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ON OSCILLATION OF SOLUTIONS OF DIFFERENTIAL
INEQUALITIES WITH RETARDED ARGUMENT

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We consider the following differential inequality

$$(1) \quad \{[r(t) y^{(n-1)}(t)]' + f(t, y(t), y[h(t)])\} \operatorname{sgn} y[h(t)] \leq 0, \quad n \geq 2,$$

where

$$(2) \quad r : [0, \infty) \rightarrow (0, \infty); \quad h : [0, \infty) \rightarrow \mathbb{R};$$

$$f : [0, \infty) \times \mathbb{R}^2 \rightarrow \mathbb{R} \quad \text{are continuous functions,}$$

$$(3) \quad h(t) \leq t, \quad \lim_{t \rightarrow \infty} h(t) = \infty \quad \text{for } t \rightarrow \infty,$$

$$(4) \quad y f(t, x, y) > 0 \quad \text{for } (t, x, y) \in [0, \infty) \times \mathbb{R}^2, \quad xy > 0;$$

$$|f(t, x_1, y_1)| \leq |f(t, x_2, y_2)| \quad \text{for } |x_1| \leq |x_2|, \quad |y_1| \leq |y_2|, \quad x_1 x_2 > 0, \\ y_1 y_2 > 0, \quad x_1 y_1 > 0.$$

Denote by \mathcal{W} the set of all solutions $y(t)$ of the differential inequality (1), which exist on a ray $[t_0, \infty) \subset [0, \infty)$ and satisfy

$$\sup \{|y(s)| : s \geq t\} > 0$$

for every $t \in [t_0, \infty)$.

A solution $y(t) \in \mathcal{W}$ is said to be *oscillatory* if the set of zeros of $y(t)$ is not bounded from the right. Otherwise the solution $y(t) \in \mathcal{W}$ is said to be *nonoscillatory*.

Definition 1. We shall say that the inequality (1) has the property *A* if every solution $y(t) \in \mathcal{W}$ is oscillatory for n even, while for n odd is either oscillatory or $y^{(i)}(t)$ ($i = 0, 1, \dots, n - 2$) and $r(t) y^{(n-1)}(t)$ tend monotonically to zero as $t \rightarrow \infty$.

Definition 2. Let $m \in \{0, 1, \dots, n - 1\}$. We shall say that the inequality (1) has the property *A_m* if every solution $y(t) \in \mathcal{W}$ is either oscillatory or $y^{(i)}(t)$ ($i = m, m + 1, \dots, n - 2$) and $r(t) y^{(n-1)}(t)$ tend monotonically to zero as $t \rightarrow \infty$.

The oscillatory properties of solutions of differential equations of the n -th order with the term $[r(t) y^{(n-1)}(t)]'$ ($n = 2, n \geq 2, r(t) > 0$) are studied, for example, in [1, 2, 4, 7, 9-12]. In this paper we shall prove sufficient conditions for the inequality (1) to have either the property A or A_0 .

Finally, with the help of the inequality (1) we shall prove a sufficient condition for the equation (r) to have the property $A_m, m \in \{0, 1, \dots, n-1\}$. Our results generalize some of those in the papers [1-3, 6, 9, 12].

Let us denote

$$\begin{aligned} \bar{r}(t) &= \max \{r(s) : t/2 \leq s \leq t\}, \\ b(t) &= \frac{r(t)}{\bar{r}(t)}, \quad b_0 = \inf \{b(t) : t \geq t_0\}, \\ R_k(t) &= \int_T^t \frac{x^k}{r(x)} dx, \quad k = 0, 1, \dots, n-2, \quad T \in [0, \infty), \\ R_k(t, u) &= \int_u^t \frac{(x-u)^k}{r(x)} dx, \quad k = 0, 1, \dots, n-2, \quad u \leq t, \\ \varrho(t) &= \frac{r[h(t)]}{\min \{r(s) : h(t) \leq s \leq t\}}, \quad H(t) = \frac{t}{h(t)} \quad \text{for } h(t) > 0. \end{aligned}$$

Let $m \in \{0, 1, \dots, n-1\}, t_0 \in [0, \infty)$. Put

$$\begin{aligned} D_{t_0}^{(m)} &= \{(t, x_1, y_1, \dots, x_n, y_n) \in [0, \infty) \times R^{2n} : t_0 \leq h(t), \\ &\frac{(n-m-1)!}{(n-j)!} \left(\frac{b_0}{2} t\right)^{m-j+1} \leq \frac{x_j}{x_{m+1}}, \quad \frac{(n-m+1)!}{(n-j)!} \left(\frac{b_0}{2} h(t)\right)^{m-j+1} \leq \\ &\leq \frac{y_j}{y_{m+1}}, \quad (j = 1, 2, \dots, m+1), \quad x_{m+1} y_{m+1} > 0, \quad x_i, y_i \in R, \\ &\quad (i = m+2, \dots, n)\}. \end{aligned}$$

