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A NOTE ON NORM-ATTAINING FUNCTIONALS

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Let X be a real Banach space. We say that $x^* \in X^*$ is a norm-attaining functional if there exists $x \in X$ such that $\|x\| = 1$ and $x^*(x) = \|x^*\|$. It is well-known that X is reflexive iff all $x^* \in X^*$ are norm-attaining. On the other hand, E. Bishop and R. R. Phelps [1] proved that in X^* there exists always a dense subset of norm-attaining functionals. In the Problem Book of the 5th Winter School in Abstract Analysis (Krkonoše, 1978), V. Zizler raised the following problem.

Problem. *Let X be an arbitrary Banach space and $y \in X^*$ an arbitrary functional. Do there exist norm-attaining functionals y_n, $n = 1, 2, \ldots$, such that $y_n \to y$ and all y_n lie on one line?*

In the present note we give the negative answer to this Problem. Thus the Bishop-Phelps theorem cannot be strengthened in the sense of the Problem. The only result of the present note is the following theorem.

Theorem. *Let $M(\langle 0, 1\rangle) = (C(\langle 0, 1\rangle))^*$ be the space of Radon measures on $\langle 0, 1\rangle$. Then the set of all $\mu \in M(\langle 0, 1\rangle)$ for which there exists $\nu \neq 0$ and $\lambda_n \searrow 0$ such that $\mu + \lambda_n \nu$ are norm-attaining functionals on $C(\langle 0, 1\rangle)$ is a set of the first category in $M(\langle 0, 1\rangle)$.*

In the following we use the terminology of N. Bourbaki [2]. The support of a measure μ will be denoted by $S(\mu)$. We shall need the following easy well-known proposition. Since I have not been able to find a reference, I give a proof.

Proposition. *Let $\mu \in M(\langle 0, 1\rangle)$ and $S(\mu^+) \cap S(\mu^-) \neq \emptyset$. Then μ is not a norm-attaining functional on $C(\langle 0, 1\rangle)$.*

Proof. Suppose on the contrary that for an $f \in C(\langle 0, 1\rangle)$ we have $\|f\| = 1$ and $\mu(f) = \|\mu\|$. Let $a \in S(\mu^+) \cap S(\mu^-)$. Then either $f(a) < 1$ or $f(a) > -1$. We shall distinguish these two cases.

(i) If $f(a) < 1$, then

$$\mu(f) = \mu^+(f^+) - \mu^+(f^-) - \mu^-(f^+) + \mu^-(f^-) \leqq \mu^+(f^+) + \mu^-(f^-) .$$

By Proposition 9, Chap. III, § 3 of [2] we have $\mu^+(1) - \mu^+(f^+) = \mu^+(1 - f^+) > 0$ and therefore $\mu(f) \leqq \mu^+(f^+) + \mu^-(f^-) < \mu^+(1) + \mu^-(1) = \|\mu\|$. This is a contradiction.

(ii) If $f(a) > 1$ then $\mu^-(1 - f^-) = \mu^-(1) - \mu^-(f^-) > 0$ and we obtain a contradiction similarly as in the preceding case.

Proof of Theorem. (i) First we shall prove that if $\mu \in M(\langle 0, 1\rangle)$ and $S(\mu^+) = S(\mu^-) = \langle 0, 1\rangle$, then ν and (λ_n) from the statement of Theorem do not exist. Suppose on the contrary that μ, ν, (λ_n) with the properties mentioned above are given. For a sufficiently large n we obtain easily that

$$S((\mu + \lambda_n \nu)^+) \neq \emptyset \quad \text{and} \quad S((\mu + \lambda_n \nu)^-) \neq \emptyset .$$

By Proposition, $S((\mu + \lambda_n \nu)^+) \cap S((\mu + \lambda_n \nu)^-) = \emptyset$ and therefore there exists an open interval $I \subset \langle 0, 1\rangle$ such that $I \cap S((\mu + \lambda_n \nu)^+) = \emptyset$ and $I \cap S((\mu + \lambda_n \nu)^-) = \emptyset$. Let $f \in C(\langle 0, 1\rangle)$ be a function with its support in I. If $k \neq n$, then

$$(1) \qquad (\mu + \lambda_k \nu)(f) = (\mu + \lambda_n \nu)(f) + (\lambda_k - \lambda_n)\nu(f) = (\lambda_k - \lambda_n)\nu(f) .$$

Since $(\mu + \lambda_n \nu)(f) = \mu(f) + \lambda_n \nu(f) = 0$ we have $\nu(f) = -\mu(f)/\lambda_n$. Thus we obtain from (1) $(\mu + \lambda_k \nu)(f) = \lambda_n^{-1}(\lambda_n - \lambda_k)\mu(f)$. Therefore we have $S((\mu + \lambda_k \nu)^+) \cap I = S((\mu + \lambda_k \nu)^-) \cap I = I$ and this is a contradiction with Proposition.

(ii) We shall prove that the set

$$A = M(\langle 0, 1\rangle) \setminus \{\mu \in M(\langle 0, 1\rangle);\ S(\mu^+) = S(\mu^-) = \langle 0, 1\rangle\}$$

is a set of the first category in $M(\langle 0, 1\rangle)$. In fact,

$$A = \bigcup \{A_{rs}^+ \cup A_{rs}^-;\ r < s \text{ and } r, s \text{ are rational}\} ,$$

where A_{rs}^+ and A_{rs}^- are the sets of all measures $\mu \in M(\langle 0, 1\rangle)$ for which $S(\mu^+) \cap (r, s) = \emptyset$ and $S(\mu^-) \cap (r, s) = \emptyset$, respectively. The sets A_{rs}^+, A_{rs}^- are obviously closed nowhere dense subsets of $M(\langle 0, 1\rangle)$. Theorem is proved.

References

[1] *E. Bishop, R. R. Phelps:* A proof that every Banach space is subreflexive, Bull. Amer. Math Soc. *67* (1961), 97—98.

[2] *N. Bourbaki:* Éléments de Mathématique, Livre VI, Intégration, Paris.

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