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CUTS IN SIMPLE CONNECTED REGIONS AND THE CYCLIC ORDERING OF THE SYSTEM OF ALL BOUNDARY ELEMENTS

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1. In the present paper we shall work in the *extended complex plane* $\mathbf{S}$; the *open complex plane* will be denoted by $\mathbf{E}$. By a *neighbourhood of a point* $z \in \mathbf{E}$ we mean any circle $U(z, \varepsilon) = \{\zeta \in \mathbf{E}; |\zeta - z| < \varepsilon\}$ (where $\varepsilon \in (0, \infty)$); *neighbourhoods of the point* ∞ will be the sets $U(\infty, \varepsilon) = \{\zeta \in \mathbf{S}; |\zeta| > 1/\varepsilon\}$ (where $\varepsilon \in (0, \infty)$ again). Neighbourhoods of points $z \in \mathbf{S}$ will be denoted briefly by $U(z)$ also. $\mathbf{N}$ will always denote *the set of all natural numbers*, $\mathbf{U} = U(0, 1)$ will be the *unit circle*, $\mathbf{C} = \partial \mathbf{U}$ the *unit circumference*. By ϱ^* we denote the *metric* in $\mathbf{S}$ obtained by transferring the cartesian metric of the threedimensional euclidean space by means of the stereographical projection of the unit sphere onto $\mathbf{S}$ (see [4], p. 24).

We shall use the common definition of the *topological limes superior* of a sequence of points $z_n \in \mathbf{S}$ or non-empty sets $M_n \subset \mathbf{S}$: $\operatorname{ls} z_n$ denotes the set of all accumulation points of $\{z_n\}$, $\operatorname{ls} M_n = \bigcup \operatorname{ls} z_n$ where the summation extends over all sequences of points $z_n \in M_n$ (cf. [1]). Note that $\operatorname{ls} z_n$ and $\operatorname{ls} M_n$ are non-empty compact sets (as $\mathbf{S}$ is compact). Besides, in what follows, we shall use the following two simple assertions:

(1) $$\emptyset \neq M_n \subset N_n \quad \text{for all} \quad n \Rightarrow \operatorname{ls} M_n \subset \operatorname{ls} N_n\,,$$

(2) if $\{M_n\}$ is a nonincreasing sequence of (non-empty) sets M_n, then
$$\operatorname{ls} M_n = \bigcap_{n=1}^{\infty} \overline{M}_n\,.$$

The following implication concerns one of the basic properties of any conformal mapping F of Ω onto G (where Ω, G are open subsets of $\mathbf{S}$):

(3) $$z_n \in \Omega\,, \quad \operatorname{ls} z_n \subset \partial\Omega \Rightarrow \operatorname{ls} F(z_n) \subset \partial G\,.$$

It immediately implies that

(4) $$\emptyset \neq M_n \subset \Omega\,, \quad \operatorname{ls} M_n \subset \partial\Omega \Rightarrow \operatorname{ls} F(M_n) \subset \partial G\,.$$

(See [4], pp. 488–489.)

By a *curve in M* we understand any continuous mapping $\varphi : \langle\alpha, \beta\rangle \to M$ (where $-\infty < \alpha < \beta < +\infty$); a *curve* will be a curve in $\mathbf{S}$. If $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$ is a curve, we denote

$$\text{(5)} \qquad i.p.\,\varphi = \varphi(\alpha)\,, \quad e.p.\,\varphi = \varphi(\beta)\,,$$

$$\text{(6)} \qquad \langle\varphi\rangle = \varphi(\langle\alpha, \beta\rangle),\ (\varphi\rangle = \varphi((\alpha, \beta\rangle),\ \langle\varphi) = \varphi(\langle\alpha, \beta)),\ (\varphi) = \varphi((\alpha, \beta))\,.$$

We say that the curve $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$ is *simple*, iff the mapping φ is one-one. By a *closed curve* we understand as usual a curve φ with $i.p.\,\varphi = e.p.\,\varphi$. A *Jordan curve* will be any closed curve $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$ such that both restrictions $\varphi \mid \langle\alpha, \beta)$, $\varphi \mid (\alpha, \beta\rangle$ are one-one. If φ is a Jordan curve in $\mathbf{E}$, we denote by Int φ (Ext φ) *the bounded (unbounded) component of* $\mathbf{S} - \langle\varphi\rangle$. A *Jordan region* will be any region (connected open set) Ω the boundary $\partial\Omega$ of which has the form $\langle\varphi\rangle$ where φ is a Jordan curve (in $\mathbf{S}$).

We introduce *the index of a point* $z \in \mathbf{S} - \langle\varphi\rangle$ *with respect to a closed curve* φ as usual (see e.g. [4], p. 214); notation: $\mathrm{ind}_\varphi\, z$. We say that a Jordan curve φ in $\mathbf{E}$ is *positively (negatively) oriented*, iff $\mathrm{ind}_\varphi = 1$ ($\mathrm{ind}_\varphi = -1$) on Int φ.

If $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$ is a curve, we define the curve $\dot{-}\varphi$ by

$$\text{(7)} \qquad (\dot{-}\varphi)(t) = \varphi(-t)\,, \quad t \in \langle-\beta, -\alpha\rangle\,.$$

If $\psi : \langle\gamma, \delta\rangle \to \mathbf{S}$ is another curve and if $\varphi(\beta) = \psi(\gamma)$, we define the *oriented sum* $\varphi \dot{+} \psi$ of the curves φ, ψ by setting

$$\text{(8)} \qquad (\varphi \dot{+} \psi)(t) = \begin{cases} \varphi(t) & \text{for } t \in \langle\alpha, \beta\rangle\,, \\ \psi(t - \beta + \gamma) & \text{for } t \in \langle\beta, \beta + \delta - \gamma\rangle\,. \end{cases}$$

We write $\varphi \dot{-} \psi$ for $\varphi \dot{+} (\dot{-}\psi)$ and speak of the *oriented difference* of φ, ψ. Sure it is clear what we mean by $\varphi_1 \dot{+} \ldots \dot{+} \varphi_n$ (where $n \geqq 2$).

We say *the curves* $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$, $\psi : \langle\gamma, \delta\rangle \to \mathbf{S}$ *differ only unsubstantially*, iff there is a continuous increasing mapping ω of $\langle\gamma, \delta\rangle$ onto $\langle\alpha, \beta\rangle$ with $\psi = \varphi \circ \omega$.

It is clear that $\langle\dot{-}\varphi\rangle = \langle\varphi\rangle$, that $\langle\varphi\rangle = \langle\psi\rangle$, if the curves φ, ψ differ only unsubstantially, and that $\langle\varphi \dot{\pm} \psi\rangle = \langle\varphi\rangle \cup \langle\psi\rangle$, if the oriented sum (difference) exists.

In what follows we shall use the following "*θ-curve theorem*":

Theorem 1,1. *Suppose* $\varphi = \varphi_1 \dot{+} \varphi_2$ *is a positively oriented Jordan curve,* λ *a simple curve such that* $i.p.\,\lambda = i.p.\,\varphi_1$, $e.p.\,\lambda = e.p.\,\varphi_1$, $(\lambda) \subset \mathrm{Int}\,\varphi$. *Then* $\omega_1 = \varphi_1 \dot{-} \lambda$, $\omega_2 = \varphi_2 \dot{+} \lambda$ *are positively oriented Jordan curves and*

$$\text{(9)} \qquad \mathrm{Int}\,\varphi - (\lambda) = \mathrm{Int}\,\omega_1 \cup \mathrm{Int}\,\omega_2$$

where the sets on the right are disjoint.

(This theorem is an immediate consequence of the well known "*topological θ-curve theorem*" – see e.g. [1] – and of the basic properties of the index of a point with respect to a curve.)

If Ω is a region and φ a curve such that *i.p.* $\varphi \in \partial\Omega$, $(\varphi\rangle \subset \Omega$ we speak of a *curve* φ *from the boundary* $\partial\Omega$ *of the region* Ω *into* Ω (or: *from i.p.* φ *into* Ω). The following theorem is well known in the theory of conformal mappings (see [4], p. 531).

Theorem 1,2. *Suppose that* F *is a conformal mapping of* Ω *onto* $\mathbf{U}$ *and* $\varphi : \langle\alpha, \beta\rangle \to \to \mathbf{S}$ *is a curve from* $\partial\Omega$ *into* Ω. *Then the limit* $(F \circ \varphi)(\alpha+)$ *exists. Besides, if* $\varphi^* : \langle\alpha^*, \beta^*\rangle \to \mathbf{S}$ *is another curve from* $\partial\Omega$ *into* Ω, *we have*

$$(F \circ \varphi)(\alpha+) = (F \circ \varphi^*)(\alpha^*+),$$

iff the following condition is satisfied:

(10) $\varphi(\alpha) = \varphi^*(\alpha^*)$, *and for each neighbourhood* $U(\varphi(\alpha))$ *there is a curve* λ *in* $U(\varphi(\alpha)) \cap \Omega$ *with i.p.* $\lambda \in \langle\varphi\rangle$, *e.p.* $\lambda \in \langle\varphi^*\rangle$.

Remark 1. An assertion analogous to the first part of Theorem 1,2 holds, of course, for any curve $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$ with $\varphi(\beta) \in \partial\Omega$, $\langle\varphi) \subset \Omega$. Instead of $(F \circ \varphi)(\alpha+)$ we investigate, naturally, the limit $(F \circ \varphi)(\beta-)$.

This implies immediately that for any curve $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$ with $(\varphi) \subset \Omega$ and for any conformal mapping F of Ω onto $\mathbf{U}$ it is consistent to define a curve $\psi : \langle\alpha, \beta\rangle \to \mathbf{S}$ by

$$\psi(t) = \begin{cases} (F \circ \varphi)(\alpha+), & \text{if } t = \alpha, \\ (F \circ \varphi)(t), & \text{if } t \in (\alpha, \beta), \\ (F \circ \varphi)(\beta-), & \text{if } t = \beta. \end{cases} \tag{11}$$

(If, e.g., $\varphi(\alpha) \in \Omega$, we have $(F \circ \varphi)(\alpha+) = F(\varphi(\alpha))$, of course.) We shall say that (under above conditions) the curve (11) is the *F-image of* φ.

Let Ω be a fixed region. Let us write, for a moment, $\varphi \sim \psi$, iff $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$, $\varphi^* : \langle\alpha^*, \beta^*\rangle \to \mathbf{S}$ are curves from $\partial\Omega$ into Ω satisfiing (10). It is obvious that the binary relation $\sim$ is reflexive, symmetric, and transitive, hence an equivalence. It partitions the set of all curves from $\partial\Omega$ into Ω into equivalence classes, which we call *bundles of curves* (*from* $\partial\Omega$ *into* Ω). (Cf. [4], p. 527.) By $\mathfrak{S}(\Omega)$ we denote *the set of all bundles* of curves from $\partial\Omega$ into Ω. If $\mathcal{S} \in \mathfrak{S}(\Omega)$, $\varphi \in \mathcal{S}$, $\varphi^* \in \mathcal{S}$, then *i.p.* $\varphi =$ $=$ *i.p.* φ^*. Thus, it is consistent to define $o(\mathcal{S}) =$ *i.p.* φ, where $\varphi \in \mathcal{S}$. The point $o(\mathcal{S})$ will be called *the origin of* $\mathcal{S}$.

In what follows we shall use that

(12) in any bundle $\mathcal{S} \in \mathfrak{S}(\Omega)$ there are simple curves.

