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SOME NEW GOOD CHARACTERIZATIONS FOR DIRECTED GRAPHS

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Suppose a class $\mathscr{G}$ of graphs (directed or undirected) is characterized in two ways: by the nonexistence of a homomorphism *from* some fixed graph A (this fact is denoted by $\mathscr{G} = A \nrightarrow$) and by the existence of a homomorphism *to* some fixed graph B (this is denoted by $\mathscr{G} = \rightarrow B$). The equality $A \nrightarrow\ = \ \rightarrow B$ we shall then call a *good characterization*. The importance of good characterizations consists in the fact that every theorem of this type makes possible the efficient solution of the appertaining problem.

We shall solve here the problem (Nešetřil, Pultr, 1975) of an existence of non-trivial good characterizations in the class of all digraphs by finding an infinite number of examples of good characterizations.

1. INTRODUCTION

Let us start with some definitions.

Let G and H be both undirected or directed graphs. A *homomorphism* h from G to H is a mapping $h : V(G) \rightarrow V(H)$ such that

$$(x, y) \in E(G) \Rightarrow (h(x), h(y)) \in E(H),$$

where $E(G)$ is the set of edges and $V(G)$ the set of vertices of G.

Further, we introduce the following classes of graphs.

$A \nrightarrow$ is the class of all graphs G such that there is no homomorphism from A to G.

$\rightarrow B$ is the class of all graphs G such that there exists a homomorphism from G to B.

For instance, for $B = K_n$ (the undirected complete graph on n vertices) we have $G \in \rightarrow K_n$ if and only if $\chi(G) \leqq n$.

The other concept to be defined is the *good characterization* (Edmonds, 1965). In this article we will understand a good characterization as a relation $A \nrightarrow\ = \ \rightarrow B$ (see [3]).

It means that a certain class of graphs is defined on the one hand by nonexistence (or existence) of a homomorphism *from* a special graph, and on the other hand by existence (or nonexistence) of a homomorphism *to* a special graph.

Some good characterizations for homomorphisms of special kinds have been found, but it is not trivial to find examples for the general type of homomorphism.

The following theorem holds in the class of all undirected graphs: *The only good characterizations in the class of all undirected graphs are* $K_2 \nrightarrow = \rightarrow K_1, K_1 \nrightarrow = \rightarrow K_0$, *where* K_n *is the complete graph and* K_0 *is the empty graph* (see [3]).

All the graphs considered in the sequel will be directed.

Denote $\mathscr{D}$ the class of all digraphs without loops and multiple edges. The following theorem is known in the class $\mathscr{D}$ (see [1]):

Let n be a positive integer. Then

$$P_n \nrightarrow = \rightarrow U_n ,$$

where

$$P_n = ([0, n], \{(i, i + 1);\ i = 0, \ldots, n - 1\}) ,$$

$$U_n = ([1, n], \{(i, j);\ i < j\}) .$$

One example see in Fig. 1.

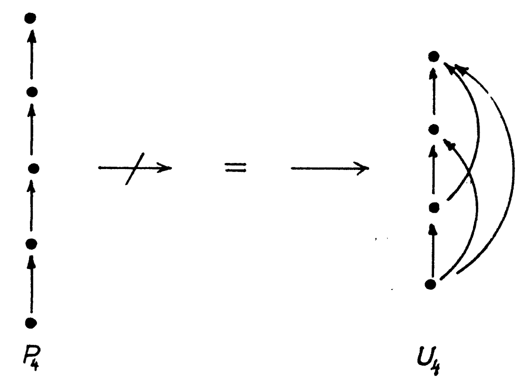


Fig. 1.

The following problem has not been solved yet (see [1], [2]):

Problem (Nešetřil, Pultr, 1975). *Are there other good characterizations in the class* $\mathscr{D}$ *besides*

$$P_n \nrightarrow = \rightarrow U_n ?$$

Definition 1. Let m, n be positive integers, $m, n \geqq 2$. We define the *digraph* $A_{m,n} = (V, E)$, where

$$V = \{0, \ldots, m,\ 0', \ldots, n'\} ,$$

$$E = \{(i, i + 1),\ i = 0, \ldots, m - 1\} \cup \{(i', i + 1'),\ i = 0, \ldots, n - 1\} \cup \{0', m\} .$$

In Fig. 2 see the graph $A_{m,n}$ with $m = 5$, $n = 2$.

