the condition CF holds for $x(t)$.*

Proof. Let us suppose $x(t)$ is an F-solution of the equation $\dot{x} = f(t, x)$ on an interval $T = \langle t_1, t_2 \rangle$. Then there exists a subset $T_1 \subset T$, $\mu(T_1) = \mu(T)$ such that $\dot{x}(t) \in$ $\in K^F(f, t, x(t))$ provided $t \in T_1$. We shall prove that $\alpha \vee \beta$ holds for any index i, for every $t \in T_1$, and with respect to any base B. According to Definition 3 it holds $\dot{x}(t) \in \overline{\text{conv}}\, f(t, U(x(t), \delta) - N_\delta(t))$ for every $t \in T_1$ and $\delta > 0$. The set $N_\delta(t)$, $\mu(N_\delta(t)) = 0$ has the same meaning as the set N_0 in Lemma 5. For the sake of brevity, let us denote $A_\delta(t) = \overline{\text{conv}}\, f(t, U(x(t), \delta) - N_\delta(t))$. Let us choose a base (cf. Remark 3). Now let $\mu(M^i_{\varepsilon,I_1,t_1}) = 0$ hold for some $t_1 \in T_1$ and fixed i and certain I_1 and $\varepsilon > 0$. Consequently, the condition α does not hold at t_1. At the same time, let there exist I_2 such that, for instance, $\mu(N^i_{2,I_2,t_1}) = 0$. Let us choose $\delta > 0$ such that $U(x(t_1), \delta) \subset I_1 \cap I_2$ where the intersection is a non-empty set because it contains the point $x(t_1)$. Consequently, the inequality $\dot{x}_i(t_1) \leqq f_i(t_1, x) - \varepsilon$ must be valid for every $x \in U(x(t_1), \delta) - N(t_1)$ where the set $N(t_1) = (N_\delta(t_1) \cup M^i_{\varepsilon,I_1,t_1} \cup N^i_{2,I_2,t_1})$ has measure zero. Hence $\varepsilon + \dot{x}_i(t_1) \leqq y_i$ for every $y = (y_1, \ldots, y_n)$ where $y \in$ $\in \overline{\text{conv}}\, f(t_1, U(x(t_1), \delta) - N(t_1)) = A_\delta(t_1)$. This implies $\dot{x}(t_1) \notin A_\delta(t_1)$, which is a contradiction. This yields that the condition $\alpha \vee \beta$ is satisfied on the whole set T_1 for every base B and $i = 1, \ldots, n$. The argument is analogous for $\mu(N^i_{1,I_2,t_1}) = 0$.

It remains to prove $CF \Rightarrow F$. Let us suppose CF holds, i.e. $\exists(T_2 \subset T : \mu(T_2) = \mu(T))\ \forall(B_j)\ \forall(i)\ \forall(t \in T_2)\ \{\alpha \vee \beta\}$. For sufficiently small $\delta_0 > 0$ the set $\overline{\bigcup_{t\in T} (t, U(x(t), \delta_0))}$ is a compact subset of G. Hence, cf. Remark 5, there exists $T' \subset T$, $\mu(T') = \mu(T)$ such that $\|f(t, x)\| \leqq m(t)$ for every $t \in T'$. Then for every $\delta \in (0, \delta_0)$ the sets $f(t, U(x(t), \delta) - N_\delta(t))$ where $\mu(N_\delta(t)) = 0$ are bounded for every fixed $t \in T'$. In the sequel we consider the set $T_2' = T_2 \cap T'$ for which again $\mu(T_2') = \mu(T_2)$. Let $\alpha \vee \beta$ be satisfied on T_2' with respect to any base $B_j \in \{B_j\}$ and for each index $i = 1, \ldots, n$. It depends on the choice of the base B_j which of the conditions α or β holds for a given i and $t \in T_2'$. However, both α and β imply the inequality

$$\{\text{vrai min} f_i(t, x) : x \in U(x(t), \delta)\} \leqq \dot{x}_i(t) \leqq \leqq \{\text{vrai max} f_i(t, x) : x \in U(x(t), \delta)\} \tag{11}$$

for $i = 1, \ldots, n$ and $t \in T_2'$. Let us choose $\delta \in (0, \delta_0)$ and let V denote the set of all vectors $v \in E_n$ with rational coordinates in the base B. The set V is countable and dense in E_n. We shall prove that for every $v \in V$ and $t \in T_2'$ the inequality

$$(\dot{x}(t), v) \leqq \{\text{vrai max}\, (f(t, x), v) : x \in U(x(t), \delta)\} \tag{12}$$

holds. Let us choose a fixed $t_2 \in T_2'$ and a fixed $v \in V$ and let us consider an orthonormal base $(e_1, \ldots, e_n) \in \{B_j\}$ such that $v = k.e_i$ for a certain fixed i, where k is a positive rational number. With respect to this base, let the equation be represented by the system $\dot{y}_j = g_j(t, y)$, $j = 1, \ldots, n$ (cf. Remark 3). The inequality (11) is satisfied in every base B_j. Furthermore, $(\dot{x}(t_2), v) = k.\dot{y}_i(t_2)$, hence $(f(t_2, x), v) = k.g_i(t_2, y)$ and, therefore, the inequality (12) holds. This inequality can be rewritten into the form $(\dot{x}(t), v) \leqq \{\sup (x, v) : x \in f(t, U(x(t), \delta) - N_\delta(t))\}$ where $\mu(N_\delta(t)) = 0$. Let us denote $A(t) = f(t, U(x(t), \delta) - N_\delta(t))$. The inequality $(\dot{x}(t), v) \leqq \{\sup (x, v) : x \in A(t)\}$ has been proved for arbitrary fixed $t \in T_2'$ and arbitrary $v \in V$. We shall prove that inequality for every $v \in E_n$. There exists a sequence $\{v_n\} \subset V$ such that $v_n \to v$ for $n \to \infty$. In the inequality $(\dot{x}(t), v_n) \leqq \{\sup (x, v_n) : x \in A(t)\}$ let $n \to \infty$. The continuity of scalar product yields $(\dot{x}(t), v_n) \to (\dot{x}(t), v)$ for $n \to \infty$. Further, $\{\sup (x, v_n) : x \in A(t)\} \to \{\sup (x, v) : x \in A(t)\}$ for $n \to \infty$ because $|\{\sup (x, v) - \sup (x, v_n) : x \in A(t)\}| \leqq |\{\sup ((x, v) - (x, v_n)) : x \in A(t)\}| = |\{\sup (x, (v - v_n)) : x \in A(t)\}| \leqq \{\sup |(x, v - v_n)| : x \in A(t)\} \leqq \{\sup \|x\|\, \|v - v_n\| : x \in A(t)\} = \{\sup \|x\| : x \in A(t)\}.\|v - v_n\| \leqq c.\|v - v_n\|$ where c is a positive constant because the set $A(t)$ is bounded. To complete the proof, it suffices to show that the implication