(Proof. Let $\varphi \in \mathcal{S}$, $\varphi : \langle\alpha, \beta\rangle \to \mathbf{S}$. Then $\varphi(\alpha) \neq \varphi(\beta)$ and, by a well known theorem — see, e.g., [1] — there is a simple curve $\psi : \langle \alpha, \beta \rangle \to \langle \varphi \rangle$ such that $\psi(\alpha) = \varphi(\alpha)$, $\psi(\beta) = \varphi(\beta)$. Obviously, $\psi \in \mathscr{S}$.)

By Theorem 1,2, we have: If F is a conformal mapping of Ω onto $\boldsymbol{U}$, $\mathscr{S} \in \mathfrak{S}(\Omega)$, $\varphi \in \mathscr{S}$ a curve defined in $\langle \alpha, \beta \rangle$, then the number $(F \circ \varphi)(\alpha+)$ is independent of the choise of the curve $\varphi \in \mathscr{S}$. We denote it by $W_F(\mathscr{S})$. (Cf. with [4], p. 537.)

Thus, for any region Ω conformally equivalent to $\boldsymbol{U}$ and for any conformal mapping F of Ω onto $\boldsymbol{U}$, we have defined the function $W_F : \mathfrak{S}(\Omega) \to \boldsymbol{C}$. By Theorem 1,2,

$$W_F \text{ is one-one (on } \mathfrak{S}(\Omega).) . \tag{13}$$

In what follows it will be important that

$$\overline{W_F(\mathfrak{S}(\Omega))} = \boldsymbol{C} . \tag{14}$$

(The proof of this assertion see, e.g., in [3], p. 402; of course, a little different terminology is used there.)

On the unit circumference $\boldsymbol{C}$ we define a *cyclic ordering* of triples of distinct points: We write $w_1 \prec w_2 \prec w_3$, iff there is a positively oriented Jordan curve $\omega : \langle \alpha, \beta \rangle \to \boldsymbol{S}$ with $\langle \omega \rangle = \boldsymbol{C}$, and a triple of points $t_j \in \langle \alpha, \beta)$ $(j = 1, 2, 3)$ such that $t_1 < t_2 < t_3$ and $w_j = \omega(t_j)$ for $j = 1, 2, 3$. Further, we write $w_1 \preceq w_2 \preceq w_3$, iff either $w_1 \prec w_2 \prec w_3$, or $w_1 = w_2$, or $w_2 = w_3$. Symbols like $w_1 \preceq w_2 \prec w_3$, $w_1 \prec w_2 \preceq w_3$ have an analogous meaning. The symbol

$$w'_1 \prec w'_2 \prec \ldots \prec w'_n \prec \ldots \prec w_0 \prec \ldots \prec w''_n \prec \ldots \prec w''_2 \prec w''_1$$

will also appear; it will mean that there is a positively oriented Jordan curve $\omega :$ $: \langle \alpha, \beta \rangle \to \boldsymbol{S}$ with $\langle \omega \rangle = \boldsymbol{C}$, and points $t'_n, t''_n, t_0 \in \langle \alpha, \beta)$ such that $w'_n = \omega(t'_n)$, $w''_n = \omega(t''_n)$ for each $n \in \boldsymbol{N}$, $w_0 = \omega(t_0)$, and

$$t'_1 < t'_2 < \ldots < t'_n < \ldots < t_0 < \ldots < t''_n < \ldots < t''_2 < t''_1 .$$

If $w_1 \neq w_2$, the set $C_1 = \{w \in \boldsymbol{C};\ w_1 \preceq w \preceq w_2\}$ is an arc of the circumference $\boldsymbol{C}$ joining w_1 with w_2; $\{w \in \boldsymbol{C};\ w_1 \prec w \prec w_2\}$ is the corresponding open arc, $C_2 =$ $= \{w \in \boldsymbol{C};\ w_2 \preceq w \preceq w_1\}$ the complementary arc.

Any mapping

$$f(z) = \begin{cases} \dfrac{az + b}{cz + d} & \text{for} \quad z \in \boldsymbol{E} , \\ a/c & \text{for} \quad z = \infty \end{cases} \tag{15}$$

where $a, b, c, d \in \boldsymbol{E}$ are numbers with $ad - bc \neq 0$ will be called a *linear fractional function*[1]).

[1]) We define $A/0 = \infty$ for $A \in \boldsymbol{S}$, $A \neq 0$.

Besides very familiar properties of linear fractional functions (see [4]) we need, in what follows, the following two ones:

(16) If F is a conformal mapping of $\mathbf{U}$ onto itself, then there is a linear fractional function f such that $f = F$ on $\mathbf{U}$.

(17) If f is a linear fractional function satisfiing $f(\mathbf{U}) = \mathbf{U}$, then the relation $w_1 \prec \prec w_2 \prec w_3$ implies the relation $f(w_1) \prec f(w_2) \prec f(w_3)$.

(For proofs of (16) and (17) see, e.g., [4], p. 470 and 543 resp.)

2. Suppose that Ω is a region and let $\varphi = \varphi_1 \dot{-} \varphi_2$ be a simple or Jordan curve such that the curves φ_1, φ_2 belong to distinct bundles from $\mathfrak{S}(\Omega)$. (The last condition is, obviously, independent of the decomposition of φ into the oriented difference of two curves.) Then the curve φ will be called a *cut in* Ω.

Theorem 2,1. *If φ is a cut in a region Ω conformally equivalent to $\mathbf{U}$, we have*

$$\Omega - (\varphi) = \Omega_1 \cup \Omega_2 \tag{18}$$

where Ω_1, Ω_2 are disjoint regions conformally equivalent to $\mathbf{U}$ and

$$\langle\varphi\rangle \subset \partial\Omega_1 \cap \partial\Omega_2\,, \quad \partial\Omega_1 \cup \partial\Omega_2 = \partial\Omega \cup (\varphi)\,. \tag{19}$$

Proof. By assumption, there is a conformal mapping F of Ω onto $\mathbf{U}$. Let ψ be the F-image of φ. By the definition of a cut and by the second part of Theorem 2,1, ψ is a simple cut in $\mathbf{U}$ [2]). By the "topological" "θ-curve theorem", this implies that

$$\mathbf{U} - (\psi) = U_1 \cup U_2 \tag{20}$$

where U_1, U_2 are disjoint Jordan regions. Besides, there exist two distinct points $w_1, w_2 \in \mathbf{C}$ such that the arcs

$$C_1 = \{w \in \mathbf{C};\ w_1 \precsim w \precsim w_2\}\,, \quad C_2 = \{w \in \mathbf{C};\ w_2 \precsim w \precsim w_1\} \tag{21}$$

have the following property:

$$\partial U_j = C_j \cup (\psi) \quad \text{for} \quad j = 1, 2\,. \tag{22}$$ [3])

This implies that

$$\langle\psi\rangle = \partial U_1 \cap \partial U_2\,, \quad \partial U_1 \cup \partial U_2 = \mathbf{C} \cup (\psi)\,. \tag{23}$$

[2]) i.e. a simple curve which is a cut in $\mathbf{U}$.

[3]) Of course, we have $\{w_1, w_2\} = \{i.p.\ \psi, e.p.\ \psi\}$.

Put

$$(24) \qquad \Omega_j = F_{-1}(U_j) \quad \text{for} \quad j = 1, 2 .$$

Then Ω_j are disjoint regions conformally equivalent to $\boldsymbol{U}$ (since the Jordan regions U_j are conformally equivalent to $\boldsymbol{U}$). The equality (18) follows immediately from (20). The rest of the assertion of theorem 2,1 is an easy consequence of (3) and of the definition of the boundary.

Remark 1. It is easy to see that in the assertion of Theorem 2,1 it is not possible to replace the inclusion $\langle\varphi\rangle \subset \partial\Omega_1 \cap \partial\Omega_2$ by equality. More detailed informations about boundaries of the regions Ω_j are contained in Theorem 6,2 which follows.

Suppose that all assumptions of Theorem 2,1 are fulfilled and let φ^* be another cut in Ω. As it is easy to see, Ω_1 is a component of both $\Omega - (\varphi)$ and $\Omega - (\varphi^*)$, iff either the curves φ, φ^*, or the curves φ, $\dot{-}\varphi^*$ differ only unsubstantially (as only then $\langle\varphi\rangle = \langle\varphi^*\rangle$).

Lemma 1. *Let Ω be a region conformally equivalent to $\boldsymbol{U}$, let φ be a cut in Ω. For each conformal mapping F of Ω onto $\boldsymbol{U}$ denote by ψ_F the F-image of φ; further, put*

$$(25) \qquad C_F^+ = \{w \in \boldsymbol{C};\ i.p.\ \psi_F \leqq w \leqq e.p.\ \psi_F\},$$

$$C_F^- = \{w \in \boldsymbol{C};\ e.p.\ \psi_F \leqq w \leqq i.p.\ \psi_F\}$$

and suppose that U_F^+ resp. U_F^- is the component of $\boldsymbol{U} - (\psi_F)$ satisfiing

$$(26) \qquad \partial U_F^+ = C_F^+ \cup (\psi_F) \quad resp. \quad \partial U_F^- = C_F^- \cup (\psi_F).$$

Then, for any two conformal mappings F, G of Ω onto $\boldsymbol{U}$ we have

$$(27) \qquad F_{-1}(U_F^+) = G_{-1}(U_G^+), \quad F_{-1}(U_F^-) = G_{-1}(U_G^-).$$

Proof. Let $\langle\alpha, \beta\rangle$ be the domain of the cut φ. If F, G are conformal mappings of Ω onto $\boldsymbol{U}$, then $F \circ G_{-1}$ is a conformal mapping of the circle $\boldsymbol{U}$ onto itself. By (16), there is a linear fractional function f such that $f = F \circ G_{-1}$ on $\boldsymbol{U}$. This implies that

$$i.p.\ \psi_F = (F \circ \varphi)(\alpha+) = ((f \circ G) \circ \varphi)(\alpha+) = f((G \circ \varphi)(\alpha+)) = f(i.p.\ \psi_G).$$

Similarly, $e.p.\ \psi_F = f(e.p.\ \psi_G)$. By (17), this implies that

$$(28) \qquad C_F^+ = f(C_G^+), \quad C_F^- = f(C_G^-).$$

Further, it follows that

$$U_F^+ = f(U_G^+), \quad U_F^- = f(U_G^-)$$

so that

$$F_{-1}(U_F^+) = G_{-1}(f_{-1}(f(U_G^+))) = G_{-1}(U_G^+)\,.$$

This is the first equality in (27); the proof of the second one is analogous.

Remark 2. If all assumptions of Lemma 1 are satisfied and if we use the same notation, then the regions $F_{-1}(U_F^+)$, $F_{-1}(U_F^-)$ are independent of the choise of the conformal mapping F (of Ω onto $\boldsymbol{U}$). They depend only of the region Ω and the cut φ. Therefore, it is consistent to define

$$\Omega_\varphi^+ = F_{-1}(U_F^+)\,, \quad \Omega_\varphi^- = F_{-1}(U_F^-)$$

(where F is any conformal mapping of Ω onto $\boldsymbol{U}$ and where U_F^+, U_F^- are as in Lemma 1). We say then that *the component* Ω_φ^+ resp. Ω_φ^- *of* $\Omega - (\varphi)$ *lies on the right side* resp. *left side of the cut* φ.

Example 1. Suppose all assumptions of Theorem 1,1 are satisfied and use the same notation. Then the region Int ω_1 (Int w_2) lies on the right (left) side of the cut λ (in the Jordan region Int φ).