Lemma 1. Let $y(t), \dots, y^{(n-1)}(t)$ be continuous functions of constant sign in the interval $[t_0, \infty) \subset [0, \infty)$. If

$$(5) \quad y(t) [r(t) y^{(n-1)}(t)]' \leq 0, \quad y(t) \neq 0 \quad \text{for } t \geq t_0;$$

$$(5') \quad y(t) y^{(n-1)}(t) \geq 0 \quad \text{for } t \geq t_0,$$

where the function r satisfies (2), then there exists an integer $k \in \{0, 1, \dots, n-1\}, n+k$ odd, such that

$$(6) \quad y^{(i)}(t) y(t) \geq 0 \quad (i = 0, 1, \dots, k), \quad t \geq t_0,$$

$$(7) \quad (-1)^{k+i} y^{(i)}(t) y(t) \geq 0 \quad (i = k+1, \dots, n-1), \quad t \geq t_0,$$

$$(8) \quad |y^{(i)}(t)| \geq L_i b(t) t^{n-i-1} |y^{(n-1)}(t)|, \quad \text{where } k \in \{1, 2, \dots, n-1\},$$

$$L_i = \frac{2^{-n^2}}{(n-i-1)!} \quad (i = 0, 1, \dots, k-1), \quad t \geq 2^{n-k} t_0,$$

$$(9) \quad |y^{(k)}(t)| \geq t^{n-k-1} b(2^{n-k-1}t) |y^{(n-1)}(2^{n-k-1}t)|, \quad t \geq t_0,$$

$$(10) \quad i! \left(\frac{b_0}{2}\right)^{j-i} t^{j-i} |y^{(k-i)}(t)| \leq j! |y^{(k-j)}(t)| \quad (j = 0, 1, \dots, k, \quad i = 0, 1, \dots, j),$$

$$t \geq 2t_0.$$

Proof. Under the assumption (5'), assertions (6) and (7) follow from Kiguradze's lemma 14.2 in [5]. Further, we may suppose, without loss of generality, that $y(t) > 0$ for $t \geq t_0$.

(a) Let $k = n - 1$. Then (6) implies

$$y^{(i)}(t) \geq 0, \quad i = 1, 2, \dots, n-1, \quad t \geq t_0.$$

Using Taylor's theorem, the last inequality, and the monotonicity of $[r(t) y^{(n-1)}(t)]$, we get

$$(11) \quad y^{(i)}(t) = \sum_{j=0}^{n-i-2} \frac{y^{(i+j)}(t/2)}{j!} \left(\frac{t}{2}\right)^j + \int_{t/2}^t y^{(n-1)}(s) \frac{(t-s)^{n-i-2}}{(n-i-2)!} ds \geq$$

$$\geq \int_{t/2}^t y^{(n-1)}(s) \frac{(t-s)^{n-i-2}}{(n-i-2)!} ds \geq \frac{b(t) y^{(n-1)}(t)}{(n-i-1)!} \left(\frac{t}{2}\right)^{n-i-1}$$

$$(i = 0, 1, \dots, n-2), \quad t \geq 2t_0.$$

From (11), we obtain (8) for $k = n - 1$.

The inequality (9) for $k = n - 1$ is evident.

(b) Let $k \in \{0, 1, \dots, n-3\}$ and let $n+k$ be an odd integer. Then, in view of Lemma 1 in [8], we get

$$(12) \quad y^{(i)}(t) \geq \bar{L}_i t^{n-i-3} y^{(n-3)}(t), \quad t \geq 2^{n-k-2} t_0,$$

$$\bar{L}_i = \frac{2^{-(n-2)^2}}{(n-i-3)!}, \quad (i = 0, 1, \dots, k-1),$$

and

$$(13) \quad y^{(k)}(t) \geq t^{n-k-3} y^{(n-3)}(2^{n-k-3}t), \quad t \geq t_0.$$

With the help of (7) and (5), we get

$$(14) \quad -y^{(n-2)}(t/2) \geq \int_{t/2}^t y^{(n-1)}(s) ds \geq \frac{t}{2} b(t) y^{(n-1)}(t), \quad t \geq 2t_0.$$

For $t \geq 4t_0$, using (6), (7) and (14), we obtain

$$(15) \quad y^{(n-3)}\left(\frac{t}{4}\right) \geq y^{(n-3)}\left(\frac{t}{4}\right) - y^{(n-3)}\left(\frac{t}{2}\right) \geq -\frac{t}{4} y^{(n-2)}\left(\frac{t}{2}\right) \geq \frac{t^2}{8} b(t) y^{(n-1)}(t).$$

The inequalities (15), (12) and (13) imply (8) and (9).

If $k \in \{0, 1, \dots, n-3\}$, then the inequality (10) follows from Kiguradze's lemma in [5]. It remains to prove (10) for $k = n-1$.

