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COMPLEMENTED TOLERANCES ON LATTICES

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The aim of this paper is to study the complementarity on the tolerance lattice $LT(\mathfrak{L})$ of a lattice $\mathfrak{L}$. A tolerance on an algebra is defined similarly as a congruence, only the requirement of transitivity is omitted; see [6]. Thus, let $\mathfrak{L}$ be a lattice. A tolerance T on $\mathfrak{L}$ is a reflexive and symmetric binary relation on $\mathfrak{L}$ such that, if $\langle x_1, x_2\rangle \in T$ and $\langle y_1, y_2\rangle \in T$, then

$$\langle x_1 \vee x_2, y_1 \vee y_2\rangle \in T, \quad \langle x_1 \wedge x_2, y_1 \wedge y_2\rangle \in T.$$

All tolerances on a given lattice $\mathfrak{L}$ form an algebraic (i.e. compactly generated) lattice $LT(\mathfrak{L})$ with respect to set inclusion; see [3], [4], [7]. This lattice $LT(\mathfrak{L})$ is called the tolerance lattice of $\mathfrak{L}$. Clearly the meet $T_1 \wedge T_2$ in $LT(\mathfrak{L})$ coincides with the intersection $T_1 \cap T_2$. The join $T_1 \vee T_2$ in $LT(\mathfrak{L})$ is the set of all pairs $\langle a, b\rangle$, where $a = p(a_1, \ldots, a_n)$, $b = p(b_1, \ldots, b_n)$ for some n-ary lattice polynomial p and elements $a_1, \ldots, a_n, b_1, \ldots, b_n$ of $\mathfrak{L}$ such that $\langle a_i, b_i\rangle \in T_1 \cup T_2$ for $i = 1, \ldots, n$. The greatest element of $LT(\mathfrak{L})$ is the Cartesian square $\mathfrak{L} \times \mathfrak{L}$, its least element is the diagonal (i.e. the identity relation) Δ on $\mathfrak{L}$.

Some properties of $LT(\mathfrak{L})$ have been studied: distributivity [2], [5], 0-modularity (H.-J. Bandelt), atomicity etc. The present paper is devoted to an investigation of complementary elements of $LT(\mathfrak{L})$.

M. F. Janowitz [6] has proved that a congruence Θ on $\mathfrak{L}$ is complementary in the lattice $Con(\mathfrak{L})$ of all congruences on $\mathfrak{L}$ if and only if $\Theta = \Theta(o, z)$, where z is a central element of $\mathfrak{L}$ (o is the least element of $\mathfrak{L}$). We shall try to extend these considerations to tolerances.

Definition 1. An element d of a lattice $\mathfrak{L}$ is called *neutral*, if for each $x \in \mathfrak{L}$ and $y \in \mathfrak{L}$ the sublattice of $\mathfrak{L}$ generated by the three elements $x, y. d$ is distributive.

Definition 2. Let $\mathfrak{L}$ be a lattice with the least and the greatest element. An element c of $\mathfrak{L}$ is called *central*, if it is neutral and if there exists a complement c' of c in $\mathfrak{L}$ which is also neutral.

By $T(a, b)$ we denote the least tolerance of $LT(\mathfrak{L})$ containing the given pair $\langle a, b\rangle$ of elements a, b of L; it is called the principal tolerance on $\mathfrak{L}$ (generated by $\langle a, b\rangle$);

see [4]. Similarly, by $\Theta(a, b)$ we denote the least congruence on $\mathfrak{L}$ containing the pair $\langle a, b\rangle$.

Theorem 1. *Let z be a neutral element of a lattice $\mathfrak{L}$ with the least element o. Then*

$$T(o, z) = \Theta(o, z)$$

and $\langle x, y\rangle \in \Theta(o, z)$ if and only if $z \vee (x \wedge y) \geqq x \vee y$.