$$\forall(v \in E_n)\, ((\dot{x}(t), v) \leqq \{\sup (x, v) : x \in A(t)\}) \Rightarrow \dot{x}(t) \in \overline{\text{conv}}\, A(t)$$

holds for fixed $t \in T_2'$. To this end, assume that $\dot{x}(t) \notin \overline{\text{conv}}\, A(t)$. Then there exists a hyperplane Γ dividing the space E_n into two open parts Γ^+, Γ^- such that $\overline{\text{conv}}\, A(t) \subset \Gamma^+ \cup \Gamma$ and $\dot{x}(t) \in \Gamma^-$. Let us substitute v by a vector v_Γ perpendicular to Γ and directed into Γ^-. We get $(\dot{x}(t), v_\Gamma) > \{\sup (x, v_\Gamma) : x \in A(t)\}$ which is a contradiction. Thus we have proved that $\dot{x}(t) \in \overline{\text{conv}}\, A(t) = A_\delta(t)$ on T_2'. Since $\dot{x}(t) \in A_\delta(t)$ holds on T_2' for arbitrary $\delta \in (0, \delta_0)$, the function $x(t)$ is an F-solution of the equation $\dot{x} = f(t, x)$ on T.

Remark 6. The condition CF

$$\exists(T_1 \subset T : \mu(T_1) = \mu(T))\ \forall(B_j)\ \forall(i)\ \forall(t \in T_1)\ \{\alpha \vee \beta\}$$

where the number of B_j's is countable while that of indices i is finite, can be rewritten into the form

$$\forall(B_j)\ \forall(i)\ \exists(T_1 \subset T : \mu(T_1) = \mu(T))\ \forall(t \in T_1)\ \{\alpha \vee \beta\}\,,$$

since for every B_j and i there exists $T_1^{j,i} \subset T$, $\mu(T_1^{j,i}) = \mu(T)$ and we can put $T_1 = \bigcap_{j,i} T_1^{j,i}$, measure of T_1 being equal to $\mu(T)$.

Theorem 2. ($F \Rightarrow MF$) *If an absolutely continuous function $x(t)$ defined on $T = \langle t_1, t_2 \rangle$ is an F-solution to the equation $\dot{x} = f(t, x)$, then it is an MF-solution as well.*

Proof. The function $x(t)$ is an F-solution on T which means $\dot{x}(t) \in K^F(f, t, x(t))$ holds a. e. on T. Let us choose an arbitrary base B. With respect to this base, let the system be expressed in the form $\dot{x}_i = f_i(t, x_1, \ldots, x_n)$, $i = 1, \ldots, n$. From Definition 5 we get $K_B^{MF}(f, t, x(t)) = \prod_{i=1}^{n} K^F(f_i, t, x(t))$. Hence the inclusion $K^F(f, t, x(t)) \subset K_B^{MF}(f, t, x(t))$ can be derived from the inequality (11) in the proof of Theorem 1. Thus obviously $\dot{x}(t) \in K_B^{MF}(f, t, x(t))$ holds a. e. on T and, therefore, $x(t)$ is an MF-solution of the equation $\dot{x} = f(t, x)$ on T.

Corollary 1. *For $n = 1$ we have $K^F(f, t, x) = K_B^{MF}(f, t, x)$ and hence Definitions* 3 *and 5 are equivalent.*

This equivalence is introduced in [1] without proof.

Example 1. In this example it will be shown that for $n > 1$ there exist equations whose MF-solutions need not be F-solutions.

Let an equation $\dot{x} = f(x)$ be given on E_2 which has, in a given base B, the form $f_1(x) = 2 - \text{sign}\, x_2$, $f_2(x) = -\text{sign}\, x_2$. Each trajectory in the sense of Definitions 3 and 5 reaches the axis x_1 after a certain time, and continues along this axis. On the axis x_1 we obtain the F-solution $x_1(t) = 2t$, $x_2(t) = 0$, unique in the sense of increasing t. There are infinitely many MF-solutions on the axis x_1 with a given base B, their form being $x_1(t) = ct$, $x_2(t) = 0$, where c is an arbitrary constant from the interval $\langle 1, 3 \rangle$.

Theorem 3. *Let a function $x(t)$ be defined and absolutely continuous on T and let it be an MF-solution of the equation $\dot{x} = f(t, x)$ on T in a given base B but not an F-solution of this equation on T. Then there exists a base B' such that $x(t)$ is not an MF-solution on T of the equation $\dot{x} = f(t, x)$ in the base B'.*

Proof. Let us consider all regular matrices C_i with rational elements of the type (n, n). Transforming the given base B by means of these matrices we obtain a countable system of bases $B_i = C_i B$ for which $K^F(f, t, x) = \bigcap_{B_i} K^{MF}_{B_i}(f, t, x)$. The last identity follows from the separability of a closed convex set from a point that does not belong to the set. Hence the countable system of bases B_i includes a base B' such that $x(t)$ is not an MF-solution of the equation $\dot{x} = f(t, x)$ on the interval T with respect to this base.