The main result of the present note is the following Theorem 1 stating that besides the good characterization $P_n \nrightarrow = \rightarrow U_n$ mentioned above, there exists an infinite number of examples of good characterizations in the class $\mathcal{D}$.

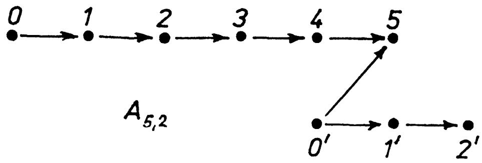


Fig. 2.

Theorem 1. *For every digraph $A_{m,n}$ we can find a digraph $B_{m,n}$ such that*

$$A_{m,n} \nrightarrow = \rightarrow B_{m,n} .$$

The proof of Theorem 1 will be given in section 3.

2. NECESSARY DEFINITIONS AND LEMMAS

Definition 2. Let G be a digraph without directed cycles (*directed cycle* $C_n = ([1, n], \{(i, i + 1), i = 1, \ldots, n - 1\} \cup \{n, 1\}))$. Define the *function* $\mu(x)$, $x \in V(G)$ by $\mu(x) = (d_1, d_2)$, where $d_1(d_2)$ is the length of the longest directed path ending (starting, respectively) in the vertex x.

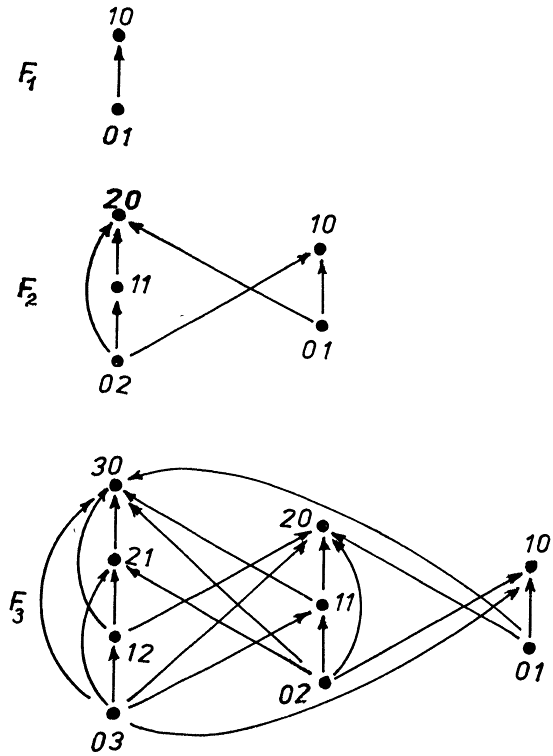


Fig. 3.

Definition 3. Let us define (for $r \geqq 1$) the *digraph* $F_r = (V_r, E_r)$ by

$$V_r = \{ij;\ i \geqq 0,\ j \geqq 0,\ 1 \leqq i + j \leqq r\}\,,$$

$$E_r = \{(ij, pq);\ ij, pq \in V_r,\ i < p,\ j > q\} \quad \text{(see Fig. 3)}\,.$$

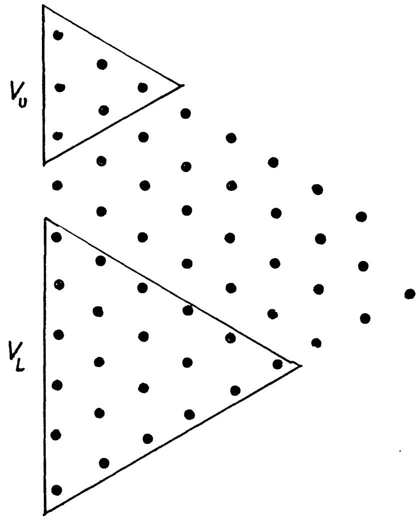


Fig. 4.