Let $k = n-1$. Using (6) we can show that

$$y^{(n-2-i)}(t) \geq \frac{t}{2} y^{(n-i-1)}\left(\frac{t}{2}\right), \quad (i = 1, 2, \dots, n-2), \quad t \geq 2t_0.$$

Utilizing the last inequality and (6), we can easily verify the correctness of the following relation

$$(16) \quad (1+i) y^{(n-2-i)}(t) - \int_{t/2}^t \left[i y^{(n-i-1)}(s) - \frac{b_0}{2} s y^{(n-i)}(s) \right] ds \geq \\ \geq (1-b_0) y^{(n-2-i)}(t) + \frac{b_0}{2} t y^{(n-i-1)}(t) + \frac{b_0}{2} \left[y^{(n-i-2)}(t) - \frac{t}{2} y^{(n-i-1)}\left(\frac{t}{2}\right) \right] \geq \\ \geq \frac{b_0}{2} t y^{(n-i-1)}(t), \quad (i = 1, 2, \dots, n-2), \quad t \geq 2t_0.$$

For $i = n-2$,

$$(17) \quad y^{(n-2)}(t) \geq \frac{b(t)}{2} t y^{(n-1)}(t) \geq \frac{b_0}{2} t y^{(n-1)}(t), \quad t \geq 2t_0$$

follows from (11).

Further, (16) and (17) imply

$$(1+i) y^{(n-2-i)}(t) \geq \frac{b_0}{2} t y^{(n-i-1)}(t), \quad t \geq 2t_0, \quad (i = 0, 1, \dots, n-2).$$

For $k = n-1$, (10) follows from the last inequality.

This completes the proof of Lemma 1.

Lemma 1 is an extension of Lemma 2 in [9].

Lemma 2. Let (2)–(4) hold.

(a) If

$$(18) \quad \int_{t_0}^{\infty} \frac{ds}{r(s)} = \infty,$$

then conditions (5) and (5') are satisfied for every nonoscillatory solution $y(t) \in W$ of (1).

(b) If

$$(19) \quad \int^{\infty} \left(\frac{1}{r(t)} \int_T^t |f(s, c, c)| ds \right) dt = \infty$$

for every $c \neq 0$ and $T \geq 0$, then conditions (5) and (5') hold for every nonoscillatory solution $y(t) \in W$ of (1) such that $\lim_{t \rightarrow \infty} y(t) \neq 0$.

Proof. We assume, without loss of generality, that $y(t) > 0$ for $t \geq t_0$. Then, in view of (3), there exists $t_1 \geq t_0$ such that $y[h(t)] > 0$ for $t \geq t_1$. From (1), with regard to (4), we obtain

$$(20) \quad [r(t) y^{(n-1)}(t)]' \leq -f(t, y(t), y[h(t)]) < 0 \quad \text{for } t \geq t_1.$$

(a) If (18) holds, then using the same method as in Lemma 1 in [9], we get $y^{(n-1)}(t) > 0$ for $t \geq t_1$.

(b) Via contradiction we prove that $y^{(n-1)}(t) > 0$ for $t \geq t_1$. We suppose that for some $t_2 \geq t_1$ we have $y^{(n-1)}(t_2) \leq 0$. Then (20) implies $y^{(n-1)}(t) < 0$ for $t \geq t_2$.

If $\lim_{t \rightarrow \infty} y(t) > 0$, then there exist $\varepsilon > 0$ and $t_3 \geq t_2$ such that $y(t) \geq \varepsilon$ and $y[h(t)] \geq \varepsilon$ hold for every $t \geq t_3$. Thus (20), under the assumption (4), yields

$$[r(t) y^{(n-1)}(t)]' \leq -f(t, \varepsilon, \varepsilon) < 0 \quad \text{for } t \geq t_3.$$

Integrating the last inequality from T ($T \geq t_3$) to t and using $y^{(n-1)}(t) \leq 0$ for $t \geq t_2$ we have

$$y^{(n-1)}(t) \leq \frac{1}{r(t)} \int_T^t f(s, \varepsilon, \varepsilon) ds.$$

Integrating the last relation from T to t , with regard to (19) we get $\lim_{t \rightarrow \infty} y^{(n-2)}(t) = -\infty$ which contradicts the positivity of $y(t)$ for $t \geq t_0$.

The proof of Lemma 2 is complete.

Theorem 1. Let r, h, f be functions satisfying conditions (2), (3), (4). Let K, α, δ be constants ($K > 0, 0 \leq \alpha < 1, \delta > 0$) and $g : [0, \infty) \rightarrow [K, \infty)$ a continuous function such that

$$(21) \quad |f(t, g(t)x, g(t)y)| = [g(t)]^\alpha |f(t, x, y)|$$

holds for every $t \geq 0$ and $|y| \geq \delta, |x| \geq \delta$.

(a) If (18) and

$$(22) \quad \int^{\infty} |f(t, \pm \bar{r}^{-1}(t) t^{n-1}, \pm \bar{r}^{-1}[h(t)] (h(t))^{n-1})| dt = \infty$$

hold, then the inequality (1) has the property A.

(b) When (19) and (22) hold, then the inequality (1) has the property A_0 .