Proof. Let R be a binary relation on L defined so that $\langle x, y\rangle \in R$ if and only if $z \vee (x \wedge y) \geqq x \vee y$. Evidently R is reflexive and symmetric. We shall prove that R satisfies the Substitution Property. Let $\langle x, y\rangle \in R$, $\langle w, t\rangle \in R$. Evidently

$$z \vee ((x \vee w) \wedge (y \vee t)) \geqq z \vee (x \wedge y),$$
$$z \vee ((x \vee w) \wedge (y \vee t)) \geqq z \vee (w \wedge t),$$

which implies

$$z \vee ((x \vee w) \wedge (y \vee t)) \geqq (z \vee (x \wedge y)) \vee (z \vee (w \wedge t)).$$

But

$$z \vee (x \wedge y) \geqq x \vee y, \quad z \vee (w \wedge t) \geqq w \vee t,$$

therefore

$$z \vee ((x \vee w) \wedge (y \vee t)) \geqq (x \vee y) \vee (w \vee t) = (x \vee w) \vee (y \vee t),$$

thus $\langle x \vee w, y \vee t\rangle \in R$.

Further, z is neutral in $\mathfrak{L}$, therefore the sublattice of $\mathfrak{L}$ generated by the elements $z, x \wedge y, w \wedge t$ is distributive. This implies

$$z \vee ((x \wedge w) \wedge (y \wedge t)) = z \vee ((x \wedge y) \wedge (w \wedge t)) =$$
$$= [z \vee (x \wedge y)] \wedge [z \vee (w \wedge t)] \geqq (x \vee y) \wedge (w \vee t) \geqq x \wedge w;$$
$$z \vee ((x \wedge w) \wedge (y \wedge t)) = z \vee ((x \wedge y) \wedge (w \wedge t)) =$$
$$= [z \vee (x \wedge y)] \wedge [z \vee (w \wedge t)] \geqq (x \vee y) \wedge (w \vee t) \geqq y \wedge t.$$

From these two inequalities we obtain

$$z \vee ((x \wedge w) \wedge (y \wedge t)) \geqq (x \wedge w) \vee (y \wedge t)$$

and hence $\langle x \wedge w, y \wedge t\rangle \in R$. We have proved $R \in LT(\mathfrak{L})$, $T(o, z) \subseteq R$.

Now let $\langle x, y\rangle \in R$. Then $z \vee (x \wedge y) \geqq x \vee y$, therefore

$$(*) \qquad x \vee y = [z \vee (x \wedge y)] \wedge (x \vee y).$$

But $\langle z, o\rangle \in R$, therefore

$$\langle [z \vee (x \wedge y)] \wedge (x \vee y), x \wedge y\rangle =$$
$$= \langle [z \vee (x \wedge y)] \wedge (x \vee y), [o \vee (x \wedge y)] \wedge (x \vee y)]\rangle \in T(o, z).$$

From $(*)$ it follows that

$$\langle x \vee y, x \wedge y \rangle \in T(o, z),$$

which implies $\langle x, y \rangle \in T(o, z)$, hence $R \subseteq T(o, z)$.

We have proved $R = T(o, z)$. It remains to prove that $\Theta(o, z) = T(o, z)$, i.e. to prove the transitivity of $T(o, z)$. Let

$$\langle a, b \rangle \in T(o, z), \quad \langle b, c \rangle \in T(o, z).$$

Then

$$z \vee (a \wedge b) \geqq a \vee b, \quad z \vee (b \wedge c) \geqq b \vee c.$$

Further, $\langle a \vee b, a \wedge b \rangle \in T(o, z)$, $\langle b \vee c, b \wedge c \rangle \in T(o, z)$, therefore

$$(**) \qquad \langle a \wedge b \wedge c, a \wedge b \rangle = \langle (a \wedge b) \wedge (b \wedge c), (a \wedge b) \wedge (b \vee c) \rangle \in T(o, z)$$

and analogously

$$\langle a \wedge b \wedge c, b \wedge c \rangle \in T(o, z).$$

The identity $(**)$ implies

$$z \vee (a \wedge b \wedge c) \geqq a \wedge b$$

and therefore

$$z \vee (a \wedge b \wedge c) \geqq z \vee (a \wedge b) \geqq a \vee b.$$

Analogously we obtain

$$z \vee (a \wedge b \wedge c) \geqq b \vee c,$$

hence

$$z \vee (a \wedge b \wedge c) \geqq a \vee b \vee c.$$

This means

$$\langle a \vee b \vee c, a \wedge b \wedge c \rangle \in T(o, z),$$

which implies $\langle a, c \rangle \in T(o, z)$. Hence $T(o, z)$ is transitive and $T(o, z) = \Theta(o, z)$.