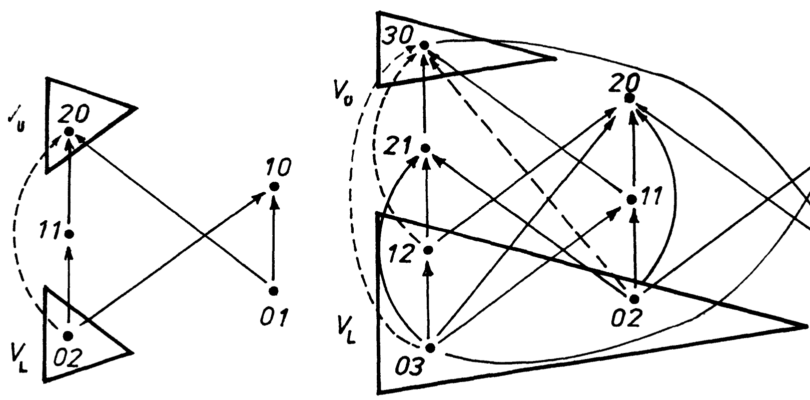


Fig. 5.

Lemma 1. *Let G be a digraph without directed cycles, let its longest directed path contain at most r edges (it means, that for every $x \in V(G)$, $\mu(x) = (d_1, d_2)$, $d_1 + d_2 \leqq r$ holds).*

There exists a homomorphism $h : G \to F_r$ satisfying the following conditions: Let $x \in V(G)$, $\mu(x) = (d_1, d_2)$.

a) *if $d_1 = d_2 = 0$, then $h(x) = 01$.*

b) *otherwise $(1 \leqq d_1 + d_2 \leqq r)$ $h(x) = d_1 d_2$.*

Proof. For every $x \in V(G)$ the vertex $h(x)$ exists (see the definition of V_r). Let $v_1, v_2 \in V(G)$, $\mu(v_1) = (d_1, d_2)$, $\mu(v_2) = (d_1', d_2')$. Let $(v_1, v_2) \in E(G)$. Then $d_1 < d_1'$ and $d_2 > d_2'$ (see the definition of μ). The definition of E_r implies that $(d_1 d_2, d_1' d_2') \in E_r$. It means that the implication $(v_1, v_2) \in E(G) \Rightarrow (h(v_1), h(v_2)) \in E_r$ holds. The proof is complete. ∎

In the proof of Theorem 1 we shall see that for every digraph $A_{m,n}$ there exists a digraph B' such that $A_{m,n} \nrightarrow \subset \rightarrow B'$. The digraph B' is one of the digraphs F_r (for $r = m + n - 2$). By removing some edges from F_{m+n-2} we shall obtain a digraph $B_{m,n}$ such that $A_{m,n} \nrightarrow = \rightarrow B_{m,n}$.

Definition 4. Let m, n be integers, $m, n \geqq 2$, let $F_{m+n-2} = (V_{m+n-2}, E_{m+n-2})$ be the digraph F_r for $r = m + n - 2$. Let

$$V_L = \{ij \in V_{m+n-2};\ j \geqq n\},$$

$$V_U = \{ij \in V_{m+n-2};\ i \geqq m\},$$

$$\bar{E} = \{(v_1, v_2) \in E_{m+n-2};\ v_1 \in V_L,\ v_2 \in V_U\}.$$

Define the *digraph* $B_{m,n} = (V', E')$ by $V' = V_{m+n-2}$, $E' = E_{m+n-2} \setminus \bar{E}$.

In Fig. 4 the vertex sets V_{m+n-2}, V_L, V_U are shown for $m = 7$, $n = 4$ (the edges from V_L to V_U are removed).

Examples for $m = n = 2$ and $m = 3$, $n = 2$ see in Fig. 5 (dashed lines are the removed edges).

3. PROOF OF THEOREM 1

Theorem 1. *Let m, n be integers, $m, n \geqq 2$. Then in the class $\mathscr{D}$*

$$A_{m,n} \nrightarrow = \rightarrow B_{m,n}.$$

Proof.

a) We will prove (for every $G \in \mathscr{D}$) that

$$G \in A_{m,n} \nrightarrow \Rightarrow G \in \rightarrow B_{m,n}.$$

Let $G \in A_{m,n} \nrightarrow$. It means that there is no homomorphism from $A_{m,n}$ to G. Specially, G contains no directed path P_k, where $k \geqq m + n - 1$ (and, of course, no directed cycle). We know that the homomorphism h from G to F_{m+n-2} exists (see Lemma 1). Suppose that there is no homomorphism from G to $B_{m,n}$. It means that there exists an edge $(v_1, v_2) \in E(G)$ such that $h(v_1) \in V_L$, $h(v_2) \in V_U$. There exists a directed path P_s $(s \geqq m)$ ending in v_2 and also a directed path P_t $(t \geqq n)$ starting in v_1 (this follows from the definition of V_L, V_U). However, this is a contradiction with the supposition $G \in A_{m,n} \nrightarrow$. It means that the impliction $G \in \nrightarrow \Rightarrow G \in \rightarrow B_{m,n}$ holds.