Definition 8. We shall say that an absolutely continuous function $x(t)$ defined on $T = \langle t_1, t_2 \rangle$ fulfils condition CMF if $\exists (T_1 \subset T : \mu(T_1) = \mu(T)) \, \forall(i) \, \forall(t \in T_1) \, . \, \{\alpha \vee \beta\}$, where $i = 1, \ldots, n$ and the condition $\alpha \vee \beta$ is from Definition 7. The index i is the index of the coordinate in a given base B.

Theorem 4. $(MF \Leftrightarrow CMF)$ *An absolutely continuous function $x(t)$ defined on $T = \langle t_1, t_2 \rangle$ is an MF-solution of the system $\dot{x} = f(t, x)$ from Remark 5 with respect to a given base B if and only if the condition CMF holds for $x(t)$.*

Proof. Let us suppose $x(t)$ is an MF-solution of the equation $\dot{x} = f(t, x)$ on the interval $T = \langle t_1, t_2 \rangle$ with respect to a given base B. The equation has the form $\dot{x}_i = f_i(t, x_1, \ldots, x_n)$, $i = 1, \ldots, n$ with respect to B. Then there exists a subset $T_1 \subset T$, $\mu(T_1) = \mu(T)$ on which $\dot{x}_i(t) \in K^F(f_i, t, x(t))$ holds for every $i = 1, \ldots, n$.

Hence, $\dot{x}_i(t) \in \overline{\operatorname{conv}} f_i(t, U(x(t), \delta) - N^i_\delta(t)) = A^i_\delta(t)$ for every $t \in T_1$, $i = 1, \ldots, n$ and $\delta > 0$ where $N^i_\delta(t)$, $\mu(N^i_\delta(t)) = 0$ has the same meaning as the set N_0 in Lemma 5. Now let $\mu(M^i_{\varepsilon, I_1, t_1}) = 0$ hold for some $t_1 \in T_1$ and fixed i and I_1 and $\varepsilon > 0$. Simultaneously, let there exist I_2 such that, for instance, $\mu(N^i_{2, I_2, t_1}) = 0$. Let us choose $\delta > 0$ such that $U(x(t_1), \delta) \subset I_1 \cap I_2$. Consequently, the inequality $\dot{x}_i(t_1) \leqq f_i(t_1, x) - \varepsilon$ is valid for every $x \in U(x(t_1), \delta) - N(t_1)$ where the set $N(t_1) = (N^i_\delta(t_1) \cup M^i_{\varepsilon, I_1, t_1} \cup N^i_{2, I_2, t_1})$ has measure zero. This implies $\dot{x}_i(t_1) \notin A^i_\delta(t_1)$, which is a contradiction. This yields that the condition $\alpha \vee \beta$ is satisfied on the whole set T_1 for each $i = 1, \ldots, n$.

It remains to prove $CMF \Rightarrow MF$. Let α or β be satisfied on some $T_2 \subset T$, $\mu(T_2) = \mu(T)$ for each index $i = 1, \ldots, n$. We shall prove $\dot{x}_i(t) \in A^i_\delta(t)$ for every $\delta > 0$ and for each index i and all $t \in T_2$, which implies that $\dot{x}_i(t) \in K^F(f_i, t, x(t))$ is satisfied for every $t \in T_2$ and i. Then $\dot{x}(t) \in K^{MF}_B(f, t, x(t))$ is satisfied for every $t \in T_2$. Let us choose a fixed $\delta > 0$ and an index i and let $I(t) = U(x(t), \delta)$. Further, let α hold for a given $t_2 \in T_2$. Then there exists $x \in I(t_2) - N^i_\delta(t_2)$, $|\dot{x}_i(t_2) - f_i(t_2, x)| < \varepsilon$ for an arbitrary ε-neighbourhood of the point $\dot{x}_i(t_2)$. We get $\dot{x}_i(t_2) \in A^i_\delta(t_2)$ because the set $A^i_\delta(t_2)$ is closed. Now let β hold for a given $t_2 \in T_2$; then there exists $x_1, x_2 \in I(t_2) -$

$- N_\delta^i(t_2)$ and it holds $f_i(t_2, x_1) < \dot{x}_i(t_2) < f_i(t_2, x_2)$. Consequently $\dot{x}_i(t_2) \in A_\delta^i(t_2)$ since the set $A_\delta^i(t_2)$ is convex. $\dot{x}_i(t_2) \in A_\delta^i(t_2)$ holds for each $i = 1, \ldots, n$ and for arbitrary $\delta > 0$ and $t_2 \in T_2$, $\mu(T_2) = \mu(T)$. Hence $\dot{x}(t_2) \in K_B^{MF}(f, t_2, x(t_2))$ for all $t_2 \in T_2$. Then $x(t)$ is an MF-solution of the equation $\dot{x} = f(t, x)$ on T in a given base B.

Lemma 7. *Let a function $x(t)$ be defined and absolutely continuous on $T = \langle t_1, t_2 \rangle$, mapping the interval T into E_n. Let a real function $f(t, x)$ be defined a. e. and measurable on the set $M = \bigcup_{t \in T} (t, R_t)$, where $R_t = \prod_{i=1}^{n} R_i$, $R_i = \langle x_i(t) - \delta, x_i(t) + \delta \rangle$ and δ is a fixed positive number. Let T' denote a subset of T with the following properties: For every $t \in T'$ and every $\varepsilon > 0$, $\mu\{x \in R_t : \|x - x(t)\| < \varepsilon, f(t, x) < \varepsilon\} > 0$ holds and for every $t \in T - T'$, there exists $\varepsilon > 0$ such that $\mu\{x \in R_t : \|x - x(t)\| < \varepsilon, f(t, x) < \varepsilon\} = 0$. Then the set T' is measurable and there exists a measurable function $h(t)$ on T with the properties: $h(t)$ equals zero on T' and is positive or $+\infty$ on $T - T'$; for every $t \in T - T'$,*

$$\mu\{x \in R_t : \|x - x(t)\| < h(t), f(t, x) < h(t)\} = 0 .$$

Proof. Let a measurable function $\lambda(t)$ be given on T. Let us choose a function $g(t, x)$ on the set M:

$$g(t, x) = 1 \quad \text{for} \quad f(t, x) > \lambda(t)$$

and

$$g(t, x) = 0 \quad \text{for} \quad f(t, x) \leqq \lambda(t) .$$

This function $g(t, x)$ is measurable and integrable on M. By virtue of Fubini's theorem it holds $\int_M g(t, x)\, dt\, dx = \int_T dt \int_{R_t} g(t, x)\, dx$; therefore, $\int_{R_t} g(t, x)\, dx = \mu\{x \in R_t : f(t, x) > \lambda(t)\}$ is a measurable function on T with respect to the variable t. Then the sets

$$\{t \in T : \mu\{x \in R_t : f(t, x) > \lambda(t)\} = 0\} \tag{13}$$

$$\{t \in T : \mu\{x \in R_t : f(t, x) > \lambda(t)\} > 0\}$$

are measurable. Let us choose a sequence $\{r_n\}_{n=1}^{\infty}$, where r_n are all positive rational numbers and define a function $h_n(t)$ on the interval T by

$$h_n(t) = r_n \quad \text{for every } t \text{ satisfying} \quad \mu\{x \in R_t : \|x - x(t)\| < r_n, f(t, x) < r_n\} = 0 ,$$

$$h_n(t) = 0 \quad \text{for every } t \text{ satisfying} \quad \mu\{x \in R_t : \|x - x(t)\| < r_n, f(t, x) < r_n\} > 0 .$$

From (13) we derive that the function $h_n(t)$ is measurable on T, hence also the function $\limsup_{n \to \infty} h_n(t)$ is measurable on T. This function will be denoted by $h(t)$.

The function $h(t)$ can be $+\infty$ for some $t \in T$. If $h_n(t) > 0$ for a certain fixed $t \in T$ and for a certain n, then $\mu\{x \in R_t : \|x - x(t)\| < r_n, f(t, x) < r_n\} = 0$, therefore $\mu\{x \in R_t : \|x - x(t)\| < r_k, f(t, x) < r_k\} = 0$ for every r_k, where $0 < r_k < r_n$. For every positive integer l there exists $k(l) > \max(n, l)$ such that $\frac{1}{2}r_n < r_{k(l)} < r_n$. Then it holds $h_{k(l)}(t) = r_{k(l)} > \frac{1}{2}r_n > 0$ and this yields $h(t) > \frac{1}{2}r_n$. Thus we have proved that $h_n(t) > 0 \Rightarrow h(t) > 0$ for every $t \in T$. Hence $h(t) = 0$ for a certain $t \in T$ implies $h_n(t) = 0$ for every positive integer n. Then $h(t) = 0$ yields $\mu\{x \in R_t : : \|x - x(t)\| < r_n, f(t, x) < r_n\} > 0$ for every positive integer n and this implies that the function $h(t)$ is positive or $+\infty$ on the set $T - T'$. The identity $h(t) = 0$ on T' follows from the definition of $h_n(t)$. Now we shall prove the last assertion of this lemma. If $h(t) > 0$ for a certain $t \in T$, then there exists a nondecreasing sequence $h_{n(i)}(t) \to h(t)$, therefore $r_{n(i)} \to h(t)$ and $\mu\{x \in R_t : \|x - x(t)\| < r_{n(i)}, f(t, x) < < r_{n(i)}\} = 0$ implies $\mu\{x \in R_t : \|x - x(t)\| < h(t), f(t, x) < h(t)\} = 0$. Let us choose $h(t) = \{\sup h_n(t) : n = 1, \ldots\}$ for every $t \in T$; then we reach the same result.

Lemma 8. *Let a measurable function $f(t, x)$ be given on an open connected set $G \subset E_{n+1}$. Let a mapping $M_0(t)$ into sets from E_n be defined on an interval $T = \langle t_1, t_2 \rangle$, such that for every $t \in T$ the sets $M_0(t)$ are subsets of E_n, $\mu(M_0(t)) > 0$, $(t, M_0(t)) \subset G$ and $M_0 = \bigcup_{t \in T} (t, M_0(t))$ is a measurable set in E_{n+1}. Then there exists a measurable function $\psi(t)$ defined on T such that $f(t, \psi(t))$ is measurable on T and $\psi(t) \in M_0(t)$ for every $t \in T$.*

Proof. We shall proceed similarly as in [2]. Let us denote $N_0 = \{x \in G' : f(t, x)$ is not measurable on T with respect to the variable $t\}$, where G' is the projection of the set G into E_n. We shall leave the set $T \times N_0$ out of the set G. In this way we have reduced the sets $M_0(t)$ at most by sets of measure zero. We shall keep the same notation $M_0(t)$ for the new sets. The set M_0 is measurable, therefore we can write $\mu(M_0) = \int_T \mu(M_0(t))\,dt > 0$. Let us denote $X = \{x \in E_n : \mu(M'_0(x)) > 0\}$, where $M'_0(x)$ are the projections of the sections of M_0 with a fixed x into the axis t. Then we can rewrite $\mu(M_0) = \int_X \mu(M'_0(x))\,dx > 0$. Now, there exists $x_0 \in X$ such that $\mu(M'_0(x_0)) \geqq \mu(M_0)/\mu(X)$. Let us denote $M_1 = \{(t, x) \in M_0 : t \notin M'_0(x_0)\}$, then $\mu(M_1) = \int_X \mu(M'_1(x))\,dx$ holds, where $M'_1(x)$ is the projection of the section of the set M_1 with a fixed x into the axis t. Again there exists $x_1 \in X$ such that $\mu(M'_1(x_1)) \geqq \geqq \mu(M_1)/\mu(X)$. Let us write generally $M_{i+1} = \{(t, x) \in M_0 : t \notin \bigcap_{j=0}^{i} M'_j(x_j)\}$ for $i = 1, 2, \ldots$; then $\mu(M_{i+1}) = \int_X \mu(M'_{i+1}(x))\,dx$ and there exists $x_{i+1} \in X$ such that $\mu(M'_{i+1}(x_{i+1})) \geqq \mu(M_{i+1})/\mu(X)$. The set sequence $\{M_i\}_{i=0}^{\infty}$ is nonincreasing.

