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JETS IN DIFFERENTIAL SPACES

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1. INTRODUCTION

One of the most fundamental concepts of global analysis and differential geometry is the concept of a jet introduced in 1951 by C. Ehresmann [1]. In 1967 R. Sikorski [5] introduced the concept of a differential space (d.s.) as a generalization of a C^∞-differentiable manifold. A big part of the foundations of differential geometry may be delivered in terms of d.s. Independently, S. Mac Lane [3] introduced the same concept of d.s. in his lectures on modern theoretical mechanics. The concept of a jet in the category of d.s. seems to be interesting. A methodologically new approach to the foundations of differential geometry presented by I. Kolář in [2] may be then extended to the category of d.s. In the present paper we introduce the concept of a jet and the differential structure of all jets of order k from a d.s. M to a d.s. N, and establish the basic properties of these concepts. The main part of the paper was presented at Czechoslovak Conference on Differential Geometry and its Applications at Poprad.

If M is a d.s., so $PointsM$ and $F(M)$ denote the set of all points of M and the differential structure of M, respectively. Following Sikorski [5] for any set C of real functions defined on a set S and for any set $A \subset S$, the set of all functions $\beta: A \to \mathbf{R}$ such that for every $p \in A$ there exist $\alpha \in C$ and a neighbourhood B (in the weakest topology on S for which all functions of C are continuous) of p fulfilling the condition $\beta \mid A \cap B = \alpha \mid A \cap B$, is denoted by C_A. C_A is called the set of all locally C-functions on A. So, $F(M)_A$ is the set of all locally $F(M)$-functions on A. Then, $(A, F(M)_A)$ is a d.s., being a differential subspace of M. This d.s. will be denoted by M_A. So, we have $PointsM_A = A$ and $F(M_A) = F(M)_A$. The weakest topology on $PointsM$ for which all functions of $F(M)$ are continuous will be denoted by $topM$. Then we have $topM_A = topM \mid A = \{A \cap B;\ B \in topM\}$. The union of all sets $F(M_A)$, where $p \in A \in topM$, will be denoted by $F(M, p)$. If f smoothly maps the d.s. M into a d.s. N, i.e. if f maps $PointsM$ into $PointsN$ and for any $\beta \in F(N)$ we have $\beta \circ f \in F(M)$, then we write $f: M \to N$. The tangent bundle (see [4]) to the d.s. M is denoted by TM. A mapping from $PointsM$ to $PointsTN$ is called a vector field on M tangent to N. For any vector field V on a subspace of M and for any $\beta \in F(N_B)$, where $B \in topN$, we set $(\partial_V \beta)(p) = V(p)(\beta)$, because $V(p)$ is a tangent vector to N at a point p of the set B. In particular, for any $\beta \in F(N)$ we have the function $\partial_V \beta$ defined on a subspace of M. A vector field is said to be smooth on M iff the mapping $V: M \to TN$ is smooth or, evidently, for any $\beta \in F(N)$ the function $\partial_V \beta$ belongs to $F(M)$. If $\beta \in F(N_B)$, $B \in topN$ and V is smooth, so $\partial_V \beta \in F(M_A)$, where A is open in M. A vector field X on M tangent to M and satisfying the condition: $X(p)$ is a vector of $T_p M$ for any p in M is briefly called a vector field on M. The set of all smooth vector fields on M is denoted by $\mathscr{X}(M)$.

2. CONCEPT OF JET

Consider the set (MN) of all pairs (p, f), where $p \in PointsM$ and f smoothly maps a differential subspace of M, such that the set of all its points is an open neighbourhood of p, into the d.s. N. We shall say that (p, f) is equivalent to (p_1, f_1) of order k, $(p, f) \equiv_k (p_1, f_1)$, iff

(i) $(p, f), (p_1, f_1) \in (MN)$, $p = p_1$ and $f(p) = f_1(p)$,

(ii) for any d.s. L, any smooth vector fields $X_1, \ldots, X_k \in \mathscr{X}(M)$, any $\beta \in F(N)$, any smooth mapping $\varphi: L \to M$ and any $t \in PointsL$ such that $\varphi(t) = p$ we have

$$\partial_{X_1} \ldots \partial_{X_r}(\beta \circ f \circ \varphi)(t) = \partial_{X_1} \ldots \partial_{X_r}(\beta \circ f_1 \circ \varphi)(t) \quad \text{for} \quad r \leqq k . \tag{2.1}$$

