Časopis pro pěstování matematiky

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Časopis pro pěstování matematiky, Vol. 101 (1976), No. 3, 263--270

Persistent URL: `http://dml.cz/dmlcz/117919`

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ON GENERALIZED WEINGARTEN SURFACES

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(Received June 30, 1975)

Following the ideas of A. ŠVEC [1], I am going to present further generalizations of the H- and K-theorems.

1.

Theorem 1. *Let $G \subset \mathscr{R}^2$ be a bounded domain, $M : G \cup \partial G \to E^3$ a surface with a net of lines of curvature, v_1 and v_2 the unit tangent vector fields of these lines, k_1 and k_2 the corresponding principal curvatures. Let $M(\partial G)$ consist of umbilical points. Further, suppose*

$$K \geqq 0\,, \tag{1.1}$$

$$(k_1 - k_2)(v_1 v_1 - v_2 v_2)\, H \geqq 0 \tag{1.2}$$

on M. Then $M(G \cup \partial G)$ is a part of a sphere.

Proof. On M, consider a field ot tangent orthonormal moving frames $\{m; v_1, v_2, v_3\}$. Then

$$\begin{aligned} \mathrm{d}m &= \omega^1 v_1 + \omega^2 v_2\,, \\ \mathrm{d}v_1 &= \omega_1^2 v_2 + \omega_1^3 v_3\,, \\ \mathrm{d}v_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3\,, \\ \mathrm{d}v_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 \end{aligned} \tag{1.3}$$

with the usual integrability conditions. We have

$$\omega_1^3 = a\omega^1\,, \quad \omega_2^3 = c\omega^2 \tag{1.4}$$

$$\begin{aligned} \mathrm{d}a &= \alpha\omega^1 + \beta\omega^2\,, \\ (a - c)\,\omega_1^2 &= \beta\omega^1 + \gamma\omega^2\,, \\ \mathrm{d}c &= \gamma\omega^1 + \delta\omega^2\,; \end{aligned} \tag{1.5}$$

(1.6)
$$\begin{aligned}
d\alpha - 3\beta\omega_1^2 &= A\omega^1 + B\omega^2 ,\\
d\beta + (\alpha - 2\gamma)\,\omega_1^2 &= B\omega^1 + (C + aK)\,\omega^2 ,\\
d\gamma + (2\beta - \delta)\,\omega_1^2 &= (C + cK)\,\omega^1 + D\omega^2 ,\\
d\delta + 3\gamma\omega_1^2 &= D\omega^1 + E\omega^2
\end{aligned}$$

and

(1.7)
$$v_1 a = \alpha , \quad v_2 a = \beta , \quad v_1 c = \gamma , \quad v_2 a = \delta ;$$

(1.8)
$$v_1 H = \tfrac{1}{2}(\alpha + \gamma) , \qquad v_2 H = \tfrac{1}{2}(\beta + \delta) ;$$

(1.9)
$$\begin{aligned}
(a - c)\, v_1 \alpha &= 3\beta^2 + A(a - c) ,\\
(a - c)\, v_1 \beta &= 3\beta\gamma - \alpha\beta + B(a - c) ,\\
(a - c)\, v_1 \gamma &= \beta(\delta - 2\beta) + (C + cK)(a - c) ,\\
(a - c)\, v_1 \delta &= D(a - c) - 3\beta\gamma ,\\
(a - c)\, v_2 \alpha &= 3\beta\gamma + B(a - c) ,\\
(a - c)\, v_2 \beta &= 2\gamma^2 - \alpha\gamma + (C + aK)(a - c) ,\\
(a - c)\, v_2 \gamma &= \gamma(\delta - 2\beta) + D(a - c) ,\\
(a - c)\, v_2 \delta &= E(a - c) - 3\gamma^2 ;
\end{aligned}$$

(1.10)
$$\begin{aligned}
(a - c)\, v_1 v_1 H &= \tfrac{1}{2}(\beta + \delta)\,\beta + \tfrac{1}{2}(a - c)(A + C + cK) ,\\
(a - c)\, v_1 v_2 H &= -\tfrac{1}{2}(\alpha + \gamma)\,\beta + \tfrac{1}{2}(a - c)(B + D) ,\\
(a - c)\, v_2 v_1 H &= \tfrac{1}{2}(\beta + \delta)\,\gamma + \tfrac{1}{2}(a - c)(B + D) ,\\
(a - c)\, v_2 v_2 H &= -\tfrac{1}{2}(\alpha + \gamma)\,\gamma + \tfrac{1}{2}(a - c)(C + E + aK) .
\end{aligned}$$

