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THE INVERSE SPECTRAL RADIUS FORMULA AND REMOVABILITY OF SPECTRUM

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INTRODUCTION

Let A be a Banach algebra with unit. For $x \in A$ set $d_l(x) = \inf\{\|yx\|, y \in A, \|y\| = 1\}$, $d_r(x) = \inf\{\|xy\|, y \in A, \|y\| = 1\}$. Denote further by $\tau_l(x)$ the left approximate point spectrum of x, $\tau_l(x) = \{\lambda \in C, \inf\{\|y(x - \lambda)\|, y \in A, \|y\| = 1\} = 0\}$ (see [4]). Similarly one can define the right approximate point spectrum $\tau_r(x)$. Clearly, $d_l(x) = 0$ if and only if $0 \in \tau_l(x)$. It is well-known that $\partial\sigma(x) \subset \tau_l(x) \cap \cap \tau_r(x)$ and $\tau_l(x) \cup \tau_r(x) \subset \sigma(x)$.

The function $d_l : A \to \langle 0, \infty)$ (and analogously the function d_r) possesses some nice properties similar to the properties of the norm. Let us compare:

1) $d_l(x) = \inf\{\|yx\|, \|y\| = 1\}$, $\|x\| = \sup\{\|yx\|, \|y\| = \}$;

2) $d_l : A \to \langle 0, \infty)$ is continuous $\|\cdot\| : A \to \langle 0, \infty)$ is continuous;

3) $d_l(xy) \geqq d_l(x) . d_l(y)$, $\|xy\| \leqq \|x\| . \|y\|$.

The aim of this paper is to prove the analogue of the spectral radius formula for the function d_l: For $x \in A$, $\lim_{n\to\infty} d_l(x^n)^{1/n}$ exists and $\lim_{n\to\infty} d_l(x^n)^{1/n} = \sup\{d_l(x^n)^{1/n}, n \in \mathbb{N}\} = \inf\{|\lambda|, \lambda \in \tau_l(x)\} = \text{dist}\{0, \tau_l(x)\}$. (Compare with the spectral radius formula $\lim_{n\to\infty} \|x^n\|^{1/n} = \inf\{\|x^n\|^{1/n}, n \in \mathbb{N}\} = \sup\{|\lambda|, \lambda \in \sigma(x)\} = \sup\{|\lambda|, \lambda \in \tau_l(x)\}$).

An analogous result was proved independently by Makai and Zemánek [5]. They used different methods.

In Section II we apply this result to a problem of removability of spectrum in a commutative Banach algebra.

I

Lemma 1. *Let $\mathcal{P}$ be the set of all polynomials wtth complex coefficients in one variable x, let $|\cdot|$ be a pseudonorm on $\mathcal{P}$ (i.e., $|p + q| \leqq |p| + |q|$, $|\alpha p| = |\alpha| . |p|$ for every $p, q \in \mathcal{P}$, $\alpha \in C$) satisfying the following two conditions:*

1) *there exists a constant* $k > 0$ *such that* $|(x-\lambda)p| \geqq k|p|$ *for every* $p \in \mathscr{P}$ *and* $\lambda \in \mathbb{C}$, $|\lambda| = 1$;

2) *there exists a constant* $K \geqq 0$ *such that* $|xp| \leqq K|p|$ *for every* $p \in \mathscr{P}$.

Then $|x^r| \leqq 3k^{-3}K(1+K)\,n^{-1}(|x^{r-n}| + |x^{r+n}|)$ *for each pair of integers* r, n, $1 \leqq n \leqq r$.

Proof. Let $\varepsilon_1, \ldots, \varepsilon_n$ be the n-th roots of [1]. Set $p(\varepsilon) = \sum_{j=-n+1}^{n-1} x^{r+j}\varepsilon^{-j}[1-(j/n)^2]$. Then

$$x^r = \frac{1}{n}\sum_{i=1}^{n} p(\varepsilon_i) \quad \text{since} \quad \sum_{i=1}^{n} \varepsilon_i^k = 0 \quad \text{for each} \quad k \neq 0, \quad -n+1 \leqq k \leqq n-1.$$

According to 1),

$$|x^r| \leqq \frac{1}{n}\sum_{i=1}^{n} |p(\varepsilon_i)| \leqq \frac{1}{n}k^{-3}\sum_{i=1}^{n} |(x-\varepsilon_i)^3 p(\varepsilon_i)|.$$

Now for each ε,

$$(x-\varepsilon)^3 p(\varepsilon) = \sum_{j=-n+1}^{n+2} c_j x^{r+j}\varepsilon^{-j+3}.$$

For j satisfying $-n+4 \leqq j \leqq n-1$ we have

$$c_j = -\left[1-\left(\frac{j}{n}\right)^2\right] + 3\left[1-\left(\frac{j-1}{n}\right)^2\right] - 3\left[1-\left(\frac{j-2}{n}\right)^2\right] + \\ + \left[1-\left(\frac{j-3}{n}\right)^2\right] = 0.$$