Let $\lim_{n\to\infty} \mu(M_n) = \alpha > 0$, then $\mu(M'_n(x_n)) \geqq \alpha/\mu(X)$ and we derive $\mu(\bigcup_{n=0}^{\infty} M'_n(x_n)) = = +\infty$ which is a contradiction because the sets $M'_n(x_n)$ are disjoint and the union of these sets is included in the interval T with a finite measure. From the preceding result we obtain $\lim_{n\to\infty} \mu(M_n) = 0$. Further $\mu(M_n) = \int_{T - \bigcup_{i=0}^{n-1} M_i'(x_i)} \mu(M_n(t))\,dt = = \int_{T - \bigcup_{i=0}^{n-1} M_i'(x_i)} \mu(M_0(t))\,dt$ because $M_0(t) = M_n(t)$ on the integration domain. The implication $\lim_{n\to\infty} \mu(M_n) = \int_{T - \bigcup_{i=0}^{\infty} M_i'(x_i)} \mu(M_0(t))\,dt = 0 \Rightarrow \mu(T - \bigcup_{i=0}^{\infty} M'_i(x_i)) = 0$

is true because $\mu(M_0(t)) > 0$ for every $t \in T$. Thus we have proved that $\mu(\bigcup_n M'_n(x_n)) = \mu(T)$, where the union is at most countable. Now let us define a function ψ on T:

$$\psi(t) = x_n \quad \text{on each set} \quad M'_n(x_n)\,,$$
$$\psi(t) \in M_0(t) \quad \text{on} \quad T - \bigcup_n M'_n(x_n)\,.$$

Theorem 5. *(CMF $\Rightarrow$ V) Let the condition CMF from Definition 8 $\exists(T_1 \subset T : \mu(T_1) = \mu(T))\, \forall(i)\, \forall(t \in T_1)\, \{\alpha \vee \beta\}$ be satisfied in a given base B for an absolutely continuous function $x(t)$ given on an interval $T = \langle t_1, t_2 \rangle$. Then $x(t)$ is a V-solution of the equation $\dot{x} = f(t, x)$ from Remark 5 on the interval T in the base B.*

Proof. Let an absolutely continuous function $x(t)$ be given on the interval T and let $\alpha \vee \beta$ be satisfied on a certain set $T_1 \subset T$, $\mu(T_1) = \mu(T)$ in the given base B for each $i = 1, \dots, n$. Let $T'_i = \{t \in T_1 : \alpha \text{ holds at the point } t\}$, then only β holds on the sets $T''_i = T_1 - T'_i$. According to Lemma 7 the sets T'_i and T''_i are measurable. Let us choose an arbitrary $\varepsilon > 0$ and $N \subset G$, $\mu(N) = 0$. For each $i = 1, \dots, n$ we shall find a function $\psi^i(t)$ on T satisfying (1)–(5) with the norm $\|x\| = \{\max |x_i| : i = 1, \dots, n\}$. Thus we shall have proved the assertion of Theorem 5. Let us fix an index i and an interval $I_t = \prod_{j=1}^{n} I^t_j$ for every $t \in T$, where $I^t_j = (x_j(t) - \varepsilon,\ x_j(t) + \varepsilon)$ with $\varepsilon > 0$ sufficiently small so that $\overline{\bigcup_{t \in T} (t, I_t)} \subset G$. Now let us choose $\varepsilon_1 > 0$ such that

$$2\varepsilon_1\, \mu(T) < \varepsilon\,. \tag{14}$$

The condition α holds for every $t \in T'_i$, which implies $\mu(M^i_{\varepsilon_1, I_t, t}) > 0$ on the set T'_i, where

$$M^i_{\varepsilon_1, I_t, t} = \{x \in I_t : |\dot{x}_i(t) - f_i(t, x)| < \varepsilon_1\}\,.$$

The condition β holds for every $t \in T''_i$, which implies $\mu(N^i_{1, I_t, t}) > 0$ and $\mu(N^i_{2, I_t, t}) > 0$, where

$$N^i_{1, I_t, t} = \{x \in I_t : \dot{x}_i(t) - f_i(t, x) < 0\}\,,$$
$$N^i_{2, I_t, t} = \{x \in I_t : \dot{x}_i(t) - f_i(t, x) > 0\}\,.$$

Now let us define for $j = 1, 2$:

$$M_j(t) = M^i_{\varepsilon_1, I_t, t} \quad \text{on} \quad T'_i\,,$$
$$M_j(t) = N^i_{j, I_t, t} \quad \text{on} \quad T''_i\,,$$
$$M_j(t) = I_t \quad \text{on} \quad T - (T'_i \cup T''_i)\,.$$

The sets $\bigcup_{t \in T} (t, M_j(t))$ are measurable.

Evidently $\mu(M_j(t)) > 0$ for every $t \in T$ and $j = 1, 2$. The function $f(t, x)$ is defined a. e. on G and there exists a set $G_0 \subset G$, $\mu(G_0) = 0$ such that $f(t, x)$ is defined on $G - G_0$. We shall denote $N_0 = N \cup G_0$ while $N_0(t)$ is the projection of the section of the set N_0 with a fixed t into E_n. Now let new mappings be defined on T for $j = 1, 2$: $M'_j(t) = M_j(t) - N_0(t)$ for every t satisfying $\mu\{M_j(t) - N_0(t)\} > 0$, $M'_j(t) = I_t$ for every t satisfying $\mu\{M_j(t) - N_0(t)\} = 0$, where the last identity is satisfied on a set of measure zero. In virtue of Lemma 8 there exist functions $\psi^i_{-1}(t)$ and $\psi^i_1(t)$ such that $\psi^i_{(-1)^j}(t) \in M'_j(t)$ holds for $j = 1, 2$ and for every $t \in T$ because the set $\bigcup_{t \in T} (t, M'_j(t))$ is measurable which again is an immediate consequence of the measurability of the functions $x(t)$, $\dot{x}_i(t)$, $f_i(t, x)$ and the set T'_i. This implies (1) for $\psi^i_{-1}(t)$ and $\psi^i_1(t)$. The functions $f_i(t, \psi^i_{(-1)^j}(t))$ are measurable on T and $(t, \psi^i_{(-1)^j}(t)) \in \bigcup_{t \in T} (t, I_t)$ on T, where $K = \overline{\bigcup_{t \in T} (t, I_t)}$ is a compact set. There exists a locally integrable function $m(t)$ such that