Remark 3. Let φ, φ^* be cuts in a region Ω (conformally equivalent to $\boldsymbol{U}$). Then $\Omega_\varphi^+ = \Omega_{\varphi^*}^+$ (and, as a consequence, $\Omega_\varphi^- = \Omega_{\varphi^*}^+$), iff the curves φ, φ^* differ only unsubstantially. The equality $\Omega_\varphi^+ = \Omega_{\varphi^*}^-$ (and, as a consequence, also the equality $\Omega_\varphi^- = \Omega_{\varphi^*}^+$) holds, iff the curves φ, $\dot{-}\varphi^*$ differ only unsubstantially.

3. Definition. Let Ω be a region conformally equivalent to $\boldsymbol{U}$. For each $n \in \boldsymbol{N}$ let $\varphi_n = \varphi_{n,1} \dot{-} \varphi_{n,2}$ be a cut in Ω and let Ω_n be a component of $\Omega - (\varphi_n)$. Suppose that the following four conditions hold:

I. The sequence $\{\Omega_n\}$ is nonincreasing.

II. For each pair of natural numbers $m \neq n$ we have $(\varphi_m) \cap (\varphi_n) = \emptyset$ and the curves $\varphi_{m,1}, \varphi_{m,2}, \varphi_{n,1}, \varphi_{n,2}$ belong to four distinct bundles from $\mathfrak{S}(\Omega)$.

III. $\bigcap\limits_{n=1}^{\infty} \bar{\Omega}_n \subset \partial\Omega$.

IV. If $\varphi \in \mathscr{S}$, $\varphi^* \in \mathscr{S}^*$, where $\mathscr{S}, \mathscr{S}^* \in \mathfrak{S}(\Omega)$, and if $\langle\varphi\rangle \cap \Omega_n \neq \emptyset \neq \langle\varphi^*\rangle \cap \Omega_n$ for all $n \in \boldsymbol{N}$, then $\mathscr{S} = \mathscr{S}^*$.

Then we shall say that $\{\Omega_n\}$ is a *normal sequence in* Ω.

Remark 1. If Ω_n are as in the above definition, then the following condition (stronger than I) holds:

I*. For each $n \in \boldsymbol{N}$ we have $\bar{\Omega}_{n+1} \cap \Omega \subset \Omega_n$.

Condition I implies, namely, that $\bar{\Omega}_{n+1} \subset \bar{\Omega}_n$. As, by theorem 2.1,

$$\partial\Omega_n \subset \partial\Omega \cup (\varphi_n)\,, \quad \partial\Omega_{n+1} \subset \partial\Omega \cup (\varphi_{n+1})\,,$$

the inclusion $\bar{\Omega}_{n+1} \subset \bar{\Omega}_n$ implies that

$$\Omega_{n+1} \cup (\varphi_{n+1}) = \bar{\Omega}_{n+1} \cap \Omega \subset \bar{\Omega}_n \cap \Omega = \Omega_n \cup (\varphi_n).$$

By condition II, however, $(\varphi_{n+1}) \cap (\varphi_n) = \emptyset$, so that $(\varphi_{n+1}) \subset \Omega_n$. This and the inclusion $\Omega_{n+1} \subset \Omega_n$ imply that $\bar{\Omega}_{n+1} \cap \Omega \subset \Omega_n$.

Theorem 3,1. *Let F be a conformal mapping of Ω onto $\mathbf{U}$. For each $n \in \mathbf{N}$ let $\varphi_n = \varphi_{n,1} \dotplus \varphi_{n,2}$ be a cut in Ω and ψ_n the F-image of φ_n. Then the following* 3 *assertions hold*:

1. *If condition* I *holds, then the condition* II *is equivalent to the following one*: *For each $n \in \mathbf{N}$, we have $\overline{F(\Omega_{n+1})} \cap \mathbf{U} \subset F(\Omega_n)$, and the arc $\overline{F(\Omega_{n+1})} \cap \mathbf{C}$ is a subset of the open arc $\overline{F(\Omega_n)} \cap \mathbf{C} - \{i.p.\, \psi_n,\ e.p.\, \psi_n\}$.*

2. *If conditions I–III hold, then the condition* IV *is equivalent to the statement that the set $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$ contains only one point.*

3. *Suppose that the sequence $\{\Omega_n\}$ is normal in Ω (so that conditions* I–IV *hold) and denote by w the only point of the set $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$. Then the sequence $\{F(\Omega_n)\}$ is normal in $\mathbf{U}$, for each $n \in \mathbf{N}$ the arc $\overline{F(\Omega_{n+1})} \cap \mathbf{C}$ is contained in the open arc $\overline{F(\Omega_n)} \cap \mathbf{C} - \{i.p.\, \psi_n,\ e.p.\, \psi_n\}$, the point w lies, for each $n \in \mathbf{N}$, on the open arc $\overline{F(\Omega_n)} \cap \mathbf{C} - \{i.p.\, \psi_n,\ e.p.\, \psi_n\}$, and the distance of w and the component of $\mathbf{U} - (\psi_n)$ distinct from $F(\Omega_n)$ is positive. Finally,*

$$\bigcap_{n=1}^{\infty} \bar{\Omega}_n = \{z \in \partial\Omega;\ \text{there are } z_n \in \Omega \text{ with } z_n \to z,\ F(z_n) \to w\}. \tag{29}$$

Proof. As we can take $\dot{-}\varphi_n$ instead of φ_n, we may, without loss of generality, suppose that each region Ω_n lies on the right side of the cut φ_n. Then

$$\overline{F(\Omega_n)} \cap \mathbf{C} = \{w \in \mathbf{C};\ i.p.\, \psi_n \leqq w \leqq e.p.\, \psi_n\}$$

for each $n \in \mathbf{N}$.

1. If I is satisfied, we have $\overline{F(\Omega_{n+1})} \cap \mathbf{C} \subset \overline{F(\Omega_n)} \cap \mathbf{C}$ for each $n \in \mathbf{N}$. By Theorem 1,2, condition II is equivalent to the statement that, for any two distinct natural numbers m, n, the sets (ψ_m), (ψ_n) are disjoint and $i.p.\, \psi_m$, $e.p.\, \psi_m$, $i.p.\, \psi_n$, $e.p.\, \psi_n$ are four distinct points. Hence, the arc $\overline{F(\Omega_{n+1})} \cap \mathbf{C}$ is a subset of the open arc $\overline{F(\Omega_n)} \cap$ $\cap\, \mathbf{C} - \{i.p.\, \psi_n,\ e.p.\, \psi_n\}$.

The proof of the reverse assertion is similar.

2. Now suppose conditions I–III hold. Condition III may be, by (2), written

equivalently in the form $\mathrm{ls}\, \Omega_n \subset \partial\Omega$ and it implies, by (4) (where $G = \boldsymbol{U}$ must be set), that

$$\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)} = \mathrm{ls}\, F(\Omega_n) \subset \boldsymbol{C}\,.$$

By a well known theorem, $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$ is a continuum. Obviously, it is both non-empty and not equal to $\boldsymbol{C}$. Hence, it is an arc of the circumference $\boldsymbol{C}$ or a one-point set.

Suppose it is an arc. By (14), there exist two distinct points w', w'' of this arc, not equal to the end points of the arc, and belonging to the set $W_F(\mathfrak{S}(\Omega))$. Hence, there exist two distinct bundles $\mathscr{S}', \mathscr{S}'' \in \mathfrak{S}(\Omega)$ with $w' = W_F(\mathscr{S}')$, $w'' = W_F(\mathscr{S}'')$. Choose curves $\varphi' \in \mathscr{S}'$, $\varphi'' \in \mathscr{S}''$; without loss of generality we may suppose their domain is $\langle 0, 1 \rangle$. Let ψ', ψ'' be the F-image of φ', φ'', respectively. As $w' = \psi'(0)$, $w'' = \psi''(0)$ are interior points of the arc $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$ [4]), they are interior points of each arc $\overline{F(\Omega_n)} \cap \boldsymbol{C}$. Hence, $\langle \psi' \rangle \cap F(\Omega_n) \neq \emptyset \neq \langle \psi'' \rangle \cap F(\Omega_n)$ for each n, and, consequently, $\langle \varphi' \rangle \cap \Omega_n \neq \emptyset \neq \langle \varphi'' \rangle \cap \Omega_n$ for each n. Since the curves φ', φ'' belong to two distinct bundles from $\mathfrak{S}(\Omega)$, the condition IV does not hold.

Reversely, suppose the condition IV does not hold. Then there are curves φ', φ'' belonging to two distinct bundles $\mathscr{S}', \mathscr{S}'' \in \mathfrak{S}(\Omega)$ such that $\langle \varphi' \rangle \cap \Omega_n \neq \emptyset \neq \langle \varphi'' \rangle \cap \Omega_n$ for each n. This implies, as it is easy to see, the points $W_F(\mathscr{S}')$, $W_F(\mathscr{S}'')$ belong to the continuum $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$. By (13), these points are distinct (so that the continuum $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$ contains more than one point).

3. Suppose the sequence $\{\Omega_n\}$ is normal in Ω. According to what we have proved already, the continuum $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$ contains one and only one point; denote it by w. As we easily see, it remains to prove the equality (29); all remaining assertions are either proved already, or they are obvious consequences of what has been said above.

Let $z \in \bigcap_{n=1}^{\infty} \bar{\Omega}_n$. Then there is a sequence of points $z_n \in \Omega_n$ with $z_n \to z$. This implies $\mathrm{ls}\, F(z_n) \subset \mathrm{ls}\, F(\Omega_n) = \bigcap_{n=1}^{\infty} \overline{F(\Omega_n)} = \{w\}$ so that $F(z_n) \to w$. This proves that the left side of (29) is a subset of the right one. Suppose, reversely, that $z \in \partial\Omega$ and that there are points $z_n \in \Omega$ with $z_n \to z$, $F(z_n) \to w$. Since, for each $m \in \boldsymbol{N}$, the distance of the point w and the component of $\boldsymbol{U} - (\psi_m)$ distinct from $F(\Omega_m)$ is positive, there is, for each $m \in \boldsymbol{N}$, an index n_m such that $F(z_n) \in F(\Omega_m)$ for each $n > n_m$. This implies $z_n \in \Omega_m$ for all $n > n_m$ and $z = \lim z_n \in \bar{\Omega}_m$ for all $m \in \boldsymbol{N}$, hence $z \in \bigcap_{m=1}^{\infty} \bar{\Omega}_m$. This completes the proof of the equality (29).

[4]) i.e. points of this arc distinct from both end points of it.

4. We shall say two sequences $\{\Omega_n\}$, $\{\Omega_m^*\}$ (for the time being, of arbitrary non-empty sets) are *mutually inscribed*, iff the following conditions hold:

$$\bigwedge_{n}\bigvee_{m_n}\bigwedge_{m>m_n}[\Omega_m^* \subset \Omega_n], \quad \bigwedge_{m}\bigvee_{n_m}\bigwedge_{n>n_m}[\Omega_n \subset \Omega_m^*]. \tag{30}$$

This represents a binary relation between some pairs of sequences of non-empty sets. As it is easy to see, the relation is reflexive, symmetric, and transitive. If Ω is a fixed region conformally equivalent to $\mathbf{U}$, the above relation partitions the set of all normal sequences in Ω into equivalence classes; these classes will be called *boundary elements of the region* Ω.

Thus, a boundary element of Ω is any non-empty system $\mathscr{H}$ of normal sequences in Ω satisfiing the following two conditions:

A. If $\{\Omega_n\} \in \mathscr{H}$ and if normal sequences $\{\Omega_n\}$, $\{\Omega_m^*\}$ are mutually inscribed, then $\{\Omega_m^*\} \in \mathscr{H}$.