Proof. Let $y(t) \in W$ be a nonoscillatory solution of (1) such that $\lim_{t \rightarrow \infty} y(t) \neq 0$. We assume, without loss of generality, that

$$(23) \quad \lim_{t \rightarrow \infty} y(t) > 0.$$

Then, in view of (3), we can choose i_0 such that $y[h(t)] > 0$ for every $t \geq i_0$. Then (1), with regard to (4), implies $[r(t) y^{(n-1)}(t)]' < 0$ for $t \geq i_0$. If any of the conditions (18) and (19) is satisfied, Lemma 2 implies $y^{(n-1)}(t) > 0$ for $t \geq i_0$. Then by Lemma 1, there exists $t_0 \geq i_0$ such that the inequalities (6)–(9) hold for $t \geq t_0$.

Integrating (1) from t ($t \geq t_0$) to ∞ , we get

$$(24) \quad \infty > r(t) y^{(n-1)}(t) \geq \int_t^\infty f(s, y(s), y[h(s)]) ds \quad \text{for } t \geq t_0,$$

and then, in view of the monotonicity of $r(t) y^{(n-1)}(t)$, we have

$$(24') \quad r[h(t)] y^{(n-1)}[h(t)] \geq \int_t^\infty f(s, y(s), y[h(s)]) ds \quad \text{for } t \geq t_1 \geq t_0.$$

I. Let $k \in \{1, 2, \dots, n-1\}$. Then we obtain by (8) for $i = 0$

$$(25) \quad y(t) \geq L_0 b(t) t^{n-1} y^{(n-1)}(t), \quad t \geq 2^{n-k} t_0 = t_2,$$

$$(25') \quad y[h(t)] \geq L_0 b[h(t)] (h(t))^{n-1} y^{(n-1)}[h(t)] \quad \text{for } t \geq t_3,$$

where $L_0 = 2^{-n^2}/(n-1)!$ and t_3 is chosen such that

$$h(t) \geq \max\{t_2, t_1\} \quad \text{for } t \geq t_3.$$

Let us denote

$$\Phi(t) = \int_t^\infty f(s, y(s), y[h(s)]) ds.$$

From (25) or (25'), with regard to (24) or (24'), we get, respectively,

$$(26) \quad y(t) \geq L_0 \bar{r}^{-1}(t) t^{n-1} \Phi(t) \quad \text{for } t \geq t_3,$$

or

$$(26') \quad y[h(t)] \geq L_0 \bar{r}^{-1}[h(t)] (h(t))^{n-1} \Phi(t) \quad \text{for } t \geq t_3.$$

Because $k \geq 1$, there exists $\delta > 0$ such that $y(t) \geq y[h(t)] \geq \delta$ for $t \geq t_3$. Then, in view of the monotonicity of the function f , (26), (26') and (21) we have

$$(27) \quad \begin{aligned} f(t, \bar{r}^{-1}(t) t^{n-1}, \bar{r}^{-1}[h(t)] (h(t))^{n-1}) &\leq \\ &\leq f(t, y(t) \{L_0 \Phi(t)\}^{-1}, y[h(t)] \{L_0 \Phi(t)\}^{-1}) = \\ &= \{L_0 \Phi(t)\}^{-\alpha} f(t, y(t), y[h(t)]) \quad \text{for } t \geq t_3. \end{aligned}$$

By integrating (27) from t_3 to t_4 ($t_3 < t_4$) we have

(28)

$$\int_{t_3}^{t_4} f(t, \bar{r}^{-1}(t) t^{n-1}, \bar{r}^{-1}[h(t)] (h(t))^{n-1}) dt \leq \frac{L_0^{-\alpha}}{1-\alpha} \left[\left(\int_t^{\infty} f(s, y(s), y[h(s)]) ds \right)^{1-\alpha} \right]_{t_4}^{t_3}.$$

From (28), in view of (24), we obtain

$$\int_{t_3}^{\infty} f(t, \bar{r}^{-1}(t) t^{n-1}, \bar{r}^{-1}[h(t)] (h(t))^{n-1}) dt < \infty,$$

which contradicts (22).

II. Let $k = 0$ (n is an odd integer). Then (9) with $k = 0$ implies in view of (23)

$$(29) \quad y(t) \geq M_0 b(t) t^{n-1} y^{(n-1)}(t) \quad \text{for } t \geq 2^n t_0,$$

where

$$M_0 = \inf_{t \geq t_0} \left\{ \frac{y(t)}{y(2^{1-n}t)} \right\} 2^{-(n-1)^2} > 0.$$

Further, using an analogous method as in the case I, we get a contradiction with (22).

If (18) holds and $k \in \{1, 2, \dots, n-1\}$, then, with regard to (6), (36) is fulfilled. In all other cases (i.e. either (18) holds and $k = 0$ or (19) holds and $k \in \{0, 1, \dots, n-1\}$) we have to assume that (23) holds. But, as shown above, this leads to a contradiction with (22). Then $\lim y(t) = 0$ for every nonoscillatory solution $y(t) \in \mathcal{W}$. Hence it follows that $\lim_{t \rightarrow \infty} y^{(i)}(t) = 0$ ($i = 0, 1, \dots, n-2$) and $\lim_{t \rightarrow \infty} r(t) y^{(n-1)}(t) = 0$.

The proof of Theorem 1 is complete.