For

(1.11)
$$f = 2(H^2 - K) = \tfrac{1}{2}(a - c)^2 ,$$

define its covariant derivatives f_i, f_{ij} by

(1.12)
$$\begin{aligned}
df &= f_1\omega^1 + f_2\omega^2 ;\\
df_1 - f_2\omega_1^2 &= f_{11}\omega^1 + f_{12}\omega^2 ,\\
df_2 + f_1\omega_1^2 &= f_{12}\omega^1 + f_{22}\omega^2 .
\end{aligned}$$

Then

$$f_{11} = (c^2 - ac)K + (\alpha - \gamma)^2 + 4\beta^2 + (a - c)(A - C), \tag{1.13}$$
$$f_{22} = (a^2 - ac)K + (\beta - \delta)^2 + 4\gamma^2 + (a - c)(C + E),$$
$$f_{12} = (\alpha - \gamma)(\beta - \delta) + 4\beta\gamma + (a - c)(B - D).$$

Now, set

$$S = (v_1 + v_2)H; \tag{1.14}$$
$$v_1v_1H + v_1v_2H = v_1S,$$
$$v_2v_1H + v_2v_2H = v_2S,$$

i.e.,

$$\beta^2 + \beta\delta + K(ac - c^2) - \beta\gamma - \alpha\beta + (a - c)A + (a - c)B + \tag{1.15}$$
$$+ (a - c)C + (a - c)D - 2(a - c)v_1S = 0,$$
$$\beta\gamma + \gamma\delta - \gamma^2 - \alpha\gamma + (a^2 - ac)K + (a - c)B + (a - c)C +$$
$$+ (a - c)D + (a - c)E - 2(a - c)v_2S = 0.$$

Eliminating A, B, C, D, E from (1.13) and (1.15), we get

$$f_{11} + f_{22} = 4Kf + 2(a - c)(v_1S - v_2S) + \alpha^2 - 3\alpha\gamma + \tag{1.16}$$
$$+ 4\gamma^2 + 4\beta^2 - 3\beta\delta + \delta^2 + \alpha\beta + 2\beta\gamma + \gamma\delta.$$

Now,

$$v_1S - v_2S = v_1v_1H + v_1v_2H - v_2v_1H - v_2v_2H, \tag{1.17}$$

$$(a - c)(v_1S - v_2S) = (a - c)(v_1v_1H - v_2v_2H) - \tfrac{1}{2}(\beta + \delta)\gamma - \tag{1.18}$$
$$- \tfrac{1}{2}(\alpha + \gamma)\beta,$$

and (1.16) turns out to be

$$f_{11} + f_{22} - 4Kf = 2(a - c)(v_1v_1 - v_2v_2)H + (\alpha - \tfrac{3}{2}\gamma)^2 + \tag{1.19}$$
$$+ (\delta - \tfrac{3}{2}\beta)^2 + \tfrac{7}{4}(\beta^2 + \gamma^2).$$

This equation satisfies the conditions of the maximum principle because of (1.1) and (1.2). Thus $H^2 - K = 0$ on $M(\partial G)$ implies $H^2 - K = 0$ on $M(G)$. QED.

Theorem 2. *Let $G \subset \mathscr{R}^2$ be a bounded domain, $M : G \cup \partial G \to E^3$ a surface with a net of lines of curvature, v_1 and v_2 be the fields of the unit tangent vectors of these lines, k_1 and k_2 be the corresponding principal curvatures. Let $M(\partial G)$ consist of umbilical points. On M, suppose*

$$(1.20)\qquad K > 0\,,$$

$$(1.21)\qquad (k_1 - k_2)(v_1v_2 - v_2v_2)K \geqq 0\,,$$

$$(1.22)\qquad \frac{4}{11} \leqq \frac{k_2^2}{k_1^2} \leqq \frac{11}{4}\,.$$

Then $M(G \cup \partial G)$ is a part of a sphere.