B. If $\{\Omega_n\}$, $\{\Omega_m^*\} \in \mathscr{H}$, then the sequences $\{\Omega_n\}$, $\{\Omega_m^*\}$ are mutually inscribed.

By $\mathfrak{H}(\Omega)$ we denote *the system of all boundary elements of the region* Ω.

The geometric image of a boundary element $\mathscr{H} \in \mathfrak{H}(\Omega)$ will be the set

$$\langle \mathscr{H} \rangle = \bigcap_{n=1}^{\infty} \bar{\Omega}_n \quad \text{where} \quad \{\Omega_n\} \in \mathscr{H}. \tag{31}$$

(Obviously, $\bigcap_{n=1}^{\infty} \bar{\Omega}_n = \bigcap_{m=1}^{\infty} \bar{\Omega}_m^*$ for any two mutually inscribed normal sequences $\{\Omega_n\}$, $\{\Omega_m^*\}$ so that the definition is consistent.)

If any conformal mapping F of Ω onto $\mathbf{U}$ is given, we define the following mapping γ_F of the system $\mathfrak{H}(\Omega)$ into $\mathbf{C}$: For each $\mathscr{H} \in \mathfrak{H}(\Omega)$, $\gamma_F(\mathscr{H})$ is the only point of the set $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$ where $\{\Omega_n\} \in \mathscr{H}$. (The set $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$ contains, by theorem 3,1, one and only one point, which is, obviously, independent of the choise of $\{\Omega_n\} \in \mathscr{H}$.)

Theorem 4,1. *If F is a conformal mapping of Ω onto $\mathbf{U}$, then γ_F is a one-one mapping of $\mathfrak{H}(\Omega)$ onto $\mathbf{C}$.*

Proof. First we prove the mapping γ_F is one-one: Let $\gamma_F(\mathscr{H}) = \gamma_F(\mathscr{H}^*) = w$ for a pair of elements $\mathscr{H}, \mathscr{H}^* \in \mathfrak{H}(\Omega)$. Let $\{\Omega_n\} \in \mathscr{H}$, $\{\Omega_m^*\} \in \mathscr{H}^*$. Then $\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)} = \{w\}$, and, by theorem 3,1, $\operatorname{dist}(w, \bar{U} - \overline{F(\Omega_n)}) > 0$ [5]) for each $n \in \mathbf{N}$. Hence, for each $n \in \mathbf{N}$ there is a neighbourhood $U_n(w)$ such that $U_n(w) \cap \bar{U} \subset \overline{F(\Omega_n)}$.

[5]) $\mathbf{U} - \overline{F(\Omega_n)}$ is the component of $\mathbf{U} - (\psi_n)$ distinct from $F(\Omega_n)$.

Since $\{w\} = \bigcap_{m=1}^{\infty} \overline{F(\Omega_m^*)}$ so that, obviously, diam $F(\Omega_m^*) \to 0$, there is an m_n such that $\overline{F(\Omega_m^*)} \subset U_n(w)$ for all $m > m_n$. This proves that

$$\bigwedge_n \bigvee_{m_n} \bigwedge_{m>m_n} [\overline{F(\Omega_m^*)} \subset \overline{F(\Omega_n)}] .$$

Since the interior of the closure of any Jordan region is equal to this region (see [4], p. 556), it follows that

$$\bigwedge_n \bigvee_{m_n} \bigwedge_{m>m_n} [F(\Omega_m^*) \subset F(\Omega_n)] .$$

This implies the first condition of (30); the second one holds similarly. This proves the sequences $\{\Omega_n\}$, $\{\Omega_m^*\}$ are mutually inscribed so that $\mathscr{H} = \mathscr{H}^*$. This completes the proof the mapping γ_F is one-one.

For the proof of the implication

(32) $$w \in \mathbf{C} \Rightarrow \text{there is an } \mathscr{H} \in \mathfrak{H}(\Omega) \text{ with } \gamma_F(\mathscr{H}) = w$$

we need the following auxiliary assertion:

(33) For each $w \in \mathbf{C}$ and each $U(w)$ there is a cut φ in Ω with $\overline{F(\Omega_\varphi^+)} \subset U(w)$ such that the point w is an interior point of the arc $\overline{F(\Omega_\varphi^+)} \cap \mathbf{C}$.

First, let us prove the implication (32) by means of (33). The assertion (33) easily implies the existence of cuts φ_n in Ω such that:

a) $F(\Omega_{\varphi_{n+1}}^+) \subset F(\Omega_{\varphi_n}^+) \cap U(w, 1/n)$ for each $n \in \mathbf{N}$,

b) denoting by ψ_n the F-image of φ_n we have $\langle\psi_{n+1}\rangle \cap \langle\psi_n\rangle = \emptyset$ for each n and

$$i.p.\ \psi_1 \prec \ldots \prec i.p.\ \psi_n \prec \ldots \prec w \prec \ldots \prec e.p.\ \psi_n \prec \ldots \prec e.p.\ \psi_1 .$$

By Theorem 3,1, we easily prove the sequence $\{\Omega_{\varphi_n}^+\}$ is normal in Ω. Denoting by $\mathscr{H}$ the boundary element of the region Ω containing the sequence $\{\Omega_{\varphi_n}^+\}$ we obviously have $\gamma_F(\mathscr{H}) = \{w\}$.

It remains to prove (33). Suppose the point $w_0 \in \mathbf{C}$ and its neighbourhood $U(w_0)$ are given. By (14), there are points $w_1, w_2 \in U(w_0) \cap W_F(\mathfrak{S}(\Omega))$ with $w_1 \prec w_0 \prec$ $\prec w_2$. Let $\mathscr{S}_j \in \mathfrak{S}(\Omega)$ $(j = 1, 2)$ be bundles such that $W_F(\mathscr{S}_j) = w_j$. By (12), each bundle $\mathscr{S}_j$ contains a simple curve φ_j; we may suppose the domains of both curves φ_j are equal to a certain interval $\langle\alpha, \beta\rangle$. If ψ_j denotes the F-image of φ_j, then ψ_j is a simple curve from the point w_j into $\mathbf{U}$.

As we easily see, there is a simple curve $\psi : \langle\alpha, \gamma\rangle \to U(w_0)$ such that:

a) $(\psi) \subset \mathbf{U}$,

b) $\psi = \omega_1 \dotplus \omega_2 \dot{-} \omega_3$ where $\omega_1 = \psi_1 \mid \langle\alpha, \delta\rangle$, $\omega_3 = \psi_2 \mid \langle\alpha, \delta\rangle$ for an appropriately chosen $\delta \in (\alpha, \beta\rangle$.

Then the function φ defined on $\langle \alpha, \gamma \rangle$ by

$$\varphi(\alpha) = o(\mathscr{S}_1), \quad \varphi(t) = F_{-1}(\psi(t)) \quad \text{for} \quad t \in (\alpha, \gamma), \quad \varphi(\gamma) = o(\mathscr{S}_2)$$

is, obviously, a cut in Ω. Since the boundary of the Jordan region $F(\Omega_\varphi^+)$, equal to $(\psi) \cup \{w \in \mathbf{C};\ w_1 \preceq w \preceq w_2\}$, is a subset of $U(w_0)$, the same holds for the set $\overline{F(\Omega_\varphi^+)}$. Besides, we have $w_0 \in \{w \in \mathbf{C};\ w_1 \prec w \prec w_2\}$ and $\overline{F(\Omega_\varphi^+)} \cap \mathbf{C} = \{w \in \mathbf{C};\ w_1 \preceq \preceq w \preceq w_2\}$.

This completes the proof of (33).

Theorem 4,2. *Let Ω be a region conformally equivalent to $\mathbf{U}$. Then for each bundle $\mathscr{S} \in \mathfrak{S}(\Omega)$ there is one and only one boundary element $\mathscr{H} \in \mathfrak{H}(\Omega)$ such that for each $\varphi \in \mathscr{S}$ and each sequence $\{\Omega_n\} \in \mathscr{H}$ we have $\langle \varphi \rangle \cap \Omega_n \neq \emptyset$ for all n. This element $\mathscr{H}$ has, further, the following two properties:*

1. *for each conformal mapping F of Ω onto $\mathbf{U}$ we have $W_F(\mathscr{S}) = \gamma_F(\mathscr{H})$;*

2. *if $\mathscr{H}^* \neq \mathscr{H}$ is another boundary element of the region Ω, then for each curve $\varphi \in \mathscr{S}$ and each sequence $\{\Omega_m^*\} \in \mathscr{H}^*$ there is an m_0 such that $\langle \varphi \rangle \cap \Omega_m^* = \emptyset$ for all $m > m_0$.*

Proof. Suppose the conformal mapping F of Ω onto $\mathbf{U}$ is fixed. By Theorem 4,1, for each bundle $\mathscr{S} \in \mathfrak{S}(\Omega)$ there is one and only one boundary element $\mathscr{H} \in \mathfrak{H}(\Omega)$ such that $W_F(\mathscr{S}) = \gamma_F(\mathscr{H})$.

Let $\mathscr{S} \in \mathfrak{S}(\Omega)$ and let $\mathscr{H} = (\gamma_F)_{-1}(W_F(\mathscr{S}))$ be the corresponding boundary element. If $\varphi \in \mathscr{S}$ is a curve defined on $\langle \alpha, \beta \rangle$ and $\{\Omega_n\} \in \mathscr{H}$ an arbitrary sequence, we have $W_F(\mathscr{S}) = (F \circ \varphi)(\alpha+)$ and, also, $\{W_F(\mathscr{S})\} = \bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$. The point $W_F(\mathscr{S})$ is an interior point of any arc $\overline{F(\Omega_n)} \cap \mathbf{C}$. If ψ denotes the F-image of φ, then *i.p.* $\psi = \psi(\alpha) = W_F(\mathscr{S})$. Hence, for each n, $\langle \psi \rangle \cap F(\Omega_n) \neq \emptyset$ (since by Theorem 3,1, $\operatorname{dist}(\psi(\alpha), \mathbf{U} - \overline{F(\Omega_n)}) > 0$). It follows immediately that $\langle \varphi \rangle \cap \Omega_n \neq \emptyset$ for each n.

If $\mathscr{H}^* \in \mathfrak{H}(\Omega)$, $\mathscr{H}^* \neq \mathscr{H}$, $\varphi \in \mathscr{S}$, $\{\Omega_m^*\} \in \mathscr{H}^*$, then $\gamma_F(\mathscr{H}^*) \neq \gamma_F(\mathscr{H})$ and $\operatorname{dist}(\gamma_F(\mathscr{H}^*), \langle \psi \rangle) > 0$ (where ψ is the F-image of φ). Since $\operatorname{diam} F(\Omega_m^*) \to 0$ for $m \to \infty$, we have $F(\Omega_m^*) \cap \langle \psi \rangle = \emptyset$, hence $\Omega_m^* \cap \langle \varphi \rangle = \emptyset$, for all m sufficiently large.

It remains to prove that for each conformal mapping G of Ω onto $\mathbf{U}$ the following implication holds:

If $\mathscr{S} \in \mathfrak{S}(\Omega)$, $\mathscr{H} \in \mathfrak{H}(\Omega)$, $W_F(\mathscr{S}) = \gamma_F(\mathscr{H})$, then $W_G(\mathscr{S}) = \gamma_G(\mathscr{H})$.