Lemma 3. *Let the assumptions of Lemma 1 be fulfilled. Let $b_0 > 0$ and let $h : [0, \infty) \rightarrow R$ be a function such that (3) holds. Then there exists $T \geq 2t_0$ such that, for $t \geq T$, we have*

$$(30) \quad |y(t)| \leq C \varrho(t) (H(t))^{n-1} |y[h(t)]|, \quad \text{where } C \geq (2/b_0)^{n-1}.$$

Proof. The case $h(t) = t$ for $t > 2t_0$ is trivial. Consider t such that $t > h(t) \geq 2t_0$. Without loss of generality, we assume that $y(t) > 0$ for $t \geq t_0$. Then, with regard to (3), (5)–(7), there exists $t_1 \geq t_0$ such that for $t \geq t_1$ we have $h(t) \geq t_0$, and either

- (a) $y^{(i)}[h(t)] \geq 0$ ($i = 0, 1, \dots, n-1$), $(r[h(t)] y^{(n-1)}[h(t)])' \leq 0$ or
- (b) $y^{(i)}[h(t)] \geq 0$ ($i = 0, 1, \dots, k$, $k \in \{0, 1, \dots, n-3\}$, $n+k$ is odd) and $y^{(k+1)}[h(t)] \leq 0$.

Consider the case (a). Applying Taylor's theorem and (5) we get

$$(31) \quad y(t) \leq \sum_{i=0}^{n-2} \frac{y^{(i)}[h(t)]}{i!} (t - h(t))^i + \frac{r[h(t)] y^{(n-1)}[h(t)]}{(n-2)!} \int_{h(t)}^t \frac{(t-s)^{n-2}}{r(s)} ds \leq \\ \leq \varrho(t) \sum_{i=0}^{n-1} \frac{y^{(i)}[h(t)]}{i!} (t - h(t))^i.$$

Because of the assumptions of Lemma 1, (10) implies

$$(32) \quad (b_0/2)^i (h(t))^i y^{(i)}[h(t)] \leq k(k-1) \dots (k-i+1) y[h(t)] \\ (i = 0, 1, \dots, k), \quad h(t) \geq 2t_0.$$

Using (31) and (32) we get

$$(b_0/2)^{n-1} y(t) \leq \varrho(t) y[h(t)] \sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{t-h(t)}{h(t)} \right)^i = \\ = \varrho(t) y[h(t)] \left(\frac{t}{h(t)} \right)^{n-1} \quad \text{for } h(t) \geq 2t_0.$$

From the last inequality we get

$$y(t) \leq C \varrho(t) y[h(t)] (H(t))^{n-1} \quad \text{for } t \geq T \geq 2t_0,$$

where $C \geq (2/b_0)^{n-1}$ and T is chosen so that $h(t) \geq 2t_0$ for $t \geq T$.

(b) Applying Taylor's theorem and the fact that $y^{(k+1)}[h(t)] \leq 0$ for $h(t) \geq t_0$ we have

$$y(t) \leq \sum_{i=0}^{k-i} \frac{y^{(i)}[h(t)]}{i!} (t - h(t))^i.$$

Next, using the same method as in the case (a) we get

$$y(t) \leq y[h(t)] (H(t))^k \leq C \varrho(t) y[h(t)] (H(t))^{n-1} \quad \text{for } h(t) \geq 2t_0.$$

This completes the proof

Lemma 3 is an extension of Lemma 4 obtained by GRIMMER in [3].

Theorem 2. Suppose that (2)–(4) are satisfied and, in addition, suppose that

- (i) $r(t) \geq r_0 > 0$ for $t \geq 0$ and $b_0 > 0$;
- (ii) there exist a positive continuous function $\varphi_1(t)$ and positive nondecreasing continuous functions $\varphi(t)$, $\varphi_2(t)$, $\psi(t)$ for $t \geq a$ such that $\varphi(t) = \varphi_1(t) \varphi_2(t)$,

$$(33) \quad \int_a^\infty \frac{dt}{\varphi(t)} < \infty ;$$

(iii) for $x \geq y \geq a$, $t \geq b > 0$, and for every constants α, β, γ (where $0 < \alpha \leq 1$, $\beta > 1$, $\gamma > 0$) we have

$$(34) \quad \liminf_{y \rightarrow \infty} \frac{\psi(\alpha x) f(t, x, y)}{\varphi_1(x) \varphi_2(\beta \varrho(t) (H(t))^{n-1} y)} \geq d \frac{f(t, \gamma, \gamma)}{\varphi_2(\varrho(t) (H(t))^{n-1})} > 0.$$

(a) If (18) holds and

$$(35) \quad \int^{\infty} \frac{R_{n-2}(t) f(t, \gamma, \gamma)}{\psi(t^{n-1}) \varphi_2(\varrho(t) (H(t))^{n-1})} dt = \infty,$$

then inequality (1) has the property A.

(b) If (19) and (35) hold, then inequality (1) has the property A_0 .