It is easy to see that $\equiv_k$ is an equivalence in (MN). Every coset of $\equiv_k$ will be called a jet of order k from M into N. The jet containing the pair (p, f) will be denoted by $j_p^k f$ or by $j^k f(p)$. The set of all jets of order k from M into N will be denoted by $\mathbf{J}^k(M, N)$. We have then $\mathbf{J}^k(M, N) = \{j_p^k f;\ (p, f) \in (MN)\}$. From (i) it follows that for any jet $\mu \in \mathbf{J}^k(M, N)$ there is a single p such that $\mu = j_p^k f$, and a single q such that $q = f(p)$, where $(p, f) \in (MN)$. The points p and q will be denoted by $a\mu$ and $b\mu$, respectively. So, we have

$$a: \mathbf{J}^k(M, N) \to PointsM \quad \text{and} \quad b: \mathbf{J}^k(M, N) \to PointsN . \tag{2.2}$$

Let $\mu \in \mathbf{J}^k(M, N)$, $\nu \in \mathbf{J}^k(N, P)$, $b\mu = a\nu$, $\mu = j_p^k f = j_{p_1}^k f_1$ and $\nu = j_q^k g = j_{q_1}^k g_1$, $(p, f), (p_1, f_1) \in (MN)$ and $(q, g), (q_1, g_1) \in (NP)$. So, for any d.s. L, any $\varphi: L \to M$ and $t \in PointsL$ such that $\varphi(t) = p$ and for any $\gamma \in F(P)$ and $X_1, \ldots, X_k \in \mathscr{X}(M)$ we have (2.1), where $\beta = \gamma \circ g_1$. Setting in (2.1) $f \circ \varphi$ instead of φ, and γ, g, g_1 instead of β, f, f_1, respectively, we get

$$\partial_{X_1} \ldots \partial_{X_r}(\gamma \circ g \circ f \circ \varphi)(t) = \partial_{X_1} \ldots \partial_{X_r}(\gamma \circ g_1 \circ f \circ \varphi)(t) .$$

Hence $\partial_{X_1} \ldots \partial_{X_r}(\gamma \circ g \circ f \circ \varphi)(t) = \partial_{X_1} \ldots \partial_{X_r}(\gamma \circ g_1 \circ f_1 \circ \varphi)(t)$ for $r \leqq k$. Thus, $j_p^k(g \circ f) = j_p^k(g_1 \circ f_1)$. Therefore, we have a correct definition of the composition $\nu . \mu$ of jets μ and ν such that $b\mu = a\nu$, as follows:

$$\nu . \mu = j_p^k(g \circ f) , \quad \mu = j_p^k f , \quad \nu = j_q^k g , \quad p = a\mu , \quad q = b\nu .$$

Let us denote the set of all pairs $(\mu, \nu) \in \mathbf{J}^k(M, N) \times \mathbf{J}^k(N, P)$, $a\nu = b\mu$, by $\mathbf{J}^k(M, N) \dot{\times} \mathbf{J}^k(N, P)$. We then have the mapping

$$(\mu, \nu) \mapsto \nu \,.\, \mu: \mathbf{J}^k(M, N) \dot{\times} \mathbf{J}^k(N, P) \to \mathbf{J}^k(M, P)\,.$$

3. THE DIFFERENTIAL SPACE $J^n(M, N)$

(3.1) $$\xi: PointsL \to \mathbf{J}^k(M, N)$$

will be called a field of (M, N)-jets of order k on L. Assume that we have smooth mappings

(3.2) $$a \circ \xi: L \to M \quad \text{and} \quad b \circ \xi: L \to N\,.$$

Let us take any $\beta \in F(N)$, any $X_1, \ldots, X_k \in \mathscr{X}(L)$ and any $Y_1, \ldots, Y_k \in \mathscr{X}(M)$. For any $t \in PointsL$ we have

(3.3) $$\xi(t) = j^k_{a\xi(t)} f_t\,, \quad \text{where} \quad f_t: U_t \to N\,,$$

U_t is an open differential subspace of M around the point $a\xi(t)$. From the definition of jets it follows that for $l \leqq k$ we have correct definitions of functions $\xi(\beta, X_1, \ldots, X_l)$ and $\xi(\beta;\, Y_1, \ldots, Y_l)$ by the formulas

$$\xi(\beta, X_1, \ldots, X_l)\,(t) = \partial_{X_1} \ldots \partial_{Xl}(\beta \circ f_t \circ a \circ \xi)\,(t)\,,$$
$$\xi(\beta, Y_1, \ldots, Y_l)\,(t) = \partial_{Y_1} \ldots \partial_{Yl}(\beta \circ f_t)\,(a\,\xi(t))$$

for $t \in PointsL$. The set of all mappings (3.1) such that the mappings (3.2) are smooth and the condition

(*) for any $X_1, \ldots, X_k \in \mathscr{X}(L)$, any $Y_1, \ldots, Y_k \in \mathscr{X}(M)$ and any $\beta \in F(N)$ the functions $\xi(\beta, X_1, \ldots, X_l)$ and $\xi(\beta;\, Y_1; \ldots, Y_l)$ belong to $F(L)$ for $l \leqq k$

is satisfied, will be denoted by $(MN)^{(k)}\, L$. The smallest differential structure on $\mathbf{J}^k(M, N)$ containing all $\gamma: \mathbf{J}^k(M, N) \to \mathbf{R}$ such that $\gamma \circ \xi \in F(L)$ for any $\xi \in (MN)^{(k)}\, L$ and any d.s. L will be called the differential structure of the d.s. $J^k(M, N)$ and denoted then by $F(J^k(M, N))$.