Proof. Let us keep the notation of the proof of the previous theorem. Then

$$(1.23)\qquad v_1K = a\gamma + c\alpha\,, \quad v_2K = a\delta + c\beta\,;$$

$$\begin{aligned}(1.24)\qquad (a - c)\,v_1v_1K &= a[\beta(\delta - 2\beta) + K(ac - c^2)] + 3c\beta^2 + \\ &\quad + a(a - c)\,C + c(a - c)\,A + 2(a - c)\,\alpha\gamma\,, \\ (a - c)\,v_2v_2K &= c[\gamma(2\gamma - \alpha) + K(a^2 - ac)] - 3a\gamma^2 + \\ &\quad + c(a - c)\,C + a(a - c)\,E + 2(a - c)\,\beta\delta\,, \\ (a - c)\,v_1v_2K &= c\beta(2\gamma - \alpha) - 3a\beta\gamma + c(a - c)\,B + \\ &\quad + a(a - c)\,D + (a - c)(\alpha\delta - \beta\gamma)\,, \\ (a - c)\,v_2v_1K &= a\gamma(\delta - 2\beta) + 3c\beta\gamma + c(a - c)\,B + \\ &\quad + a(a - c)\,D + (a - c)(\alpha\delta + \beta\gamma)\,.\end{aligned}$$

Set

$$(1.25)\qquad (v_1 + v_2)\,K = S\,.$$

Then

$$\begin{aligned}(1.26)\qquad (v_1v_1 + v_1v_2)\,K &= v_1S\,, \\ (v_2v_1 + v_2v_2)\,K &= v_2S\,.\end{aligned}$$

From (1.24)

$$\begin{aligned}(1.27)\qquad & a(\beta\delta - 2\beta^2) + aK(ac - c^2) + 3c\beta^2 + 2(a - c)\,\alpha\gamma - \\ & - 3a\beta\gamma + 2c\beta\gamma - c\alpha\beta + (a - c)(\alpha\delta + \beta\gamma) - \\ & - (a - c)\,v_1S + (ca - c^2)\,A + (ca - c^2)\,B + (a^2 - ac)\,C + \\ & + (a^2 - ac)\,D = 0\,, \\ & a(\gamma\delta - 2\beta\gamma) + 3c\beta\gamma + (a - c)(\alpha\delta + \beta\gamma) - 3a\gamma^2 + \\ & + 2c\gamma^2 - c\alpha\gamma + Kc(a^2 - ac) + 2(a - c)\,\beta\delta - (a - c)\,v_2S + \\ & + (ca - c^2)\,B + (ca - c^2)\,C + (a^2 - ac)\,D + (a^2 - ac)\,E = 0\,.\end{aligned}$$

Eliminating A, B, C, D, E from (1.13) and (1.27), we get

$$(1.28)\qquad cf_{11} + af_{22} - 2K(a + c)f = (a - c)(v_1 - v_2)S + \\ + \{(a + c)\beta\gamma + c\alpha\beta + a\gamma\delta\} + (3a + c)\beta^2 + \\ + (a + 2c)\beta\delta - (2a + c)\alpha\gamma + (a + 3c)\gamma^2 + \\ + c\alpha^2 + a\delta^2 .$$

From (1.24) and (1.26),

$$(1.29)\qquad cf_{11} + af_{22} - 2K(a + c)f = (a - c)(v_1v_1 - v_2v_2)K + \\ + a\left[\left\{\delta - \left(\frac{1}{2} + \frac{c}{a}\right)\beta\right\}^2 + \left(\frac{11}{4} - \frac{c^2}{a^2}\right)\beta^2\right] + \\ + c\left[\left\{\alpha - \left(\frac{1}{2} + \frac{a}{c}\right)\gamma\right\}^2 + \left(\frac{11}{4} - \frac{a^2}{c^2}\right)\gamma^2\right].$$

This equation satisfies the conditions of the maximum principle because of (1.20) to (1.22). Again, $H^2 - K = 0$ on ∂G implies $H^2 - K = 0$ on G. QED.

Remark. Let us replace (1.21) and (1.22) by the condition $K = ac = \text{const.} > 0$. Then

$$(1.30)\qquad v_1K = c\alpha + a\gamma = 0\,, \quad v_2K = c\beta + a\delta = 0\,.$$

Put

$$(1.31)\qquad \alpha = pa\,, \quad \beta = qa\,, \quad \gamma = -pc\,, \quad \delta = -qc\,.$$

The equation (1.29) turns out to be

$$(1.32)\qquad cf_{11} + af_{22} - 2K(a + c)f = p^2(3a^2c + 2ac^2 + 3c^3) + \\ + q^2(3ac^2 + 2a^2c + 3a^3) = (cp^2 + aq^2)(3a^2 + 2ac + 3c^2)\,,$$

and we get the proof of the K-theorem.