Proof. Let $y(t) \in \mathcal{W}$ be a nonoscillatory solution of (1) such that $\lim_{t \rightarrow \infty} y(t) \neq 0$. We assume, without loss of generality, that

$$(36) \quad \lim_{y \rightarrow \infty} y(t) > 0.$$

Further, exactly as in the proof of Theorem 2 we prove that the conditions of Lemma 1 and Lemma 2 are satisfied and the inequalities (5)–(9) and (24) hold.

I. Let $k \in \{1, 2, \dots, n-1\}$. By virtue of (5)–(7) and the assumption $r(t) \geq r_0 > 0$, it is easy to show that there exist constants $\bar{\alpha}, \bar{\gamma}$ ($0 < \bar{\alpha} \leq 1, \bar{\gamma} > 0$) and $t_1 \geq t_0$ such that

$$(37) \quad \bar{\alpha} y(t) \leq t^{n-1}, \quad y[h(t)] \geq \bar{\gamma} \quad \text{for } t \geq t_0.$$

In view of Lemma 3, the monotonicity of the function ψ , (4), (34) and (37) we get

$$(38) \quad \frac{f(t, y(t), y[h(t)])}{\varphi(y(t))} \geq \frac{\psi(\bar{\alpha} y(t))}{\psi(t^{n-1}) \varphi_1(y(t))},$$

$$\frac{f(t, y(t), y[h(t)])}{\varphi_2(C \varrho(t) (H(t))^{n-1} y[h(t)])} \geq d \frac{f(t, \bar{\gamma}, \bar{\gamma})}{\psi(t^{n-1}) \varphi_2(\varrho(t) (H(t))^{n-1})}$$

for $t \geq T_1 \geq t_1$.

I_a . If $k \in \{2, 3, \dots, n-1\}$, then (8) and the fact that $b(t) \geq b_0 > 0$ imply

$$(39) \quad \dot{y}(t) \geq L_1 b_0 t^{n-2} y^{(n-1)}(t) \quad \text{for } t \geq 2^n t_0.$$

Let $k = 1$ and $\lim_{t \rightarrow \infty} \dot{y}(t) \neq 0$. Then (9), with regard to $b(t) \geq b_0 > 0$, yields

$$(40) \quad \dot{y}(t) \geq L_1 b_0 t^{n-2} y^{(n-1)}(t) \quad \text{for } t \geq t_1,$$

where

$$L_1 = \inf_{t \geq t_0} \left\{ \frac{\dot{y}(t)}{\dot{y}(2^{2-n}t)} \right\} 2^{-n^2} > 0.$$

Put $B = \min \{L_1 b_0, \bar{L}_1 b_0\}$. Using (24), (39) and (40) we get

$$\dot{y}(t) \geq B \frac{t^{n-2}}{r(t)} \int_t^\infty f(s, y(s), y[h(s)]) ds.$$

With regard to (38) and the monotonicity of y and φ , after multiplying the last inequality by $\{\varphi(y(t))\}^{-1}$, we obtain

$$(41) \quad \begin{aligned} \frac{\dot{y}(t)}{\varphi(y(t))} &\geq B \frac{t^{n-2}}{r(t)} \int_t^\infty \frac{f(s, y(s), y[h(s)])}{\varphi(y(s))} ds \geq \\ &\geq dB \frac{t^{n-2}}{r(t)} \int_t^\infty \frac{f(s, \bar{y}, \bar{y})}{\psi(s^{n-1}) \varphi_2(\varrho(s) (H(s))^{n-1})} ds \\ &\text{for } t \geq T \geq \max \{T_1, 2^n t_0\}. \end{aligned}$$

In view of (33), after integrating (41) from T to t ($t > T$) we get

$$\infty > \int_T^\infty \frac{\dot{y}(t)}{\varphi(y(t))} dt \geq dB \int_T^t \frac{R_{n-2}(s, T) f(s, \bar{y}, \bar{y})}{\psi(s^{n-1}) \varphi_2(\varrho(s) (H(s))^{n-1})} ds,$$

which contradicts (35).

I_b. Let $k = 1$ and $\lim_{t \rightarrow \infty} \dot{y}(t) = 0$. Integrating (24) from t ($t \geq t_0$) to ∞ we obtain

$$-y^{(n-2)}(t) \geq \int_t^\infty R_0(s, t) f(s, y(s), y[h(s)]) ds.$$

Repeating this procedure $n - 3$ times, we get

$$(42) \quad (-1)^n \dot{y}(t) \geq \int_t^\infty \frac{R_{n-3}(s, t)}{(n-3)!} f(s, y(s), y[h(s)]) ds \text{ for } t \geq t_0.$$

Multiplying (42) by $\{\varphi(y(t))\}^{-1}$, using the monotonicity of the functions y , φ , (38), and the fact that n is even ($n + k$ is odd), we obtain

$$(43) \quad \begin{aligned} \frac{\dot{y}(t)}{\varphi(y(t))} &\geq d \int_t^\infty \frac{R_{n-3}(s, t) f(s, y(s), y[h(s)])}{(n-3)! \varphi(y(s))} ds \geq \\ &\geq \frac{d}{(n-3)!} \int_t^\infty \frac{R_{n-3}(s, t) f(s, \bar{y}, \bar{y})}{\psi(s^{n-1}) \varphi_2(\varrho(s) (H(s))^{n-1})} ds \text{ for } t \geq T. \end{aligned}$$

Integrating (43) from T to t ($t \geq T$) and using (33) we get a contradiction with (35).