Then, however, $G \circ F_{-1}$ is a conformal mapping of $\mathbf{U}$ onto itself, and there is a linear fractional function f such that $f = G \circ F_{-1}$ on $\mathbf{U}$. This implies that for each curve $\varphi : \langle \alpha, \beta \rangle \to \mathbf{S}$, $\varphi \in \mathscr{S}$ we have

$$(34) \qquad W_G(\mathscr{S}) = (G \circ \varphi)(\alpha+) = ((f \circ F) \circ \varphi)(\alpha+) = f((F \circ \varphi)(\alpha+)) = f(W_F(\mathscr{S}))\,.$$

If $\{\Omega_n\} \in \mathscr{H}$, then, further,

$$(35) \qquad \{\gamma_G(\mathscr{H})\} = \bigcap_{n=1}^{\infty} \overline{G(\Omega_n)} = \bigcap_{n=1}^{\infty} \overline{f(F(\Omega_n))} = f(\bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}) = \{f(\gamma_F(\mathscr{H}))\}\,.$$

Hence, the equality $W_F(\mathscr{S}) = \gamma_F(\mathscr{H})$ implies the equality $W_G(\mathscr{S}) = \gamma_G(\mathscr{H})$, which completes the proof of Theorem 4,2.

Definition. Suppose Ω is a region conformally equivalent to $\boldsymbol{U}$ and $\mathscr{S} \in \mathfrak{S}(\Omega)$. *The boundary element* (of the region Ω) *determined by the bundle* $\mathscr{S}$ will be the boundary element $\mathscr{H} \in \mathfrak{H}(\Omega)$ with the property that the condition $\langle\varphi\rangle \cap \Omega_n \neq \emptyset$ for each n holds for a certain (hence, for each) curve $\varphi \in \mathscr{S}$ and for a certain (hence, for each) sequence $\{\Omega_n\} \in \mathscr{H}$.

Remark 1. As we easily see, for the boundary element $\mathscr{H} \in \mathfrak{H}(\Omega)$ determined by the bundle $\mathscr{S} \in \mathfrak{S}(\Omega)$ the following condition holds: If $\varphi : \langle\alpha, \beta\rangle \to \boldsymbol{S}$, $\varphi \in \mathscr{S}$, $\{\Omega_n\} \in \mathscr{H}$, then for each $n \in \boldsymbol{N}$ there is a $\delta_n > 0$ such that $\varphi((\alpha, \alpha + \delta_n)) \subset \Omega_n$.

5. It is convenient to introduce a *cyclic ordering* into the system $\mathfrak{H}(\Omega)$ (where Ω is a region conformally equivalent to $\boldsymbol{U}$) as follows: We write $\mathscr{H}_1 \prec \mathscr{H}_2 \prec \mathscr{H}_3$ [6]), iff for any conformal mapping F of Ω onto $\boldsymbol{U}$ the relation

$$(36) \qquad \gamma_F(\mathscr{H}_1) \prec \gamma_F(\mathscr{H}_2) \prec \gamma_F(\mathscr{H}_3)$$

holds.

Let us note that the validity of (36) for one conformal mapping F of Ω onto $\boldsymbol{U}$ implies the validity of a similar relation for any such mapping. Suppose, namely, G is another conformal mapping of Ω onto $\boldsymbol{U}$. Denoting by f the linear fractional function satisfiing $f = G \circ F_{-1}$ on $\boldsymbol{U}$ we have the equality $\gamma_G(\mathscr{H}) = f(\gamma_F(\mathscr{H}))$ for each boundary element $\mathscr{H} \in \mathfrak{H}(\Omega)$ (cf. (35)). By (17), the relation $\gamma_G(\mathscr{H}_1) \prec \prec \gamma_G(\mathscr{H}_2) \prec \gamma_G(\mathscr{H}_3)$ is a consequence of the relation (36).

Theorem 5,1. *For each boundary element $\mathscr{H}_0$ of the region Ω and for each open set M containing $\langle\mathscr{H}_0\rangle$ there are elements $\mathscr{H}_1, \mathscr{H}_2 \in \mathfrak{H}(\Omega)$ with $\mathscr{H}_1 \prec \mathscr{H}_0 \prec \mathscr{H}_2$ such that*

$$(37) \qquad \mathscr{H}_1 \prec \mathscr{H} \prec \mathscr{H}_2 \Rightarrow \langle\mathscr{H}\rangle \subset M\,.$$

Proof. Suppose the assertion does not hold. Then there is an element $\mathscr{H}_0 \in \mathfrak{H}(\Omega)$ and an open set M containing $\langle\mathscr{H}_0\rangle$ such that for each two elements $\mathscr{H}_1, \mathscr{H}_2 \in \in \mathfrak{H}(\Omega)$ satifiing $\mathscr{H}_1 \prec \mathscr{H}_0 \prec \mathscr{H}_2$ there is an element $\mathscr{H} \in \mathfrak{H}(\Omega)$ such that $\mathscr{H}_1 \prec \prec \mathscr{H} \prec \mathscr{H}_2$ and $\langle\mathscr{H}\rangle - M \neq \emptyset$.

[6]) The confusion of the sign $\prec$ for cyclic ordering in $\boldsymbol{C}$ with the sign now introduced will sure not take place.

Fix a conformal mapping F of Ω onto $\boldsymbol{U}$ and let $w_0 = \gamma_F(\mathscr{H}_0)$. Choose points $w_n^1, w_n^2 \in \boldsymbol{C}$ such that

(38) $$w_n^1 \to w_0\,, \quad w_n^2 \to w_0\,,$$

(39) $$w_1^1 \prec w_2^1 \prec \ldots \prec w_n^1 \prec \ldots \prec w_0 \prec \ldots \prec w_n^2 \prec \ldots \prec w_2^2 \prec w_1^2$$

and denote

(40) $$\mathscr{H}_n^j = (\gamma_F)_{-1}\,(w_n^j) \quad \text{for} \quad j = 1, 2 \quad \text{and} \quad n \in \boldsymbol{N}\,.$$

Then $\mathscr{H}_n^1 \prec \mathscr{H}_0 \prec \mathscr{H}_n^2$ for each n and, by assumption, there are elements $\mathscr{H}_n \in \mathfrak{H}(\Omega)$ with $\mathscr{H}_n^1 \prec \mathscr{H}_n \prec \mathscr{H}_n^2$ and $\langle \mathscr{H}_n \rangle - M \neq \emptyset$. Denote $w_n = \gamma_F(\mathscr{H}_n)$ and choose points $z_n \in \langle \mathscr{H}_n \rangle - M$. By (29), for each $n \in \boldsymbol{N}$ there is a point $z_n^* \in \Omega$ such that

(41) $$\varrho^*(z_n^*, z_n) < \frac{1}{n}\,, \quad \varrho^*(F(z_n^*), w_n) < \frac{1}{n}\,.$$

The relation $\mathscr{H}_n^1 \prec \mathscr{H}_n \prec \mathscr{H}_n^2$ implies that $w_n^1 \prec w_n \prec w_n^2$. Thus, by (38), we have $w_n \to w_0$; (41) implies that $F(z_n^*) \to w_0$ also. There is a convergent subsequence $\{z_{n_k}\}$ of $\{z_n\}$; denoting $z_0 = \lim z_{n_k}$ we have

$$z_{n_k}^* \to z_0\,, \quad F(z_{n_k}^*) \to w_0\,,$$

which implies $z_0 \in \langle \mathscr{H}_0 \rangle$. This is a contradiction to the fact none of the points z_{n_k} lies in the open set M containing $\langle \mathscr{H}_0 \rangle$.

This contradiction completes the proof of Theorem 5,1.

6. It is quite easy to prove the following theorem (the proof of which we do not present, since we need it only for making clear the significance of the assertion which then follows):

Theorem 6,1. *Suppose Ω, Ω^* are two regions conformally equivalent to $\boldsymbol{U}$. Let $\mathscr{H}_j \in \mathfrak{H}(\Omega)$, $\mathscr{H}_j^* \in \mathfrak{H}(\Omega^*)$ be two arbitrary triples of boundary elements such that*

(42) $$\mathscr{H}_1 \prec \mathscr{H}_2 \prec \mathscr{H}_3\,, \quad \mathscr{H}_1^* \prec \mathscr{H}_2^* \prec \mathscr{H}_3^*\,.$$

Then there exists one and only one conformal mapping F of Ω onto Ω^* such that for each $j = 1, 2, 3$ the following implication holds:

(43) $$z_n \in \Omega\,, \quad \operatorname{ls} z_n \subset \langle \mathscr{H}_j \rangle \Rightarrow \operatorname{ls} F(z_n) \subset \langle \mathscr{H}_j^* \rangle\,.$$

Theorem 6,1 shows the cyclic ordering of the system of all boundary elements plays an important role in certain fundamental questions of the theory of conformal mappings. Theorem 6,2 contains several criteria for the relation $\mathscr{H}_1 \prec \mathscr{H}_2 \prec \mathscr{H}_3$.

Note that the verifiing of this relation immediately by the definition is, excluding the most trivial cases, practically impossible, since further properties of conformal mappings F of Ω onto $\boldsymbol{U}$ are unknown. The assertions presented in what follows make it possible to decide (also in many concrete situations) which component of $\Omega - (\varphi)$ lies on the right (left) side of the cut φ in Ω. Further informations, very useful in many aplications, about boundaries of the components of $\Omega - (\varphi)$ are presented also. Besides, these assertions contain several fundamental informations connected with the possibility of a continuous extension of a conformal mapping to a certain part of the boundary of its definition domain.

Theorem 6,2. 1. *Suppose φ is a cut in a region Ω conformally equivalent to $\boldsymbol{U}$; let $\langle\alpha, \beta\rangle$ be its definition domain. Denote by $\mathscr{S}_0$ resp. $\mathscr{S}_1$ the bundle from $\mathfrak{S}(\Omega)$ containing the curve $\varphi \mid \langle\alpha, \frac{1}{2}(\alpha + \beta)\rangle$ resp. $\dot{-}\varphi \mid \langle\frac{1}{2}(\alpha + \beta), \beta\rangle$, and let $\mathscr{H}_0$ resp. $\mathscr{H}_1$ be the boundary element of the region Ω determined by the bundle $\mathscr{S}_0$ resp. $\mathscr{S}_1$.*

Then

$$(44') \qquad \bigcup_{\mathscr{H}_0 \prec \mathscr{H} \prec \mathscr{H}_1} \langle\mathscr{H}\rangle \cup (\varphi) \subset \partial\Omega_\varphi^+ \subset \bigcup_{\mathscr{H}_0 \preceq \mathscr{H} \preceq \mathscr{H}_1} \langle\mathscr{H}\rangle \cup (\varphi),$$

$$(44'') \qquad \bigcup_{\mathscr{H}_1 \prec \mathscr{H} \prec \mathscr{H}_0} \langle\mathscr{H}\rangle \cup (\varphi) \subset \partial\Omega_\varphi^- \subset \bigcup_{\mathscr{H}_1 \preceq \mathscr{H} \preceq \mathscr{H}_0} \langle\mathscr{H}\rangle \cup (\varphi).$$

Further, if both sets $\langle\mathscr{H}_0\rangle$, $\langle\mathscr{H}_1\rangle$ contain only one point, the following equalities hold:

$$(45) \qquad \partial\Omega_\varphi^+ = \bigcup_{\mathscr{H}_0 \preceq \mathscr{H} \preceq \mathscr{H}_1} \langle\mathscr{H}\rangle \cup (\varphi), \quad \partial\Omega_\varphi^- = \bigcup_{\mathscr{H}_1 \preceq \mathscr{H} \prec \mathscr{H}_0} \langle\mathscr{H}\rangle \cup (\varphi).$$

2. *Let all assumptions of the first part of the theorem hold. Let φ_1 be a simple curve satisfiing i.p. $\varphi_1 = \varphi(t_1)$, e.p. $\varphi_1 = \varphi(t_2)$, where $\alpha < t_1 < t_2 < \beta$, and $(\varphi_1) \subset \Omega - (\varphi)$. Put $\varphi_2 = \varphi \mid \langle t_1, t_2\rangle$, $\varphi^* = \varphi_1 \dot{-} \varphi_2$. Suppose the Jordan curve φ^* is positively oriented and* Int $\varphi^* \subset \Omega$.