Theorem 2. *Let $\mathfrak{L}$ be a lattice with the least element o and the greatest element i and let z be a central element of $\mathfrak{L}$, let z' be its complement. Then $\Theta(o, z)$, $\Theta(o, z')$ are complementary in $Con(\mathfrak{L})$.*

Proof. Evidently

$$\langle o, i \rangle = \langle o \vee o, z \vee z' \rangle \in \Theta(o, z) \vee \Theta(o, z'),$$

therefore

$$\Theta(o, z) \vee \Theta(o, z') = \mathfrak{L} \times \mathfrak{L}.$$

Suppose $\langle a, b \rangle \in \Theta(o, z) \cap \Theta(o, z')$. According to Theorem 1 we have

$$z \vee (a \wedge b) \geqq a \vee b,$$
$$z' \vee (a \wedge b) \geqq a \vee b,$$

hence

$$[z \vee (a \wedge b)] \wedge [z' \vee (a \wedge b)] \geqq a \vee b .$$

But z is neutral, therefore the sublattice generated by the elements $z, z', a \wedge b$ is distributive, which implies

$$(z \wedge z') \vee (a \wedge b) \geqq a \vee b ,$$

i.e.

$$a \wedge b \geqq a \vee b ,$$

which implies $a = b$. Thus $\Theta(o, z) \cap \Theta(o, z') = \Delta$ is proved and hence $\Theta(o, z)$, $\Theta(o, z')$ are complementary.

Theorem 3. *Let $\mathfrak{L}$ be a modular lattice with the least element o and the greatest element i. Let $z \in \mathfrak{L}$ and let z' be a complement of z in $\mathfrak{L}$. Then the following two assertions are equivalent:*

(a) *z is central in $\mathfrak{L}$.*

(b) *$T(o, z)$ and $T(o, z')$ are complementary in $LT(\mathfrak{L})$.*

Proof. (a) $\Rightarrow$ (b). If z is central, also z' is central and by Theorem 1 we have $T(o, z) = \Theta(o, z)$, $T(o, z') = \Theta(o, z')$. Evidently

$$\langle o, i \rangle = \langle o \vee o,\ z \vee z' \rangle \in T(o, z) \vee T(o, z')$$

and thus

$$T(o, z) \vee T(o, z') = \mathfrak{L} \times \mathfrak{L} .$$

As the meet in $LT(\mathfrak{L})$ and in $Con(\mathfrak{L})$ is the same (set intersection), Theorems 1 and 2 immediately imply

$$T(o, z) \wedge T(o, z') = \Theta(o, z) \cap \Theta(o, z') = \Delta ,$$

hence (b) holds.

(b) $\Rightarrow$ (a). Suppose that z is not central in $\mathfrak{L}$. Then either z or z' is not neutral and there exist elements a, b of $\mathfrak{L}$ such that at least one of the following assertions holds ([1], Theorem 5.3.8):

(i) $z \vee (a \wedge b) \neq (z \vee a) \wedge (z \vee b)$,

(ii) $z \wedge (a \vee b) \neq (z \wedge b) \vee (z \wedge b)$,

(iii) $a \vee (b \wedge z) \neq (a \vee b) \wedge (a \vee z)$,

(iv) $a \wedge (b \vee z) \neq (a \wedge b) \vee (a \wedge z)$,

(i') $z' \vee (a \wedge b) \neq (z' \vee a) \wedge (z' \vee b)$,

(ii') $z' \wedge (a \vee b) \neq (z' \wedge a) \vee (z' \wedge b)$,

(iii') $a \vee (b \wedge z') \neq (a \vee b) \wedge (a \vee z')$,

(iv') $a \wedge (b \vee z') \neq (a \wedge b) \vee (a \wedge z')$.