II. Let $k = 0$ (n is an odd number). In view of (36), (3) and (7), there exist constants σ, ε ($0 < \sigma \leq 1, \varepsilon > 0$) and $t_3 \geq t_0$ such that

$$\sigma y(t) \leq t^{n-1}, \quad y[h(t)] \geq y(t) \geq \varepsilon \text{ for } t \geq t_3.$$

By virtue of the monotonicity of ψ, φ_2, f , the last inequality and (30) we have

$$(44) \quad f(t, y(t), y[h(t)]) \geq \frac{\psi(\sigma y(t)) \varphi_2(y(t)/C y[h(t)]) f(t, y(t), y[h(t)])}{\psi(t^{n-1}) \varphi_2(\varrho(t) (H(t))^{n-1})} \geq \\ \geq \frac{K f(t, \varepsilon, \varepsilon)}{\psi(t^{n-1}) \varphi_2(\varrho(t) (H(t))^{n-1})} \quad \text{for } t \geq t_3,$$

where

$$K = \psi(\varepsilon\sigma) \varphi_2(C_0), \quad C_0 = \frac{1}{C} \inf_{t \geq t_3} \left\{ \frac{y(t)}{y[h(t)]} \right\} > 0.$$

It is obvious that (42) holds also for $k = 0$. Then (42) with n odd, in view of (44), implies

$$-y'(t) \geq \frac{K}{(n-3)!} \int_t^\infty \frac{R_{n-3}(s, t) f(s, \varepsilon, \varepsilon)}{\psi(s^{n-1}) \varphi_2(\varrho(s) (H(s))^{n-1})} ds \quad \text{for } t \geq t_3.$$

Integrating the last inequality from $T (\geq t_3)$ to ∞ we get

$$y(T) > y(T) - y(\infty) \geq \frac{K}{(n-2)!} \int_T^\infty \frac{R_{n-2}(s, T) f(s, \varepsilon, \varepsilon)}{\psi(s^{n-1}) \varphi_2(\varrho(s) (H(s))^{n-1})} ds,$$

which contradicts (35).

If (18) holds and $k \in \{1, 2, \dots, n-1\}$, then, with regard to (6), (36) is fulfilled. In all other cases (i.e. either (18) holds and $k = 0$ or (19) holds and $k \in \{0, 1, \dots, n-1\}$) we have to assume that (36) holds. But, as shown above, this leads to a contradiction with (35). Then $\lim_{t \rightarrow \infty} y(t) = 0$ for every nonoscillatory solution $y(t) \in \mathcal{W}$. Hence it follows that $\lim_{t \rightarrow \infty} y^{(i)}(t) = 0$ ($i = 0, 1, \dots, n-2$) and $\lim_{t \rightarrow \infty} r(t) y^{(n-1)}(t) = 0$.

The proof of Theorem 2 is complete.

Remark. If $\psi(t) \equiv 1$, it is evident from the proof that Theorem 2 holds without the assumption $r(t) \geq r_0 > 0$.

In the case that $n = 2$, $r(t) \equiv 1$, we get Theorem 2.9 in [6].

Further, consider the following equation

$$(r) \quad \{r(t) y^{(n-1)}(t)\}' + F(t, y(t), y[h_0(t)], \dots, y^{(n-1)}(t), y^{(n-1)}[h_{n-1}(t)]) = 0, \\ n \geq 2,$$

where

$$(45) \quad r : [0, \infty) \rightarrow (0, \infty), \quad h_i : [0, \infty) \rightarrow \mathbb{R} \quad (i = 0, 1, \dots, n-1), \\ F : D (\equiv [0, \infty) \times \mathbb{R}^{2n}) \rightarrow \mathbb{R} \quad \text{are continuous functions ;}$$

$$(46) \quad t \geq h_i(t) \text{ for } t \geq 0 \text{ and } \lim_{t \rightarrow \infty} h_i(t) = \infty \quad (i = 0, 1, \dots, n-1);$$

$$(47) \quad y_1 F(t, x_1, y_1, \dots, x_n, y_n) > 0 \text{ for } (t, x_1, y_1, \dots, x_n, y_n) \in D \\ \text{and } x_1 y_1 > 0.$$

The next theorem follows directly from Theorem 1 and Theorem 2.

Theorem 3. Let equation (r) fulfil conditions (45)–(47), and in addition, let there exist a function f which satisfies (2), (4) and

$$|F(t, x_1, y_1, \dots, x_n, y_n)| \geq |f(t, x_1, y_1)|$$

for every point $(t, x_1, y_1, \dots, x_n, y_n) \in D_{t_0}^{(0)}$. If inequality (1) has either the property A or A_0 then equation (r) has the same property.