Then Int $\varphi^* \cup (\varphi_1) \subset \Omega_\varphi^+$.

3. *Let all assumptions of the first part of the theorem hold. If φ is a negatively oriented Jordan curve, then $\Omega_\varphi^+ = \Omega \cap$* Int φ, *and the following two implications hold*:

$$(46) \quad \langle\mathscr{H}\rangle \subset \operatorname{Int}\varphi \Rightarrow \mathscr{H}_0 \prec \mathscr{H} \prec \mathscr{H}_1, \quad \langle\mathscr{H}\rangle \subset \operatorname{Ext}\varphi \Rightarrow \mathscr{H}_1 \prec \mathscr{H} \prec \mathscr{H}_0.$$

4. *Let all assumptions of the first part of the theorem hold. Let φ be a simple curve, λ a simple curve in $\partial\Omega$ such that* a) *i.p. λ = i.p. φ, e.p. λ = e.p. φ,* b) *the Jordan curve $\lambda \dot{-} \varphi$ is positively oriented, and* c) Int $(\lambda \dot{-} \varphi) \subset \Omega$.

Then $\Omega_\varphi^+ =$ Int $(\lambda \dot{-} \varphi)$.

Further, suppose the definition domain of the curve λ is $\langle 0, 1\rangle$. For each $t \in (0, 1)$ let $\mathscr{S}_t \in \mathfrak{S}(\Omega)$ be the bundle containing a curve from $\lambda(t)$ into $\operatorname{Int}(\lambda \dotminus \varphi)$. Denote by $\mathscr{H}_t$ (where $t \in (0, 1)$) the boundary element of the region Ω determined by the bundle $\mathscr{S}_t$. Then the following four assertions hold:

(47) $\mathscr{H}_0 \prec \mathscr{H}_t \prec \mathscr{H}_1$ *for each* $t \in (0, 1)$.

(48) *If* $t_1, t_2 \in (0, 1)$, *then* $\mathscr{H}_0 \prec \mathscr{H}_{t_1} \prec \mathscr{H}_{t_2} \prec \mathscr{H}_1$, *iff* $t_1 < t_2$.

(49) *The function $\gamma_F(\mathscr{H}_t)$ is a one-one and continuous mapping of the interval $\langle 0, 1\rangle$ onto the arc $\{w \in \mathbf{C};\ \gamma_F(\mathscr{H}_0) \leqq w \leqq \gamma_F(\mathscr{H}_1)\}$ of the circumference $\mathbf{C}$.*

(50) $\langle \mathscr{H}_t \rangle = \{\lambda(t)\}$ *for each* $t \in (0, 1)$.

5. *Let all assumptions of the first part of the theorem hold. Suppose φ is a Jordan curve in $\mathbf{E}$ and $\lambda : \langle 0, 1\rangle \to \partial\Omega \cap \mathbf{E}$ is a Jordan curve with i.p. λ = i.p. φ.*

If $(\varphi) \subset \operatorname{Int}\lambda$ (resp. $(\varphi) \subset \operatorname{Ext}\lambda$), denote $G = \operatorname{Int}\lambda \cap \operatorname{Ext}\varphi$ (resp. $G = \operatorname{Ext}\lambda \cap \cap \operatorname{Int}\varphi$) and suppose the curves φ, λ are positively (resp. negatively) oriented. Suppose, further, $G \subset \Omega$, and for each $t \in (0, 1)$ let $\mathscr{S}_t \in \mathfrak{S}(\Omega)$ be the bundle containing a curve from $\lambda(t)$ into G, $\mathscr{H}_t \in \mathfrak{H}(\Omega)$ the boundary element determined by the bundle $\mathscr{S}_t$.

Then $\Omega^+_\varphi = G$, and assertions (47)–(50) hold.

Proof. Since φ will be a fixed cut in the region Ω, we shall write Ω^+ resp. Ω^- instead of Ω^+_φ resp. Ω^-_φ. Let us fix a conformal mapping F of Ω onto $\mathbf{U}$ and denote by ψ the F-image of φ. Then

$$\mathbf{U} - (\psi) = U^+ \cup U^- \tag{51}$$

where U^+, U^- are components of $\mathbf{U} - (\psi)$, hence disjoint Jordan regions. Choose notation so that

$$\partial U^+ = (\psi) \cup C^+ , \quad \partial U^- = (\psi) \cup C^- \tag{52}$$

where

$$C^+ = \{w \in \mathbf{C};\ \psi(\alpha) \leqq w \leqq \psi(\beta)\} , \quad C^- = \{w \in \mathbf{C};\ \psi(\beta) \leqq w \leqq \psi(\alpha)\} . \tag{53}$$

Then, by definition,

$$\Omega^+ = F_{-1}(U^+) . \quad \Omega^- = F_{-1}(U^-) . \tag{54}$$

Since $\mathscr{H}_0$ resp. $\mathscr{H}_1$ is the boundary element determined by the bundle $\mathscr{S}_0$ resp. $\mathscr{S}_1$, we have, by definition of these bundles,

$$\gamma_F(\mathscr{H}_0) = W_F(\mathscr{S}_0) = \psi(\alpha) , \quad \gamma_F(\mathscr{H}_1) = W_F(\mathscr{S}_1) = \psi(\beta) . \tag{55}$$

1. We first prove (by assumptions of the first part of the theorem) the inclusions (44′). The condition $\mathscr{H}_0 \prec \mathscr{H} \prec \mathscr{H}_1$ means, by definition and by (55), that $\psi(\alpha) \prec$

$\prec \gamma_F(\mathscr{H}) \prec \psi(\beta)$, i.e. $\gamma_F(\mathscr{H}) \in \hat{C}^+$. (If L is an arc with end points a, b we denote, in what follows, by $\hat{L}$ the corresponding open arc $L - \{a, b\}$.) This implies $\operatorname{dist}(\gamma_F(\mathscr{H}), \bar{U}^-) > 0$. If $\{\Omega_n\} \in \mathscr{H}$, then $\{\gamma_F(\mathscr{H})\} = \bigcap_{n=1}^{\infty} \overline{F(\Omega_n)}$ and $\operatorname{diam} \overline{F(\Omega_n)} \to 0$. This implies $F(\Omega_n) \subset U^+$ for all sufficiently large n, so that, for such n, we have $\Omega_n \subset \Omega^+$, hence $\bar{\Omega}_n \subset \bar{\Omega}^+$. Thus, $\langle \mathscr{H} \rangle = \bigcap_{n=1}^{\infty} \bar{\Omega}_n \subset \bar{\Omega}^+ \cap \partial\Omega$, and, by Theorem 2,1, $\bar{\Omega}^+ \cap \partial\Omega \subset \partial\Omega^+$. This proves that $\bigcup_{\mathscr{H}_1 \prec \mathscr{H} \prec \mathscr{H}_2} \langle \mathscr{H} \rangle \subset \partial\Omega^+$; by Theorem 2,1, this implies

$$(56) \qquad \bigcup_{\mathscr{H}_1 \prec \mathscr{H} \prec \mathscr{H}_2} \langle \mathscr{H} \rangle \cup (\varphi) \subset \partial\Omega^+ .$$

Now suppose that $z \in \partial\Omega^+ - (\varphi)$; then, by Theorem 2,1, we have $z \in \partial\Omega$. Since $z \in \partial\Omega^+$, there are points $z_n \in \Omega^+$ with $z_n \to z$. Since there is a convergent subsequence, we may suppose that $\lim F(z_n) = w$ exists. Since $F(z_n) \in U^+$, we have $w \in \partial U^+$; since $z \in \partial\Omega$, we have $w \in \mathbf{C}$. This yields $w \in \partial U^+ \cap \mathbf{C} = C^+$.

Let $\mathscr{H} \in \mathfrak{H}(\Omega)$ be the element with $\gamma_F(\mathscr{H}) = w$. Then $\mathscr{H}_0 \leqq \mathscr{H} \leqq \mathscr{H}_1$ and $z \in \langle \mathscr{H} \rangle$. This proves the inclusion

$$(57) \qquad \partial\Omega^- - (\varphi) \subset \bigcup_{\mathscr{H}_0 \leqq \mathscr{H} \leqq \mathscr{H}_1} \langle \mathscr{H} \rangle .$$

(56) and (57) implie (44′). The proof of (44″) is analogous.

Now suppose the set $\langle \mathscr{H}_0 \rangle$ contains only one point z_0. Since $\psi(\alpha) \in \partial U^+$, there are points $w_n \in U^+$ with $w_n \to \psi(\alpha)$. By (29) and by definition of $\langle \mathscr{H}_0 \rangle$, we have $\operatorname{ls} F_{-1}(w_n) \subset \langle \mathscr{H}_0 \rangle$ $(=\{z_0\})$, so that $\lim F_{-1}(w_n) = z_0$. Since $F_{-1}(w_n) \in \Omega^+$, we have, by (3), $z_0 \in \partial\Omega^+$. Hence, the inclusion $\langle \mathscr{H}_0 \rangle \subset \partial\Omega^+$ holds. We prove similarly that $\langle \mathscr{H}_1 \rangle \subset \partial\Omega^+$, if $\langle \mathscr{H}_1 \rangle$ contains one point only. By Theorem 2,1, we have $(\varphi) \subset \partial\Omega^+$.

Thus, by (56), if both $\langle \mathscr{H}_0 \rangle$ and $\langle \mathscr{H}_1 \rangle$ contain one point only, we have

$$(58) \qquad \bigcup_{\mathscr{H}_0 \leqq \mathscr{H} \leqq \mathscr{H}_1} \langle \mathscr{H} \rangle \cup (\varphi) \subset \partial\Omega^+ .$$

This, together with (57), yields the first equality in (45); the proof of the other one is similar.

2. Let all assumptions of the second part of the theorem be fulfilled. By them, we have $\overline{\operatorname{Int} \varphi^*} \subset \Omega$. Since the curve φ^* is positively oriented and since the mapping F is holomorphic on Ω, by a well known theorem (see [4], p. 572), the curve $\psi^* = F \circ \varphi^*$ also is positively oriented.

Choose simple curves ω^+, ω^- such that $\langle \omega^+ \rangle = C^+$, $\langle \omega^- \rangle = C^-$ and that $\omega = \omega^+ + \omega^-$ is a positively oriented Jordan curve. Then, by Theorem 1,1, the curves $\omega^+ \dotminus \psi$, $\omega^- + \psi$ also are positively oriented, and $\mathrm{Int}(\omega^+ \dotminus \psi) = U^+$, $\mathrm{Int}(\omega^- + \psi) = U^-$.

The inclusion $(\psi_1) \subset U^-$ for the curve $\psi_1 = F \circ \varphi_1$ would, by Theorem 1,1, implie the curve $\dotminus\psi^* = F \circ \varphi_2 \dotminus \psi_1 = \psi \mid \langle t_1, t_2 \rangle \dotminus \psi_1$ is positively oriented, which is a contradiction. Therefore, $(\psi_1) \subset U^+$ so that $(\varphi_1) \subset \Omega^+$.

