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# PURE CIRCUITS IN CUBE GRAPHS

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A graph of the $n$-dimensional cube is the graph $Q_n$ whose vertex set is the set of all Boolean vectors (i.e. vectors whose coordinates are equal to 0 or 1) of the dimension $n$ and in which two vertices are adjacent if and only if they differ in exactly one coordinate.

A pure circuit in a graph $G$ is a circuit $C$ which is an induced subgraph of $G$ (i.e. all edges of $G$ joining two vertices of $C$ belong to $C$).

By $\lambda(n)$ we denote the maximum length of a pure circuit in $Q_n$ for $n \geqq 2$.

In [1] A. Kotzig proposed the problem to find the exact values or at least good estimates of $\lambda(n)$ for small values of $n$; the values $\lambda(2) = 4$, $\lambda(3) = 6$, $\lambda(4) = 8$ and $\lambda(5) = 14$ were presented in that paper. (These are the only values known at present.) We shall give some bounds for $\lambda(n)$.

**Theorem 1.** *Let $n$ be an integer, $n \geqq 2$. Then*

$$\lambda(n + 1) \geqq \tfrac{3}{2}\,\lambda(n)$$

*for $\lambda(n) \equiv 0 \pmod 4$ and*

$$\lambda(n + 1) \geqq \tfrac{3}{2}\,\lambda(n) - 1$$

*for $\lambda(n) \equiv 2 \pmod 4$.*

Proof. Consider the graphs $Q_n$, $Q_{n+1}$ of cubes of dimensions $n$ and $n + 1$, respectively. Let $M_0$ (or $M_1$) be the subset of the vertex set $V(Q_{n+1})$ of $Q_{n+1}$ consisting of all vectors with the last coordinate 0 (or 1, respectively). Let $G_0$ (or $G_1$) be the subgraph of $Q_{n+1}$ induced by the set $M_0$ (or $M_1$, respectively). Let $\varphi_0$ (or $\varphi_1$) be the mapping of $V(Q_n)$ into $V(Q_{n+1})$ such that for each $n$-dimensional vector $\mathbf{v}$ the image $\varphi_0(\mathbf{v})$ (or $\varphi_1(\mathbf{v})$) is the $(n + 1)$-dimensional vector obtained from $\mathbf{v}$ by adding the $(n + 1)$-th coordinate equal to 0 (or 1, respectively). Clearly $\varphi_0$ (or $\varphi_1$) is an isomorphic mapping of $Q_n$ onto $G_0$ (or $G_1$, respectively). Now let $C$ be a pure circuit in $Q_n$ of the length $\lambda(n)$. Let the vertices of $C$ be $u_0, u_1, \ldots, u_{\lambda(n)-1}$ and let the edges of $C$ be $u_i u_{i+1}$ for $i = 0, 1, \ldots, \lambda(n) - 1$, the sum $i + 1$ being taken modulo $\lambda(n)$.

The graph $Q_n$ is bipartite and all circuits in it have even lengths; hence $\lambda(n)$ is even. If $\lambda(n) \equiv 0 \pmod 4$, we construct a circuit $C^*$ in $Q_{n+1}$ in the following way. For each $i$ such that $0 \leqq i \leqq \lambda(n) - 4$ and $i \equiv 0 \pmod 4$ we construct a path $P_i$ from $\varphi_0(u_i)$ into $\varphi_0(u_{i+4})$ having the vertices $\varphi_0(u_i)$, $\varphi_0(u_{i+1})$, $\varphi_0(u_{i+2})$, $\varphi_1(u_{i+2})$, $\varphi_1(u_{i+3})$, $\varphi_1(u_{i+4})$, $\varphi_0(u_{i+4})$. The circuit $C^*$ is the union of the paths $P_i$ for all $i$ with the described property; its length is $\frac{3}{2}\lambda(n)$. If $\lambda(n) \equiv 2 \pmod 4$, we construct the paths $P_i$ in the same way for each $i$ such that $0 \leqq i \leqq \lambda(n) - 6$ and $i \equiv 0 \pmod 4$. Further, we denote by $P'$ the path from $u_{\lambda(n)-2}$ to $u_0$ of the length 2 with the inner vertex $u_{\lambda(n)-1}$. Now $C^*$ will be the union of $P_i$ for all $i$ with the described property and of $P'$; its length is $\frac{3}{2}\lambda(n) - 1$. It remains to prove that $C^*$ is a pure circuit in $Q_{n+1}$. Suppose that there are two vertices of $C^*$ which are joined by an edge not belonging to $C^*$. If they are both in $M_0$, then they are $\varphi_0(u_j)$, $\varphi_0(u_k)$ for some $j$ and $k$ such that $|j - k| \not\equiv 1 \pmod{\lambda(n)}$. As $\varphi_0$ is an isomorphism, the vertices $u_j$, $u_k$ are adjacent in $Q_n$ and the edge $u_ju_k$ joins two vertices of $C$ and does not belong to $C$, which is a contradiction with the assumption that $C$ is a pure circuit in $Q_n$. Analogously if both these vertices are in $M_1$. If one of them is in $M_0$ and the other in $M_1$, then they are $\varphi_0(u_j)$, $\varphi_1(u_j)$ for some $j$. From the construction of $C^*$ it is clear that this is not possible. Hence $C^*$ is a pure circuit in $Q_{n+1}$, which yields the assertion.

**Corollary.** *Let $n$ be an integer, $n \geqq 5$. Then*

$$\lambda(n) \geqq 12 \,.\, \left(\tfrac{3}{2}\right)^{n-5} + 2 \,.$$

This follows immediately from Theorem 1 and the fact [1] that $\lambda(5) = 14$.

**Theorem 2.** *Let $n$ be an integer, $n \geqq 2$. Then*

$$\lambda(n) \leqq 2^{n-1}(1 + 1/(n-1)) \,.$$

Proof. Let $C$ be a pure circuit in $Q_n$ of the length $\lambda(n)$. Each vertex of $C$ is adjacent to $n - 2$ vertices not belonging to $C$. Each vertex not belonging to $C$ is adjacent to at most $n$ vertices of $C$. Thus for the number $2^n - \lambda(n)$ of vertices not belonging to $C$ we have

$$2^n - \lambda(n) \geqq (n - 2)\,\lambda(n)/n \,.$$

This implies our inequality.

*Reference*


[1] *A. Kotzig:* Selected open problems in graph theory. In: Graph Theory and Related Topics, Academic Press, New York 1979.



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