Corollary. Let the function h satisfy conditions (2), (3). Let p be a continuous function and ν, σ real numbers such that $p: [0, \infty) \rightarrow (0, \infty)$, $\nu \geq 0$, $\sigma > 1$. If

$$\int_0^\infty t^{(n-1)(1-\sigma)} [h(t)]^{(n-1)\sigma} p(t) dt = \infty,$$

then the equation

$$y^{(n)}(t) + p(t) |y(t)|^\nu |y[h(t)]|^\sigma \operatorname{sgn} y[h(t)] = 0, \quad n \geq 2$$

has the property A.

Proof. If we put $F(t, x_1, y_1, \dots, x_n, y_n) = p(t) |x_1|^\nu |y_1|^\sigma \operatorname{sgn} y_1$, $\psi(x) \equiv 1$, $\varphi_1(x) = |x|^\nu$, $\varphi_2(y) = |y|^\sigma$, then the assertion follows from Theorem 3 and Theorem 2.

Theorem 4. Let $m \in \{1, 2, \dots, n-1\}$ and let the conditions (18), (45)–(47), $b_0 > 0$ be fulfilled. Further, we suppose:

$$(a) \quad h_m(t) \leq \{\min [h_0(t), h_1(t), \dots, h_{m-1}(t)]; t \geq 0\};$$

(b) there exists a function f which satisfies (2), (4), and

$$(48) \quad |F(t, x_1, y_1, \dots, x_n, y_n)| \geq |f(t, x_{m+1}, y_{m+1})|$$

for every point $(t, x_1, y_1, \dots, x_n, y_n) \in D_{t_0}^{(m)}$;

(c) the following inequality

$$(49) \quad \{[r(t) y^{(n-m-1)}(t)]' + f(t, y(t), y[h_m(t)])\} \operatorname{sgn} y[h_m(t)] \leq 0$$

has the property A.

Then equation (r) has the property A_m .

Proof. Let $y(t) \in W$ be a nonoscillatory solution of (r) such that

$$\liminf_{t \rightarrow \infty} y^{(m)}(t) = C > 0$$

(the case $\limsup_{t \rightarrow \infty} y^{(m)}(t) = C < 0$ is treated similarly).

From (49), in view of (46), we get

$$(50) \quad y^{(i)}(t) > 0, \quad y^{(i)}[h_i(t)] > 0 \quad (i = 0, 1, \dots, m) \quad \text{for } t \geq t_0 > 0.$$

Thus, with regard to (50), (47) and (18), it is obvious that the assumptions of Lemma 1 and Lemma 2 are fulfilled and therefore (5)–(10) hold, where $m \leq k \in \{0, 1, \dots, n-1\}$, $n+k$ is odd. By (10) and the assumption (a), it is easy to prove that the following inequalities

$$(51) \quad \frac{(n-m-1)!}{(n-j)!} \left(\frac{b_0 t}{2}\right)^{m-j+1} \leq \frac{y^{(j-1)}(t)}{y^{(m)}(t)},$$

$$\frac{(n-m-1)!}{(n-j)!} \left(\frac{b_0}{2} h_m(t)\right)^{m-j+1} \leq \frac{y^{(j-1)}[h_{j-1}(t)]}{y^{(m)}[h_m(t)]} \quad \text{for } t \geq t_1 \geq 2t_0$$

hold.

Evidently, $u(t) = y^{(m)}(t)$ satisfies

$$(52) \quad \liminf_{t \rightarrow \infty} u(t) = C > 0$$

and, for $t \geq t_1$, $u(t)$ is a solution of the following equation

$$(53) \quad [r(t) u^{(n-m-1)}(t)]' +$$

$$+ G(t, u(t), u[h_m(t)], \dots, u^{(n-m-1)}(t), u^{(n-m-1)}[h_{n-1}(t)]) = 0,$$

where

$$G(t, x_1, y_1, \dots, x_{n-m}, y_{n-m}) = F\left(t, \frac{y(t)}{y^{(m)}(t)} x_1, \frac{y[h_0(t)]}{y^{(m)}[h_m(t)]} y_1, \dots,$$

$$\dots, \frac{y^{(m-1)}(t)}{y^{(m)}(t)} x_1, \frac{y^{(m-1)}[h_{m-1}(t)]}{y^{(m)}[h_m(t)]} y_1, x_1, y_1, \dots, x_{n-m}, y_{n-m}\right).$$

In view of the last relation, (47), (48) and (51) we get

$$(54) \quad y_1 G(t, x_1, y_1, \dots, x_{n-m}, y_{n-m}) > 0,$$

$$(55) \quad |G(t, x_1, y_1, \dots, x_{n-m}, y_{n-m})| \geq |f(t, x_1, y_1)| \quad \text{for } x_1 y_1 > 0,$$

$$x_i, y_i \in R \quad (i = 1, 2, \dots, n-m).$$

By (c), the inequality (49) has the property A . Thus, with regard to (54), (55) and Theorem 3, the equation (53) has the same property. This leads to a contradiction with (52). Hence the equation (r) has the property A_m , $m \in \{1, 2, \dots, n - 1\}$.

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