Suppose that (i) holds. Then evidently

$$z \vee (a \wedge b) < (z \vee a) \wedge (z \vee b) .$$

We have

$$\langle (z \vee a) \wedge (z \vee b),\ a \wedge b \rangle =$$
$$= \langle (z \vee a) \wedge (z \vee b),\ (o \vee a) \wedge (o \vee b) \rangle \in T(o, z)\,.$$

As $a \wedge b \leqq z \vee (a \wedge b) < (z \vee a) \wedge (z \vee b)$, we also have

$$\langle z \vee (a \wedge b),\ (z \vee a) \wedge (z \vee b) \rangle \in T(o, z)\,.$$

On the other hand,

$$\langle z \vee (a \wedge b), i \rangle = \langle z \vee (a \wedge b) \vee o,\ z \vee (a \wedge b) \vee z' \rangle \in T(o, z')\,.$$

As

$$z \vee (a \wedge b) < (z \vee a) \wedge (z \vee b) \leqq i\,,$$

we also have

$$\langle z \vee (a \wedge b),\ (z \vee a) \wedge (z \vee b) \rangle \in T(o, z')\,.$$

Therefore

$$\langle z \vee (a \wedge b),\ (z \vee a) \wedge (z \vee b) \rangle \in T(o, z) \wedge T(o, z') \neq \Delta\,,$$

which is a contradiction with the assumption that $T(o, z)$ and $T(o, z')$ are complementary.

Now suppose that (ii) holds. Then

$$(z \wedge a) \vee (z \wedge b) < z \wedge (a \vee b)\,.$$

As $\langle z, i \rangle = \langle z \vee o,\ z \vee z' \rangle \in T(o, z')$, we have

$$\langle (z \wedge a) \vee (z \wedge b),\ a \vee b \rangle =$$
$$= \langle (z \wedge a) \vee (z \wedge b),\ (i \wedge a) \vee (i \wedge b) \rangle \in T(o, z')\,.$$

As

$$(z \wedge a) \vee (z \wedge b) < z \wedge (a \vee b) \leqq a \vee b\,,$$

we also have

$$\langle (z \wedge a) \vee (z \wedge b),\ z \wedge (a \vee b) \rangle \in T(o, z')\,.$$

On the other hand, as $\langle z', i \rangle \in T(o, z)$, we have

$$\langle z \wedge (a \vee b), o \rangle = \langle z \wedge (a \vee b) \wedge i,\ z \wedge (a \vee b) \wedge z' \rangle \in T(o, z)\,.$$

As

$$o \leqq (z \wedge a) \vee (z \wedge b) < z \wedge (a \vee b)\,,$$

we have

$$\langle (z \wedge a) \vee (z \wedge b),\ z \wedge (a \vee b) \rangle \in T(o, z) \cap T(o, z') \neq \Delta\,,$$

which is again a contradiction. The cases (i′) and (ii′) would be investigated quite analogously.

Now we may suppose that none of the assertions (i), (ii), (i′), (ii′) hold for any a, b. Suppose that there exist a, b such that (iii) holds. Then evidently $a \neq b$. Then

$$\langle a \vee (b \wedge z),\ a \vee z \rangle = \langle a \vee (b \wedge z),\ a \vee (b \wedge z) \vee z \rangle \in T(o, z)\,.$$

As

$$a \vee (b \wedge z) < (a \vee b) \wedge (a \vee z) \leqq a \vee z\,,$$

we also have

$$\langle a \vee (b \wedge z), (a \vee b) \wedge (a \vee z)\rangle \in T(o, z).$$

On the other hand,

$$\langle a \vee (b \wedge z), a \vee b \vee z'\rangle = \langle a \vee (b \wedge z), a \vee (b \wedge z) \vee z'\rangle =$$
$$= \langle a \vee (b \wedge z), a \vee [(b \vee z') \wedge (z \vee z')]\rangle =$$
$$= \langle a \vee (b \wedge z), a \vee (b \wedge z) \vee z'\rangle \in T(o, z').$$

As

$$a \vee (b \wedge z) < (a \vee b) \wedge (a \vee z) \leqq a \vee b \leqq a \vee b \vee z',$$

we have

$$\langle a \vee (b \wedge z), (a \vee b) \wedge (a \vee z)\rangle \in T(o, z')$$

and again this pair belongs to $T(o, z) \cap T(o, z')$, which is a contradiction with the assumption $T(o, z) \cap T(o, z') = \Delta$.

If (iv) holds, then we proceed dually, taking into account (as in the case (ii)) that $\langle z', i\rangle \in T(o, z)$, $\langle z, i\rangle \in T(o, z')$. Analogously we proceed in the cases (iii') and (iv').

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