It is easy to check that $c_{-n+3} = c_n = 0$. Thus we have

$$(x-\varepsilon)^3 p(\varepsilon) = x^{r+n+2}\varepsilon\frac{2n-1}{n^2} + x^{r+n+1}\varepsilon^2\frac{2n+1}{-n^2} + x^{r-n+2}\varepsilon^{n+1}\frac{2n+1}{n^2} + \\ + x^{r-n+1}\varepsilon^{n+2}\frac{2n-1}{-n^2}.$$

So

$$|x^r| \leqq \frac{1}{n}k^{-3}\sum_{i=1}^{n}|(x-\varepsilon_i)^3 p(\varepsilon_i)| \leqq \frac{1}{n}k^{-3}n\frac{3}{n}\left[|x^{r+n+2}| + |x^{r+n+1}| + |x^{r-n+2}| + \right. \\ \left. + |x^{r-n+1}|\right] \leqq 3k^{-3}K(1+K)\,n^{-1}\left[|x^{r+n}| + |x^{r-n}|\right].$$

Lemma 2. *Suppose a pseudonorm* $|\cdot|$ *on* $\mathscr{P}$ *satisfies the conditions of the previous lemma. Fix an integer* $n \geqq 1$ *such that* $t = 6k^{-3}K(1+K)\,n^{-1} < 1$. *Then*

$$|x^r| \leqq t^{2^s}\max\{|x^{r-2^s n}|, |x^{r+2^s n}|\} \quad \textit{for every} \quad s \geqq 0 \quad \textit{and} \quad r \geqq 2^s n.$$

In particular, $|x^{2^s n}| \leqq t^{2^s} \max\{|1|, |x^{2^{s+1}n}|\}$.

Proof. We shall prove Lemma 2 by induction on s. For $s = 0$ we have by Lemma 1, $|x^r| \leqq t/2\, (|x^{r-n}| + |x^{r+n}|) \leqq t \max\{|x^{r-n}|, |x^{r+n}|\}$. Suppose the statement of Lemma 2 is true for $s - 1 \geqq 0$. Then by the induction hypothesis

$$|x^r| \leqq t^{2^{s-1}} \max\{|x^{r-2^{s-1}n}|, |x^{r+2^{s-1}n}|\} \leqq$$
$$\leqq t^{2^{s-1}} \max\{t^{2^{s-1}} \max\{|x^{r-2^s n}|, |x^r|\}, t^{2^{s-1}} \max\{|x^r|, |x^{r+2^s n}|\}\} =$$
$$= t^{2^s} \max\{|x^{r-2^s n}|, |x^r|, |x^{r+2^s n}|\} =$$
$$= t^{2^s} \max\{|x^{r-2^s n}|, |x^{r+2^s n}|\} \quad \text{as} \quad t < 1\,.$$

Theorem 1. *Let A be a Banach algebra with unit and let $x \in A$. Then $\lim_{n\to\infty} d_l(x^n)^{1/n}$ exists and*

$$\lim_{n\to\infty} d_l(x^n)^{1/n} = \sup\{d_l(x^n)^{1/n}\ n \in \mathbb{N}\} = \operatorname{dist}\{0, \tau_l(x)\}\,.$$

Proof. As $d_l(x^{m+n}) \geqq d_l(x^m)\, d_l(x^n)$ (see property 3), it is well-known that $\lim_{n\to\infty} d_l(x^n)^{1/n}$ exists and equals $\sup\{d_l(x^n)^{1/n}, n \in \mathbb{N}\}$. Further, for $\lambda \in \mathbb{C}$, $|\lambda| < d_l(x)$ we have $d_l(x - \lambda) \geqq d_l(x) - |\lambda| > 0$, i.e. $\lambda \notin \tau_l(x)$. So $\operatorname{dist}\{0, \tau_l(x)\} \geqq d_l(x)$.

As $\tau_l(x^n) = \{\lambda^n, \lambda \in \tau_l(x)\}$ (see e.g. [7]) we have $\operatorname{dist}\{0, \tau_l(x)\} = \operatorname{dist}\{0, \tau_l(x^n)\}^{1/n} \geqq d_l(x^n)^{1/n}$ and $\operatorname{dist}\{0, \tau_l(x)\} \geqq \sup\{d_l(x^n)^{1/n}, n \in \mathbb{N}\} = c$.