$$(15) \qquad \|f(t, x)\| \leqq m(t) \text{ holds a. e. on } K.$$

(Cf. Remark 5.) This yields that $f_i(t, \psi^i_{(-1)^j}(t))$ are integrable functions, therefore, (2) holds. Further, (3) holds for $\psi^i_{(-1)^j}(t)$ because $\psi^i_{(-1)^j}(t) \in I_t$ on T. The definition of the mappings $M'_j(t)$ implies (5) for both functions $\psi^i_{(-1)^j}(t)$. Let us choose $\tau_0 = t_1$ and define $\beta^1_{\tau_0}(t) = \int_{T_i'' \cap \langle \tau_0, t \rangle} (\dot{x}_i(\tau) - f_i(\tau, \psi^i_1(\tau)))\, d\tau$ on T. The function $\beta^1_{\tau_0}(t)$ is nondecreasing on T and $\beta^1_{\tau_0}(\tau_0) = 0$. If the inequality $\beta^1_{\tau_0}(t) < \frac{1}{2}\varepsilon$ hold on T, then we choose $\tau_1 = t_2$. If the contrary is true, then we choose the first $\tau_1 \in T$ for which $\beta^1_{\tau_0}(\tau_1) = \frac{1}{2}\varepsilon$. If $\tau_1 < t_2$ we define

$$\beta^{-1}_{\tau_1}(t) = \int_{T_i'' \cap \langle \tau_1, t \rangle} (\dot{x}_i(\tau) - f_i(\tau, \psi^i_{-1}(\tau)))\, d\tau \quad \text{on} \quad \langle \tau_1, t_2 \rangle .$$

This function is nonincreasing and $\beta^{-1}_{\tau_1}(\tau_1) = 0$. If $\beta^{-1}_{\tau_1}(t) > -\varepsilon$ on $\langle \tau_1, t_2 \rangle$, then we choose $\tau_2 = t_2$. If this last inequality does not hold on the whole $\langle \tau_1, t_2 \rangle$, then we choose the first $\tau_2 \in \langle \tau_1, t_2 \rangle$ for which $\beta^{-1}_{\tau_1}(\tau_2) = -\varepsilon$. If again $\tau_2 < t_2$, we define an analogous function $\beta^1_{\tau_2}(t)$. If $\beta^1_{\tau_2}(t) < \varepsilon$ on $\langle \tau_2, t_2 \rangle$, then we choose $\tau_3 = t_2$. If this inequality does not hold, then we continue further analogously. Now we have defined

$$(16) \qquad \beta^{(-1)^j}_{\tau_j}(t) = \int_{T_i'' \cap \langle \tau_j, t \rangle} (\dot{x}_i(\tau) - f_i(\tau, \psi^i_{(-1)^j}(\tau)))\, d\tau \quad \text{on} \quad \langle \tau_j, t_2 \rangle$$

for $j = 0, 1, 2, \ldots$. We choose $\tau'_0 = \tau_0$ and define auxiliary functions

$$(17) \qquad \lambda^{(-1)^j}_{\tau_j'}(t) = \int_{T_i'' \cap \langle \tau_j', t \rangle} (\dot{x}_i(\tau) + (-1)^j\, m(\tau))\, d\tau \quad \text{on} \quad \langle \tau'_j, t_2 \rangle$$

by the same method as the functions (16), where $m(t)$ is the function from (15).

As

$$f_i(t, \psi_1^i(t)) \leqq m(t) \quad \text{on} \quad T$$

and

$$f_i(t, \psi_{-1}^i(t)) \geqq -m(t) \quad \text{on} \quad T,$$

the inequality $\tau_n' \leqq \tau_n$ holds for $n = 0, 1, \ldots$. If $\tau_n' \leqq t_2$ for all n, then τ_n' converge to a certain τ', $\tau' \leqq t_2$. We get

$$\sum_{n=0}^{\infty} \int_{T_i'' \cap \langle \tau'_{2n}, \tau'_{2n+1} \rangle} (\dot{x}_i(\tau) + m(\tau))\, \mathrm{d}\tau = \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} \varepsilon = +\infty$$

which is a contradiction because $\dot{x}(t)$ and $m(t)$ are absolutely integrable functions on T. Hence it holds $\tau'_{n_0} = t_2$ for a certain positive integer n_0. We have proved that a finite number of steps τ_j is sufficient in (17) and (16). Let us define a function $\beta(t)$ on T:

$$\beta(t) = \beta_{\tau_0}^1(t) \quad \text{for} \quad t \in \langle \tau_0, \tau_1 \rangle,$$

$$\beta(t) = \beta_{\tau_n}^{(-1)^n}(t) + \sum_{j=0}^{n-1} \beta_{\tau_j}^{(-1)^j}(\tau_{j+1}) \quad \text{for} \quad t \in (\tau_n, \tau_{n+1}\rangle,$$

where $n = 1, 2, \ldots$. The inequality $-\frac{1}{2}\varepsilon \leqq \beta(t) \leqq \frac{1}{2}\varepsilon$ holds on T. Let us define a function $\psi_0^i(t)$ on T:

$$\psi_0^i(t) = \psi_1^i(t) \quad \text{on} \quad (\tau_{2n}, \tau_{2n+1}\rangle$$

$$\psi_0^i(t) = \psi_{-1}^i(t) \quad \text{on} \quad (\tau_{2n+1}, \tau_{2n+2}\rangle$$

for $n = 0, 1, 2, \ldots$ and $\psi_0^i(t_1) = \psi_1^i(t_1)$.