2.

Let us consider the surfaces with nets of lines of curvature (for notation, see our Theorems) for which they are functions $P, Q, T : M \to \mathscr{R}$ such that

$$(2.1)\qquad Pv_1H + Qv_2H + T = 0\,.$$

Following the remark to Theorem 2 in [1], we wish to establish the class of operators (2.1) such that we might be able to prove by means of the maximum principle that each surface satisfying (2.1) is a part of a sphere.

Without loss of generality, (2.1) may be written as

$$v_1 H + R v_2 H = S . \tag{2.2}$$

Applying v_1 and v_2 to (2.2) and using (1.10), we get the equations of the form

$$\begin{aligned}(a - c)(A + RB + C + RD) &= \Phi_1(a, C, \alpha, \beta, \gamma, \delta) ,\\ (a - c)(B + RC + D + RE) &= \Phi_2(a, C, \alpha, \beta, \gamma, \delta) .\end{aligned} \tag{2.3}$$

Now, our task is to eliminate $A, \ldots, E$ from (2.3) and (1.13). For this, the rang of the matrix (of the coefficients at $A, \ldots, E$)

$$(a - c) . \begin{Vmatrix} 1 & R & 1 & R & 0 \\ 0 & 1 & R & 1 & R \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{Vmatrix} \tag{2.4}$$

should be <5. This implies $R = \pm 1$, and our operators are given by

$$(v_1 \pm v_2) H - S = 0 . \tag{2.5}$$

Similarly, for the operators of the form

$$P' v_1 K + Q' v_2 K + T' = 0 , \tag{2.6}$$

our class of "convenable" operators is given again by

$$(v_1 \pm v_2) K - S' = 0 . \tag{2.7}$$

3.

We might give a generalization of Theorems 1 and 2, this being, of course, not as sharp in the suppositions.

Theorem 3. *Let $G \subset \mathcal{R}^2$ be a bounded domain, $M : G \cup \partial G \to E^3$ a surface with a net of lines of curvature, v_1 and v_2 the unit tangent vector fields of these lines, k_1 and k_2 be the corresponding principal curvatures. Let $M(\partial G)$ consist of umbilical points; further, let us suppose*

$$K \geqq 0, \tag{3.1}$$

$$\frac{4}{11} \leqq \frac{k_2^2}{k_1^2} \leqq \frac{11}{4}$$

on M, and let there be a function $F : M \to \mathscr{R}^2$ *satisfying*

$$(k_1 - k_2)(v_1 v_1 - v_2 v_2) F(H, K) \geqq 0, \tag{3.3}$$

$$F_H \geqq 0, \quad F_K \geqq 0, \tag{3.4}$$

$$(k_1 - k_2)(v_1 + v_2) H . \{F_{HH}(v_2 - v_1) H + F_{HK}(v_2 - v_1) K\} \geqq 0, \tag{3.5}$$

$$(k_1 - k_2)(v_1 + v_2) H . \{F_{KH}(v_2 - v_1) H + F_{KK}(v_2 - v_1) K\} \geqq 0 \tag{3.6}$$

on $M(G \cup \partial G)$. *Then* $M(G \cup \partial G)$ *is a part of a sphere.*

Proof. The function S be defined by

$$(v_1 + v_2) F(H, K) + S = 0. \tag{3.7}$$

$$\begin{aligned} &\{F_{HH} . v_1 H + F_{HK} . v_1 K\} . (v_1 + v_2) H + (v_1 v_1 + v_1 v_2) H . F_H + \\ &+ \{F_{KH} . v_1 H + F_{KK} . v_1 K\} . (v_1 + v_2) K + (v_1 v_1 + v_1 v_2) K . F_K + \\ &+ v_1 S = 0, \\ &\{F_{HH} . v_2 H + F_{HK} . v_2 K\} . (v_1 + v_2) H + (v_2 v_1 + v_2 v_2) H . F_H + \\ &+ \{F_{KH} . v_2 H + F_{KK} . v_2 K\} . (v_1 + v_2) K + \\ &+ (v_2 v_1 + v_2 v_2) K . F_K + v_2 S = 0. \end{aligned} \tag{3.8}$$