It remains to prove $\operatorname{dist}\{0, \tau_l(x)\} \leqq c$. We may suppose $c \neq 0$ and define $y = x/c$. Then $\sup\{d_l(y^n)^{1/n}, n \in \mathbb{N}\} = 1$. Suppose on the contrary $\operatorname{dist}\{0, \tau_l(y)\} > 1$, i.e. $\tau_l(y)$ does not intersect the unit circle. As the function d_l is continuous there exists a constant $k > 0$ such that $\|z(y - \lambda)\| \geqq k\|z\|$ for every $z \in A$ and $\lambda \in \mathbb{C}$, $|\lambda| = 1$. Fix an $n \geqq 1$ such that $t = 6k^{-3}\|y\|\,(1 + \|y\|)\, n^{-1} < 1$. Since $d_l(y^{2^{s+1}n}) \leqq 1$ for every $s \in \mathbb{N}$, there exists an element $a_s \in A$, $\|a_s\| = 1$ with $\|a_s y^{2^{s+1}n}\| \leqq 2$. On the set $\mathscr{P}$ of all polynomials define a pseudonorm $|\cdot|_s$ by $|p|_s = \|a_s\, p(y)\|$. Obviously $|\cdot|_s$ satisfies all the conditions of Lemmas 1, 2 (with $K = \|y\|$), so

$$\|a_s y^{2^s n}\| \leqq t^{2^s} \max\{\|a_s\|, \|a_s y^{2^{s+1}n}\|\} \leqq 2t^{2^s}\,.$$

Thus

$$d_l(y^{2^s n}) \leqq 2t^{2^s} \quad \text{and} \quad d_l(y^{2^s n})^{1/2^s n} \leqq 2^{1/2^s n} t^{1/n}\,.$$

Hence

$$\limsup_{s\to\infty} d_l(y^{2^s n})^{1/2^s n} \leqq t^{1/n} < 1\,,$$

a contradiction with the assumption $\lim_{m\to\infty} d_l(y^m)^{1/m} = 1$.

II

In this section we shall apply the previous result to a problem of removability of spectrum. We shall deal with commutative Banach algebras with unit. In this case the functions d_l and d_r coincide and we shall denote $d(x) = d_l(x) = d_r(x)$ as well as $\tau(x) = \tau_l(x) = \tau_r(x)$ for every x.

Let A be a commutative Banach algebra with unit and B its superalgebra (i.e. there exists a unit-preserving isometric isomorphism $f : A \to B$). Then $\tau_A(x) \subset \subset \sigma_B(x) \subset \sigma_A(x)$. By a result of Arens [1], $\tau_A(x) = \bigcap_{B \supset A} \sigma_B(x)$ (the intersection is taken over all superalgebras $B \supset A$).

A natural question is whether this intersecion is attained by a single superalgebra B, i.e., whether for every A and $x \in A$ there exists a superalgebra $B \supset A$ such that $\tau_A(x) = \sigma_B(x)$. This is a problem of B. Bollobás [3] (for related topics see also [2] and [6]).

In the following we shall show that any closed disc which does not intersect $\tau(x)$ may be removed from $\sigma(x)$.

Theorem 2. *Let A be a commutative Banach algebra with unit, $x \in A$, and let $V = \{\lambda \in \mathbb{C}, |\lambda - a| \leqq r\}$ be a closed disc in the complex plane, $V \cap \tau_A(x) = \emptyset$. Then there exists a superalgebra $B \supset A$ such that $V \cap \sigma_B(x) = \emptyset$.*

Proof. We have $r < \operatorname{dist}\{a, \tau_A(x)\} = \operatorname{dist}\{0, \tau_A(x - a)\} = \lim_{n \to \infty} d((x - a)^n)^{1/n}$. Fix an $n \in \mathbb{N}$ with $d((x - a)^n)^{1/n} > r$ and consider the element $y = (x - a)^n$ for which $d(y) > r^n$. By the construction of Arens [1] there exists a superalgebra $B \supset A$ such that y is invertible in B and $\|y^{-1}\|_B = d(y)^{-1} < r^{-n}$. So $\sigma_B(y^{-1}) \subset \{\lambda \in \mathbb{C}, |\lambda| < r^{-n}\}$ and $\sigma_B(y) \subset \{\lambda \in \mathbb{C}, |\lambda| > r^n\}$, hence $\sigma_B(x) \subset \{\lambda \in \mathbb{C}, |\lambda - a| > r\}$, $\sigma_B(x) \cap V = \emptyset$.

Remark. If we replace the words "closed disc" in Theorem 2 by "open disc" the result remains true. It is also possible to prove this by using Theorem 1 but the proof is more complicated.

References

[1] *R. Arens:* Extensions of Banach algebras, Pacific J. Math. *10* (1960), 1—16.

[2] *B. Bollobás:* Adjoining inverses to commutative Banach algebras. TAN TAMS *181* (1973), 165—173.

[3] *B. Bollobás:* Adjoining inverses to commutative Banach algebras. Algebras in analysis, ed. J. H. Williamson. Acad. Press 1975, 256—257.

[4] *R. Harte:* Spectral mapping theorems. Proc. Roy. Irish Acad. ser. A, vol. *72* (1972), 89—107.

[5] *E. Makai, J. Zemánek:* The surjectivity radius, packing numbers and boundedness below of linear operators, (to appear).

[6] *G. J. Murphy, T. T. West:* Removing the interior of the spectrum. CMUC *21*, 3 (1980), 421—431.

[7] *Z. Slodkowski, W. Zelazko:* On joint spectra of commuting families of operators. Studia Math. *50* (1974), 127—148.

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