This function $\psi_0^i(t)$ satisfies (1), (2), (3), (5) because the functions $\psi_1^i(t)$ and $\psi_{-1}^i(t)$ satisfy the same conditions. Now we must prove that $\psi_0^i(t)$ satisfies the inequality (4), too. It holds

$$\left| x_i(t) - x_i(t_1) - \int_{t_1}^{t} f_i(\tau, \psi_0^i(\tau))\, \mathrm{d}\tau \right| = \left| \int_{t_1}^{t} (\dot{x}_i(\tau) - f_i(\tau, \psi_0^i(\tau)))\, \mathrm{d}\tau \right| \leqq$$

$$\leqq \int_{T_i' \cap \langle t_1, t \rangle} |\dot{x}_i(\tau) - f_i(\tau, \psi_0^i(\tau))|\, \mathrm{d}\tau + \left| \int_{T_i'' \cap \langle t_1, t \rangle} (\dot{x}_i(\tau) - f_i(\tau, \psi_0^i(\tau)))\, \mathrm{d}\tau \right| \leqq$$

$$\leqq \mu(T)\, \varepsilon_1 + |\beta(t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

on T because $x(t)$ is absolutely continuous and owing to the definition of the mappings $M_j'(t)$ on T_i' for $j = 1, 2$. Thus we proved that the inequality (4) holds for the function $\psi_0^i(t)$. Hence the function $x(t)$ is a V-solution of the equation $\dot{x} = f(t, x)$ on T in the given base B.

Theorem 6. ($V \Rightarrow CMF$) *If an absolutely continuous function $x(t)$ is a V-solution of the equation $\dot{x} = f(t, x)$ from Remark 5 on the interval $T = \langle t_1, t_2 \rangle$ in the given base B, then the condition CMF from Definition 8 holds for $x(t)$ on T.*

Proof. The condition CMF can be written in the form

$$\forall(i)\, \exists(T_1 \subset T : \mu(T_1) = \mu(T))\, \forall(t \in T_1)\, \{\alpha \vee \beta\}$$

because CMF obviously implies this condition and also the proof of the converse implication follows easily: Let us choose for $i = 1, \ldots, n$ a set $T_1^i \subset T$, $\mu(T_1^i) = \mu(T)$ where the condition $\alpha \vee \beta$ holds and denote $T_1 = \bigcap_{i=1}^{n} T_1^i$. The set T_1 so constructed can be used in CMF. The converse to this new condition has the form: $\exists(i)\, \forall(T_1 \subset T :$

$: \mu(T_1) = \mu(T))\, \exists(t \in T_1)\, \{\text{non}\, (\alpha \vee \beta)\}$. This coincides with the condition

$$\exists(i)\, \exists(T' \subset T : \mu^*(T') > 0)\, \forall(t \in T')\, \{\text{non}\, (\alpha \vee \beta)\},$$

where μ^* is the outer measure. Suppose that $\alpha \vee \beta$ does not hold for a certain index i on $T' \subset T$ with $\mu^*(T') > 0$ while $\alpha \vee \beta$ holds on $T - T'$. We shall prove that $x(t)$ is not a V-solution of the equation $\dot{x} = f(t, x)$ on T in the given base B. Now there exists $\varepsilon_t > 0$ and a certain open interval I'_t, $x(t) \in I'_t$ such that

$$\mu\{x \in I'_t : |\dot{x}_i(t) - f_i(t, x)| < \varepsilon_t\} = 0 \quad \text{for every} \quad t \in T'$$

and at the same time there exists a certain open interval I''_t, $x(t) \in I''_t$ such that, for example, $\mu\{x \in I''_t : \dot{x}_i(t) - f_i(t, x) < 0\} = 0$ for every $t \in T'$. To every $t \in T$ let us choose an open interval I_t^0 with the following properties: $x(t) \in I_t^0$ and $\bigcup_{t \in T} (t, I_t^0)$ is bounded and $\overline{I_t^0} \subset G(t)$, where $G(t)$ is the projection of the section of the set G with a fixed t into E_n. Moreover, $I_t^0 \subset I'_t \cap I''_t$ for every $t \in T'$. The intervals I_t^0 can be written on T by the formula $I_t^0 = \prod_{j=1}^{n} I_t^j$, where I_t^j has the form $(x_j(t) - \delta'_{t,j}, x_j(t) + \delta''_{t,j})$. Let us denote $\delta_t = \{\min(\delta'_{t,j}, \delta''_{t,j}) : j = 1, \ldots, n\}$ and $I_t = \prod_{j=1}^{n} (x_j(t) - \delta_t, x_j(t) + \delta_t)$ in the space E_n for every $t \in T$. By virtue of Lemma 7 the set T' is measurable with a positive measure and we can choose ε_t, δ_t positive on T' and measurable on T' with respect to the variable t and such that $\mu\{x \in I_t : |\dot{x}_i(t) - f_i(t, x)| < \varepsilon_t\} = 0$, $\mu\{x \in I_t : \dot{x}_i(t) - f_i(t, x) < 0\} = 0$ for every $t \in T'$. Now there exists a measurable subset $T'' \subset T'$, $\mu(T'') > 0$ such that there exists a positive integer k for which the inequality $1/k < \delta_t$ holds for every $t \in T''$. Let us denote $B(t) = \{x \in I_t : |\dot{x}_i(t) - f_i(t, x)| < \varepsilon_t\} \cup \{x \in I_t : \dot{x}_i(t) - f_i(t, x) < 0\}$ and $N' = \bigcup_{t \in T''} (t, B(t))$. The set N' is measurable because $f_i(t, x)$ is measurable with respect to (t, x) and ε_t, δ_t, $\dot{x}_i(t)$ are measurable with respect to t. Then the measure of the set N' can be written in the form $\mu(N') = \int_{T''} \mu(B(t))\, dt$. Hence $\mu(N') = 0$ because $\mu(B(t)) =$