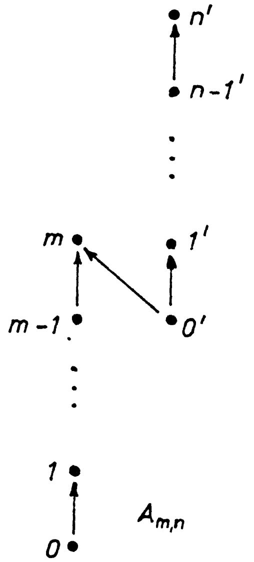
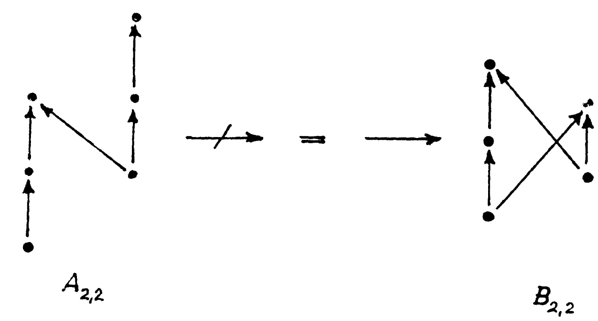


Fig. 6.



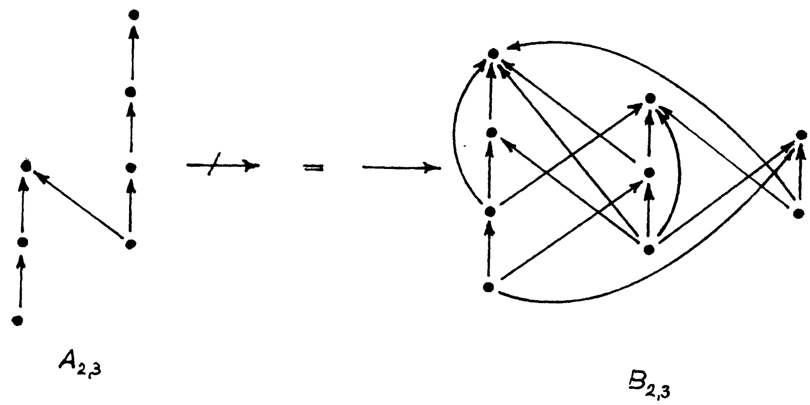


Fig. 7.

b) It remains to prove that for every $G \in \mathscr{D}$ the following implication holds:

$$G \in \rightarrow B_{m,n} \Rightarrow G \in A_{m,n} \nrightarrow .$$

Let $G \in \rightarrow B_{m,n}$. Suppose that $G \notin A_{m,n} \nrightarrow$. From these two statements it follows that there exists a homomorphism h from $A_{m,n}$ to $B_{m,n}$.

There is a directed path P_m ending in the vertex m (see Fig. 6); it means that $h(m) \in V_U$. There is a directed path P_n starting in the vertex $0'$; it means that $h(0') \in \in V_L$. We have $(0', m) \in E(A_{m,n})$, but there are no edges between V_L and V_U. This contradiction implies that if $G \in \rightarrow B_{m,n}$, then $G \in A_{m,n} \nrightarrow$. The proof is complete. ∎

Some examples of good characterizations see in Fig. 7.

References

[1] *J. Nešetřil, A. Pultr:* On Classes of Graphs Determined by Factor Objects and Subobjects. Discrete Math. 22 (1978), s. 287—300.

[2] Combinatorics, edited by A. Hajnal and Vera T. Sós, s. 1209—1210, Amsterdam, North Holland (1978).

[3] *J. Nešetřil:* Theory of Graphs (Czech). Praha, SNTL (1979).

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