As $\langle \varphi_2 \rangle = \varphi(\langle t_1, t_2 \rangle)$ is a subset of the boundary of the region $\mathrm{Int}\,\varphi^*$, we have

$$\varphi(\langle t_1, t_2 \rangle) \cap \mathrm{Int}\,\varphi^* = \emptyset\,. \tag{59}$$

From the inclusion $\overline{\mathrm{Int}\,\varphi^*} \subset \Omega$ it follows that $\partial\Omega \subset \mathbf{S} - \Omega \subset \mathrm{Ext}\,\varphi^*$. Since the sets $\varphi(\langle \alpha, t_1))$, $\varphi((t_2, \beta\rangle)$ are connected and disjoint with $\langle \varphi^* \rangle = \partial(\mathrm{Ext}\,\varphi^*)$, and since the sets $\varphi(\langle \alpha, t_1)) \cap \partial\Omega$, $\varphi((t_2, \beta\rangle) \cap \partial\Omega$ (containing $\varphi(\alpha)$, $\varphi(\beta)$, respectively) are non-empty, we have

$$\varphi(\langle \alpha, t_1)) \cup \varphi((t_2, \beta\rangle) \subset \mathrm{Ext}\,\varphi^*\,. \tag{60}$$

(59) and (60) implie $\langle \varphi \rangle \cap \mathrm{Int}\,\varphi^* = \emptyset$. Thus, the connected set $\mathrm{Int}\,\varphi^* \subset \Omega$ is a subset of one of the components Ω^+, Ω^- of the set $\Omega - (\varphi)$, whereas $\overline{\mathrm{Int}\,\varphi^*}$ is disjoint with the other one. Since $(\varphi_1) \subset \Omega^+ \cap \overline{\mathrm{Int}\,\varphi^*}$, we have $\mathrm{Int}\,\varphi^* \cup (\varphi_1) \subset \Omega^+$, which completes the proof of the second part of the theorem.

3. Let all assumptions of the third part of the theorem be fulfilled. It is not too difficult to prove the sets

$$\Omega \cap \mathrm{Int}\,\varphi\,, \quad \Omega \cap \mathrm{Ext}\,\varphi \tag{61}$$

are components of the set $\Omega - (\varphi)$. (The proof will be left to the reader.) In order to prove $\Omega^+ = \Omega \cap \mathrm{Int}\,\varphi$ it is sufficient, by the 2. part of the present theorem, to find a curve φ_1 with properties mentioned there and such that $(\varphi_1) \subset \Omega \cap \mathrm{Int}\,\varphi$.

We prove easily that

$$\partial(\Omega \cap \mathrm{Int}\,\varphi) = (\partial\Omega \cap \mathrm{Int}\,\varphi) \cup \langle \varphi \rangle \tag{62}$$

and that the set (φ) is open in $\partial(\Omega \cap \mathrm{Int}\,\varphi)$. By a well known theorem (see e.g. [4], p. 527), the set of all points $z \in (\varphi)$ accessible from $\Omega \cap \mathrm{Int}\,\varphi$ (i.e. all points $z \in (\varphi)$ such that there is a simple curve from z into $\Omega \cap \mathrm{Int}\,\varphi$) is dense in (φ). From this it follows easily there are numbers $t_1, t_2 \in (\alpha, \beta)$, $t_1 < t_2$, and a simple curve φ_1 such that

$$i.p.\ \varphi_1 = \varphi(t_1)\,, \quad e.p.\ \varphi_1 = \varphi(t_2)\,, \quad (\varphi_1) \subset \Omega \cap \mathrm{Int}\,\varphi\,. \tag{63}$$

Put $\varphi_2 = \varphi \mid \langle t_1, t_2 \rangle$. Since the curve φ, by assumptions, is negatively oriented, the curve $\varphi^* = \varphi_1 \dotminus \varphi_2$ is, by Theorem 1,1, oriented positively.

By the same theorem, $\mathrm{Int}\,\varphi^* \subset \mathrm{Int}\,\varphi$. Obviously, $\varphi(\alpha) \in \mathrm{Ext}\,\varphi^*$. Since $\langle \varphi^* \rangle \subset \Omega$, we have $\langle\varphi^*\rangle \cap (\mathbf{S} - \Omega) = \emptyset$, so that the connected set $\mathbf{S} - \Omega$ [7]) is disjoint either with Int φ^*, or with Ext φ^*. Since the set $(\mathbf{S} - \Omega) \cap \operatorname{Ext} \varphi^*$ contains $\varphi(\alpha)$, we have $(\mathbf{S} - \Omega) \cap \operatorname{Int} \varphi^* = \emptyset$ so that $\operatorname{Int} \varphi^* \subset \Omega$.

Thus, we have $\operatorname{Int} \varphi^* \subset \Omega \cap \operatorname{Int} \varphi$. This proves, by the 2. part of the theorem, that $\Omega \cap \operatorname{Int} \varphi = \Omega^+$.

It remains to prove the implications (46). Let $\mathscr{H} \in \mathfrak{H}(\Omega)$, $\langle\mathscr{H}\rangle \subset \operatorname{Int} \varphi$, $\{\Omega_n\} \in \mathscr{H}$. As $\langle\mathscr{H}\rangle = \bigcap_{n=1}^{\infty} \bar{\Omega}_n$, it follows from the inclusion $\langle\mathscr{H}\rangle \subset \operatorname{Int} \varphi$ that $\bar{\Omega}_n \subset \operatorname{Int} \varphi$ for all n sufficiently large. For such n we have, further, $F(\Omega_n) \subset F(\Omega \cap \operatorname{Int} \varphi) = F(\Omega^+) = U^+$ so that the arc $\overline{F(\Omega_n)} \cap \mathbf{C}$ is a subset of C^+. By Theorem 3,1, this implies $\gamma_F(\mathscr{H})$ is a point of the open arc $\hat{C}^+$, i.e. $\psi(\alpha) \prec \gamma_F(\mathscr{H}) \prec \psi(\beta)$, which means that $\mathscr{H}_0 \prec \mathscr{H} \prec \mathscr{H}_1$.

This completes the proof of the first implication (46); the proof of the second one is analogous.

4. Now let all assumptions of the fourth part of the theorem hold. Since the region $G = \operatorname{Int}(\lambda \dotplus \varphi)$, by these assumptions, is contained in Ω and since it is disjoint with $\langle\varphi\rangle$, it is contained in a certain component Ω^* of the set $\Omega - (\varphi)$. Provided that $G \neq \Omega^*$, the region Ω^* would intersect both G and $\mathbf{S} - G$, hence ∂G also. This, however, is a contradistion, as

$$\Omega^* \cap \partial G = \Omega^* \cap (\langle\lambda\rangle \cup \langle\varphi\rangle) \subset (\Omega - (\varphi)) \cap (\partial\Omega \cup (\varphi)) = \emptyset\,.$$

Hence $G = \Omega^*$, which means G is a component of the set $\Omega - (\varphi)$.

Let us prove that $G = \Omega^+$. The conformal mapping $F \mid G$ (of the Jordan region G onto one component of the set $\mathbf{U} - (\psi)$, hence onto a Jordan region) may be, by a well known theorem (see [4], p. 538), extended to a homeomorphic mapping F^* of $\bar{G}$ onto $\overline{F(G)}$. According to another well known theorem (see [4], p. 541) the curve $F^* \circ (\lambda \dotplus \varphi) = F^* \circ \lambda \dotplus \psi$ has the same orientation as the curve $\lambda \dotplus \varphi$, hence the positive one. The curve $F^* \circ \lambda$ is simple, and $\langle F^* \circ \lambda\rangle$ is equal either to C^+ or to C^-. Provided that $\langle F^* \circ \lambda\rangle = C^-$, the curve $F^* \circ \lambda \dotplus \psi$ would, obviously, be negatively oriented. Hence $\langle F^* \circ \lambda\rangle = C^+$, which implies $F(G) = U^+$ and $G = \operatorname{Int}(\lambda \dotplus \varphi) = \Omega^+$, as we had to prove.

Let us note that in consequence of what has been said above also the following assertion holds:

(64) The mapping $F^* \circ \lambda$ admits of an extension to a positively oriented Jordan curve χ such that $i.p.\ \chi = i.p.\ (F^* \circ \lambda) = \psi(\alpha)$, $\langle\chi\rangle = \mathbf{C}$.

Now let us suppose the curve λ is defined on the interval $\langle 0, 1\rangle$. $\mathscr{S}_t$, $\mathscr{H}_t$ (where $t \in \langle 0, 1\rangle$) let be defined as in assumptions. Let $t \in (0, 1)$ and suppose $\mu_t \in \mathscr{S}_t$ is a curve from $\lambda(t) = o(\mathscr{S}_t)$ into $G = \operatorname{Int}(\lambda \dotplus \varphi)$ defined on $\langle 0, 1\rangle$. Then

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[7]) The region Ω is conformally equivalent to $\mathbf{U}$, hence its complement is connected.

$$\gamma_F(\mathscr{H}_t) = W_F(\mathscr{S}_t) = (F \circ \mu_t)(0+) = F^*(\mu_t(0)) = F^*(\lambda(t))\,;$$

besides, obviously,

$$\gamma_F(\mathscr{H}_0) = \psi(\alpha) = F^*(\lambda(0))\,, \quad \gamma_F(\mathscr{H}_1) = \psi(\beta) = F^*(\lambda(1))\,.$$

From this it follows that

$$\gamma_F(\mathscr{H}_t) = F^*(\lambda(t)) \quad \text{for each} \quad t \in \langle 0, 1\rangle\,. \tag{65}$$

Since $F^* \circ \lambda$ is one-one and continuous on $\langle 0, 1\rangle$ and $\langle F^* \circ \lambda\rangle = C^+$, (49) holds. By (64), (47) and (48) also hold.

Thus, it remains to prove (50). If $t \in (0, 1)$ and $z \in \langle \mathscr{H}_t\rangle$, there are points $z_n \in \Omega$ such that $z_n \to z$, $F(z_n) \to \gamma_F(\mathscr{H}_t) = F^*(\lambda(t))$ (cf. (29)). Since $F^*(\lambda(t)) \in \hat{C}^+$, we have $F(z_n) \in U^+$ for all n sufficiently large. Since the mapping $(F^*)_{-1}$ is continuous on $\bar{U}^+$, the relation $F(z_n) \to F^*(\lambda(t))$ implies $z_n \to \lambda(t)$. Hence $z = \lambda(t)$. Thus, $\lambda(t)$ is the only point of the set $\langle \mathscr{H}_t\rangle$.