$= 0$ holds for every $t \in T''$. Given $\Delta > 0$, there exists an open set G_Δ such that $T'' \subset G_\Delta \subset T$ and $\mu(G_\Delta - T'') < \Delta$. For brevity, let us write $b = \int_{T''} \varepsilon_t \, dt$. The value b is positive and obviously b is independent of Δ and ψ, N from (1)–(5). Let us denote by $s_i(t)$ the minorant function of the functions $\dot{x}_i(t) - f_i(t, \psi(t))$ on $\langle t_1, t_2 \rangle$, where $\psi(t)$ is an arbitrary function on T with the property $\psi(t) \in I_t$. The inequality

$$s_i(t) \leqq \dot{x}_i(t) - f_i(t, \psi(t)) \tag{18}$$

holds for such functions ψ and the existence of an integrable minorant $s_i(t)$ follows (cf. Remark 5) from the compactness of the set $\overline{\bigcup_{t \in T} (t, I_t)} \subset G$. Further, let us choose $\Delta > 0$ small enough so that the inequality

$$\int_{G_\Delta \cap T''} \varepsilon_t \, dt + \int_{G_\Delta - T''} s_i(t) \, dt > \frac{b}{2} > 0$$

holds. Let G_Δ be written in the form $G_\Delta = \bigcup_j G_j$, where G_j are the components of the set G_Δ; then

$$\sum_j \left(\int_{G_j \cap T''} \varepsilon_t \, dt + \int_{G_j - T''} s_i(t) \, dt \right) > \frac{b}{2} > 0 \, .$$

This implies the existence of a certain G_j and $b' \in (0, \frac{1}{2}b)$ such that

$$\int_{G_j \cap T''} \varepsilon_t \, dt + \int_{G_j - T''} s_i(t) \, dt > b' > 0 \, . \tag{19}$$

The value b' depends only on ε_t, $s_i(t)$ and G_Δ and does not depend on N and ψ. Now we have defined the functions ε_t, δ_t, $s_i(t)$, the sets T'', G_Δ, N', G_j and the constants $b' > 0$ and k. In the sequel we shall use the fact that the function $x(t)$ is a V-solution of the equation $\dot{x} = f(t, x)$. Let us choose $\varepsilon > 0$ satisfying

$$\varepsilon < \min \left(\frac{1}{k}, \frac{b'}{2} \right) \tag{20}$$

and $N \subset G$, $\mu(N) = 0$, and let us denote the union $N \cup N'$ again by N. From the definition of the set N' and from (5) the inequality

$$\dot{x}_i(t) - f_i(t, \psi(t)) \geqq \varepsilon_t > 0 \quad \text{a. e. on} \quad T'' \tag{21}$$

follows for every function $\psi(t)$ satisfying the conditions (1), (2), (3), (5) and we shall prove that the inequality (4) does. This inequality can be written in the form

$$\left| \int_{t_1}^{t} (\dot{x}_i(\tau) - f_i(\tau, \psi(\tau))) \, d\tau \right| < \varepsilon \quad \text{on} \quad T \, .$$

Further,

$$\int_{G_\Delta} (\dot{x}_i(t) - f_i(t, \psi(t)))\,\mathrm{d}t = \int_{G_\Delta \cap T''} (\dot{x}_i(t) - f_i(t, \psi(t)))\,\mathrm{d}t +$$

$$+ \int_{G_\Delta - T''} (\dot{x}_i(t) - f_i(t, \psi(t)))\,\mathrm{d}t \geqq \int_{G_\Delta \cap T''} \varepsilon_t\,\mathrm{d}t + \int_{G_\Delta - T''} s_i(t)\,\mathrm{d}t$$

holds for every $\psi(t)$ satisfying (1), (2), (3), (5) (cf. (18), (21)). The inequality (19) implies

$$\text{(22)} \qquad \int_{G_j} (\dot{x}_i(t) - f_i(t, \psi(t)))\,\mathrm{d}t > b' > 0 \quad \text{for every} \quad \psi(t)$$

satisfying (1), (2), (3), (5). We assume that $x(t)$ is V-solution. Therefore for a given $\varepsilon > 0$, $\varepsilon < \min(1/k, \frac{1}{2}b')$ and $N \subset G$, $\mu(N) = 0$ there exists a function $\psi(t)$ satisfying (1)–(5). Let $\hat{\psi}(t)$ satisfy the properties (1), (2), (3), (5). Denoting $\varphi(t) = \dot{x}_i(t) - f_i(t, \hat{\psi}(t))$ we can write the last inequality (22) in the form $\int_{G_j} \varphi(t)\,\mathrm{d}t > b' > 0$. Let $G_j = (\tau_1, \tau_2) \subset T$. Now due to (4) $\left|\int_{t_1}^{\tau_1} \varphi(\tau)\,\mathrm{d}\tau\right| < \varepsilon$ must hold for τ_1, hence $-\varepsilon < \int_{t_1}^{\tau_1} \varphi(\tau)\,\mathrm{d}\tau < \varepsilon$ and $\int_{t_1}^{\tau_2} \varphi(\tau)\,\mathrm{d}\tau = \int_{t_1}^{\tau_1} \varphi(\tau)\,\mathrm{d}\tau + \int_{\tau_1}^{\tau_2} \varphi(\tau)\,\mathrm{d}\tau > \int_{t_1}^{\tau_1} \varphi(\tau)\,\mathrm{d}\tau + b' > -\varepsilon + b' > \varepsilon$ holds because the value $\varepsilon > 0$ was defined in (20) so that $2\varepsilon < b'$. The inequality $\left|\int_{t_1}^{\tau_2} \varphi(\tau)\,\mathrm{d}\tau\right| < \varepsilon$ does not hold for τ_2 which is a contradiction to the inequality (4).

This contradiction completes the proof of the theorem.

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Author's address: 199 05 Praha 9, Beranových 130 (Výzkumný a zkušební letecký ústav).