From (1.8), (1.10), (1.23) and (1.24),

$$\begin{aligned} &(a - c)\{F_{HH}(\tfrac{1}{2}\alpha + \tfrac{1}{2}\gamma) + F_{HK}(a\gamma + c\alpha)\}(\tfrac{1}{2}\alpha + \tfrac{1}{2}\gamma + \tfrac{1}{2}\beta + \tfrac{1}{2}\delta) + \\ &+ \tfrac{1}{2}F_H[\beta^2 + \beta\delta - \beta\gamma - \alpha\beta + (a - c)(A + C + cK + B + D)] + \\ &+ (a - c)\{F_{KH}(\tfrac{1}{2}\alpha + \tfrac{1}{2}\gamma) + F_{KK}(\alpha\gamma + c\alpha)\}(a\gamma + c\alpha + a\delta + c\beta) + \\ &+ F_K[a\{\beta\delta - 2\beta^2 + cK(a - c)\} + (a - c)\gamma\alpha + 3c\beta^2 + (a - c)\alpha\gamma + \\ &+ a(a - c) C + c(a - c) A + a(a - c) D + c(a - c) B - 3a\beta\gamma + \\ &+ (a - c)\alpha\delta + 2c\beta\gamma - c\alpha\beta + (a - c)\beta\gamma] + (a - c) v_1 S = 0, \\ &(a - c)\{F_{HH}(\tfrac{1}{2}\beta + \tfrac{1}{2}\delta) + F_{HK}(a\delta + c\beta)\}(\tfrac{1}{2}\alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\gamma + \tfrac{1}{2}\delta) + \end{aligned} \tag{3.9}$$

$$+ \tfrac{1}{2}F_H[\beta\gamma + \gamma\delta + (a - c)(B + D) - \gamma^2 - \gamma\alpha + (a - c)(C +$$

$$+ aK + E)] + (a - c)\{F_{KH}(\tfrac{1}{2}\beta + \tfrac{1}{2}\delta) + F_{KK}(a\delta + c\beta)\}(a\gamma +$$

$$+ c\alpha + a\delta + c\beta) + F_K\{a\gamma\delta - 2a\beta\gamma + (a - c)\beta\gamma + 3c\beta\gamma +$$

$$+ (a - c)\alpha\delta + a(a - c)D + c(a - c)B + a(a - c)E +$$

$$+ c(a - c)C - 3a\gamma^2 + (a - c)\beta\delta + 2c\gamma^2 - c\alpha\gamma + c(a^2 - ac)K +$$

$$+ (a - c)\beta\delta\} + (a - c)v_2S = 0\,.$$

Multiplying the first two equations (1.13) by $(-\tfrac{1}{2}F_H - cF_K)$ and $(-\tfrac{1}{2}F_H - aF_K)$ resp., and using (3.9), we can eliminate A, B, C, D and E, and we get

$$(3.10)\quad (\tfrac{1}{2}F_H + cF_K)f_{11} + (\tfrac{1}{2}F_H + aF_K)f_{22} - \{\tfrac{1}{2}F_HK + 2F_KH\}\,2f =$$

$$= (a - c)(v_1v_1 - v_2v_2)F + \tfrac{1}{2}F_H[3(\beta - \tfrac{1}{2}\delta)^2 + 3(\gamma - \tfrac{1}{2}\alpha)^2 +$$

$$+ \beta^2 + \gamma^2 + \tfrac{1}{4}\alpha^2 + \tfrac{1}{4}\delta^2] + F_K\left[a\left\{\delta - \left(\frac{1}{2} + \frac{c}{a}\right)\beta\right\}^2 +\right.$$

$$\left.+ a\left(\frac{11}{4} - \frac{c^2}{a^2}\right)\beta^2 + c\left\{\alpha - \left(\frac{1}{2} + \frac{a}{c}\right)\gamma\right\}^2 + c\left(\frac{11}{4} - \frac{a^2}{c^2}\right)\gamma^2\right] +$$

$$+ (a - c)F_{HH}(v_1 + v_2)H\,.\,(v_2 - v_1)H +$$

$$+ (a - c)F_{HK}(v_1 + v_2)H\,.\,(v_2 - v_1)K +$$

$$+ (a - c)F_{KH}(v_1 + v_2)K\,.\,(v_2 - v_1)H +$$

$$+ (a - c)F_{KK}(v_1 + v_2)K\,.\,(v_2 - v_1)K\,.$$

The result follows.

Bibliography

[1] *A. Švec:* Several new characterizations of the sphere. Czech. Math. J., *25* (*100*) 1975, 645—652.

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