5. Let all assumptions of the fifth part of the theorem hold. Suppose first $(\varphi) \subset \subset \operatorname{Int} \lambda$. Let

$$G = \operatorname{Int} \lambda \cap \operatorname{Ext} \varphi \subset \Omega \tag{66}$$

and suppose the curves φ, λ are positively oriented. As $\langle\lambda\rangle \subset \partial\Omega$, we have either $\Omega \subset \operatorname{Int}\lambda$ or $\Omega \subset \operatorname{Ext}\lambda$. Hence, the inclusion $(\varphi) \subset \Omega \cap \operatorname{Int}\lambda$ implies $\Omega \subset \operatorname{Int}\lambda$. As we easily see, the set G is a component of the set $\operatorname{Int}\lambda - (\varphi)$. As $G \subset \Omega$, the inclusion $G \subset \operatorname{Int}\lambda$ implies G is a component of the set $\Omega - (\varphi)$ also. The other component of the set $\Omega - (\varphi)$ equals to $\Omega \cap \operatorname{Int}\varphi$. Besides, obviously,

$$\partial G = \langle\lambda\rangle \cup \langle\varphi\rangle \text{ [8]}\,. \tag{67}$$

By the theorem on accessibility of points of the boundary of any Jordan region from this region (see [4], p. 196), it immediately follows there is a simple curve λ^* and numbers $t^* \in (0, 1)$, $T^* \in (\alpha, \beta)$ such that

$$i.p.\,\lambda^* = \lambda(t^*)\,, \quad e.p.\,\lambda^* = \varphi(T^*)\,, \quad (\lambda^*) \subset G\,. \tag{68}$$

Take

$$\lambda_1 = \lambda \,|\, \langle 0, t^*\rangle\,, \quad \lambda_2 = \lambda \,|\, \langle t^*, 1\rangle\,, \quad \varphi_1 = \varphi \,|\, \langle \alpha, T^*\rangle\,, \quad \varphi_2 = \varphi \,|\, \langle T^*, \beta\rangle\,. \tag{69}$$

Then the Jordan curves

$$\nu_1 = \lambda_1 \dotplus \lambda^* \dot{-} \varphi_1\,, \quad \nu_2 = \lambda_2 \dot{-} \varphi_2 \dot{-} \lambda^* \tag{70}$$

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[8]) This is an analogy of the topological θ-curve theorem (see [1]). Instead of a topological circumference (a set homeomorphic to **C**) and an arc the end points of which are the only points common with the circumference, here we have two topological circumferences with one and only one point common.

are, by Theorem 1,1, positively oriented. Besides, it is obvious that

$$\operatorname{ind}_{\nu_1} + \operatorname{ind}_{\nu_2} = \operatorname{ind}_\lambda - \operatorname{ind}_\varphi \tag{71}$$

on $\boldsymbol{S} - (\langle\lambda\rangle \cup \langle\varphi\rangle \cup \langle\lambda^*\rangle)$.

If $z \in \operatorname{Int} \nu_j$ for $j = 1$ or $j = 2$, then (71) implies that $\operatorname{ind}_\lambda z - \operatorname{ind}_\varphi z = \operatorname{ind}_{\nu_1} z + \operatorname{ind}_{\nu_2} z \geqq \operatorname{ind}_{\nu_j} z = 1$. From this it follows that $\operatorname{ind}_\lambda z = 1$, $\operatorname{ind}_\varphi z = 0$, which means that $z \in \operatorname{Int} \lambda \cap \operatorname{Ext} \varphi = G$. This proves the inclusion

$$\operatorname{Int} \nu_1 \cup \operatorname{Int} \nu_2 \subset G\,. \tag{72}$$

Thus, any curve going from the point $\lambda(t)$, where $0 < t < t^*$ resp. $t^* < t < 1$, into $\operatorname{Int} \nu_1$ resp. $\operatorname{Int} \nu_2$ goes into G also.

Taking into account that $\lambda^* \in \mathscr{S}_{t^*}$, $\varphi_1 \in \mathscr{S}_0$, $\dot{-}\varphi_2 \in \mathscr{S}_1$ we see that, by the 4. part of the theorem (aplied to the curves $\nu_1 = \lambda_1 \dot{-} (\varphi_1 \dot{-} \lambda^*)$, $\nu_2 = \lambda_2 \dot{-} (\lambda^* \dot{+} \varphi_2)$), we have

$$\mathscr{H}_0 \prec \mathscr{H}_t \prec \mathscr{H}_{t^*} \quad \text{for} \quad t \in (0, t^*)\,, \quad \mathscr{H}_{t^*} \prec \mathscr{H} \prec \mathscr{H}_1 \quad \text{for} \quad t \in (t^*, 1)\,. \tag{73}$$

From this (47) and (48) follow easily.

By the 4. part of the present theorem, the function $\gamma_F(\mathscr{H}_t)$ where $t \in \langle 0, t^*\rangle$ resp. $t \in \langle t^*, 1\rangle$, is one-one and continuous, and

$$\{\gamma_F(\mathscr{H}_t);\ t \in \langle 0, t^*\rangle\} = \{w \in \mathbf{C};\ \gamma_F(\mathscr{H}_0) \leqq w \leqq \gamma_F(\mathscr{H}_{t^*})\} \tag{74'}$$

resp.

$$\{\gamma_F(\mathscr{H}_t);\ t \in \langle t^*, 1\rangle\} = \{w \in \mathbf{C};\ \gamma_F(\mathscr{H}_{t^*}) \leqq w \leqq \gamma_F(\mathscr{H}_1)\}\,. \tag{74''}$$

This proves (49). It is obvious also that $F(G) = U^+$, which implies $G = \Omega^+$.

The assertion (50) will be proved similarly as in the proof of the fourth part of the theorem.

This completes the proof of the 5. part of the theorem in case that $(\varphi) \subset \operatorname{Int} \lambda$. If $(\varphi) \subset \operatorname{Ext} \lambda$ (and if corresponding assumptions of the 5. part hold), we proof analogously the components of the set $\Omega - (\varphi)$ are the sets $G = \operatorname{Ext} \lambda \cap \operatorname{Int} \varphi$, $\Omega \cap \operatorname{Ext} \varphi$; (67) also holds.

Defining the curves ν_1, ν_2 by (70) we prove once more they are positively oriented. The rest of the proof also is similar as in case $(\varphi) \subset \operatorname{Int} \lambda$.

This completes the proof of Theorem 6,2.

Remark 1. Theorem 6,2 yields some informations of the relations between $\langle\mathscr{H}_0\rangle$ resp. $\langle\mathscr{H}_1\rangle$ and $\partial\Omega_\varphi^+$, $\partial\Omega_\varphi^-$. In the general case, however, not much can be said. Of course, it is e.g. $o(\mathscr{S}_0) \in \langle\mathscr{H}_0\rangle \cap \partial\Omega_\varphi^+ \cap \partial\Omega_\varphi^-$ and $\langle\mathscr{H}_0\rangle \subset \partial\Omega_\varphi^+ \cup \partial\Omega_\varphi^-$. In what follows we show by examples that the relations between $\langle\mathscr{H}_0\rangle$ (and, similarly, $\langle\mathscr{H}_1\rangle$) and $\partial\Omega_\varphi^+$, $\partial\Omega_\varphi^-$ may be rather complicated.

Example 1. The inclusion $\langle\mathscr{H}_0\rangle \subset \partial\Omega_\varphi^+ \cap \partial\Omega_\varphi^-$, as we know, holds if $\langle\mathscr{H}_0\rangle$ contains one point only. However, it also may hold in case $\langle\mathscr{H}_0\rangle$ is a proper continuum. If e.g.

$$(76)\qquad \Omega = \{z \in \mathbf{E};\ 0 < \operatorname{Re} z < 2,\ |\operatorname{Im} z| < 1\} - \left(\langle 0, 1\rangle \cup \bigcup_{n=2}^{\infty} \overline{\frac{i}{n};\ 1 + \frac{i}{n}}\right)^{9)}$$

and

$$(77)\qquad \varphi(t) = 1 + (1 + i)\,t \quad \text{for} \quad t \in \langle 0, 1\rangle\,,$$

then

$$\langle\mathscr{H}_0\rangle = \langle 0, 1\rangle \subset \partial\Omega_\varphi^+ \cap \partial\Omega_\varphi^-\,.$$

Example 2. If

$$(78)\qquad \Omega = \{z \in \mathbf{E};\ 0 < \operatorname{Re} z < 2,\ 0 < \operatorname{Im} z < 1\} - \bigcup_{n=2}^{\infty} \overline{\frac{i}{n};\ 1 + \frac{i}{n}}$$

and if φ is as in (77), we have $o(\mathscr{S}_0) = 1$, $\langle\mathscr{H}_0\rangle = \langle 0, 1\rangle$, and

$$\langle\mathscr{H}_0\rangle \subset \partial\Omega_\varphi^-\,,\quad \langle\mathscr{H}_0\rangle \cap \partial\Omega_\varphi^+ = \{o(\mathscr{S}_0)\}\,.$$

Example 3. In examples 1 and 2 both sets $\langle\mathscr{H}_0\rangle \cap \partial\Omega_\varphi^+$, $\langle\mathscr{H}_0\rangle \cap \partial\Omega_\varphi^-$ were connected. In the general case, nothing like this holds. Take, namely,

$$(79)\qquad \Omega = \{z \in \mathbf{E};\ 0 < \operatorname{Re} z < 2,\ |\operatorname{Im} z| < 1\} -$$
$$- \left(\langle 0, 1\rangle \cup \bigcup_{n=2}^{\infty} \overline{\frac{i}{n};\ 1 + \frac{i}{n}} \cup \{z \in \mathbf{E};\ \tfrac{1}{3} \leqq \operatorname{Re} z \leqq \tfrac{2}{3},\ -\tfrac{1}{2} \leqq \operatorname{Im} z \leqq 0\}\right),$$

and let φ be as in (77). Then $\langle\mathscr{H}_0\rangle = \langle 0, 1\rangle$ and

$$\langle\mathscr{H}_0\rangle \subset \partial\Omega_\varphi^-\,,\quad \langle\mathscr{H}_0\rangle \cap \partial\Omega_\varphi^+ = \langle 0, \tfrac{1}{3}\rangle \cup \langle \tfrac{2}{3}, 1\rangle\,.$$

It is easy to see an analogous example may be given with $\langle\mathscr{H}_0\rangle \cap \partial\Omega_\varphi^+$ equal e.g. to the Cantor discontinuum.

Example 4. Examples 1–3 may sugest the set $\langle\mathscr{H}_0\rangle$ always is a subset either of $\partial\Omega_\varphi^+$ or of $\partial\Omega_\varphi^-$. In general, however, nothing like this hold. Take, namely,

$$(80)\qquad J = \{z \in \mathbf{E};\ |\operatorname{Re} z| < 4,\ 0 < \operatorname{Im} z < 8\}\,;$$

for each $n \in \mathbf{N}$ let

$$(81')\qquad A_n = \partial U\left(\overline{0; 6i} \cup \overline{2i - 2; 2i},\ \frac{1}{n}\right) \cap \{z;\ \operatorname{Re} z \leqq 0,\ 0 \leqq \operatorname{Im} z \leqq 6\}\,,$$

[9]) If $a, b \in \mathbf{E}$, $a \neq b$, then $\overline{a; b}$ denotes the set $\{z;\ z = a + t(b - a),\ t \in \langle 0, 1\rangle\}$.

$$(81'') \qquad B_n = \partial U\left(\overline{0; 6i} \cup \overline{4i; 4i + 2}, \frac{1}{n}\right) \cap \{z; \operatorname{Re} z \geqq 0,\ 0 \leqq \operatorname{Im} z \leqq 6\}.$$

Put

$$(82) \qquad \Omega = J - \left(\bigcup_{n=1}^{\infty} (A_n \cup B_n) \cup \overline{0; 6i} \cup \overline{2i - 2; 2i} \cup \overline{4i; 4i + 2}\right)$$

and

$$(83) \qquad \varphi(t) = 6i + 2it, \quad t \in \langle 0, 1 \rangle.$$

Then $\langle \mathscr{H}_0 \rangle = \overline{0, 6i} \cup \overline{2i - 2; 2i} \cup \overline{4i; 4i + 2}$, $\langle \mathscr{H}_0 \rangle \cap \partial\Omega_\varphi^- = \overline{0; 6i} \cup \cup \overline{2i - 2; 2i}$, $\langle \mathscr{H}_0 \rangle \cap \partial\Omega_\varphi^+ = \overline{0; 6i} \cup \overline{4i; 4i + 2}$.

References

[1] *Kuratowski K.:* Topologie I, II (1948, 1952).
[2] *Carathéodory C.:* Über die Begrenzung einfach zusammenhängender Gebiete (Math. Annalen 73, 1913).
[3] *Markuševič A. I.:* Teorija analitičeskich funkcij (1950).
[4] *Černý I.:* Základy analysy v komplexním oboru (1967).

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