Proposition. *For any d.s. M and N there are smooth mappings*

(3.4) $$a: J^k(M, N) \to M \quad \text{and} \quad b: J^k(M, N) \to N\,.$$

For any smooth mapping

(3.5) $$g: N \to P$$

we have a smooth mapping

(3.6) $$j^k g: N \to J^k(N, P)\,, \quad \textit{where} \quad j^k g(q) = j^k_q g \quad \textit{for} \quad q \in PointsL\,.$$

Setting $g_*(\mu) = j^k_p(g \circ f)$, where $\mu = j^k_p f$ $f\colon U \to N$, U being an open differential subspace of M around p, we obtain the smooth mapping

$$(3.7) \qquad g_*\colon J^k(M, N) \to J^k(M, P).$$

The correspondence $g \mapsto g_*$ defines a covariant functor from the full category of d.s. into it self.

For any d.s. N and any diffeomorphism

$$(3.8) \qquad h\colon P \to M$$

we have the diffeomorphism

$$(3.9) \qquad h^*\colon J^k(M, N) \to J^k(P, N)$$

defined by the formulas $h^*(\mu) = j^k(f \circ h)(h^{-1}(p))$, $\mu = j^k_p f$. The correspondence $h \mapsto h^*$ gives a contravariant functor from the category Diff(d.s.) of all d.s. together with all diffeomorphisms between d.s. into the same category.

Proof. The smothness of the mappings (3.4) follows from the smoothness of (3.2) for any d.s. L and any $\xi \in (MN)^{(k)} L$. To prove that (3.6) is smooth we check that $j^k g \in (NP)^{(k)} N$. To this aim take any $\beta \in F(P)$ and any $X_1, \ldots, X_k \in \mathscr{X}(N)$. We have

$$j^k g(\beta, X_1, \ldots, X_l)(t) = \partial_{X_1} \ldots \partial_{X_l}(\beta \circ g \circ a \circ j^k g)(t) = \partial_{X_1} \ldots \partial_{X_l}(\beta \circ g)(t)$$

and

$$j^k g(\beta; X_1, \ldots, X_l)(t) = \partial_{X_1} \ldots \partial_{X_l}(\beta \circ g)(a j^k g(t)) = \partial_{X_1} \ldots \partial_{X_l}(\beta \circ g)(t)$$

for any point t of N. This yields that $j^k g(\beta, X_1, \ldots, X_l)$ and $j^k g(\beta; X_1, \ldots, X_l)$ belong to $F(N)$ for $l \leqq k$. Let us take any $\gamma\colon \mathbf{J}^k(N, P) \to \mathbf{R}$ such that for each d.s. L and any $\xi \in (NP)^{(k)} L$ we have $\gamma \circ \xi \in F(L)$. We then get $\gamma \circ j^k g \in F(N)$.

To prove that for any smooth mapping (3.5) the mapping (3.7) is smooth take $\gamma\colon \mathbf{J}^k(M, P) \to \mathbf{R}$ such that for any d.s. L and any $\eta \in (MP)^{(k)} L$ we have $\gamma \circ \eta \in F(L)$. We set $\gamma_1 = \gamma \circ g_*$. Let $\xi \in (MN)^{(k)} L$ and $\eta = g_* \circ \xi$ Then, for any $X_1, \ldots, X_k \in \mathscr{X}(L)$, any $Y_1, \ldots, Y_k \in \mathscr{X}(M)$, $\beta \in F(P)$, $l \leqq k$ and $\alpha = \beta \circ g$ we have successively (3.3), $\eta(t) = g_*(\xi(t)) = j^k_{a\xi(t)}(g \circ f_t)$, $\eta(\beta, X_1, \ldots, X_l)(t) = \partial_{X_1} \ldots \partial_{X_l}(\beta \circ (g \circ f_t) \circ a \circ \eta)(t) = \partial_{X_1} \ldots \partial_{X_l}(\alpha \circ f_t \circ a \circ \xi)(t) = \xi(\alpha, X_1, \ldots, X_l)(t)$ and $\eta(\beta; Y_1, \ldots, Y_l)(t) = \partial_{Y_1} \ldots \partial_{Y_l}(\beta \circ (g \circ f_t))(a\,\eta(t)) = \partial_{Y_1} \ldots \partial_{Y_l}(\alpha \circ f_t)(a\,\xi(t)) = \xi(\alpha; Y_1, \ldots, Y_l)(t)$ for $t \in$ *PointsL*. Hence $\eta(\beta, X_1, \ldots, X_l) = \xi(\alpha, X_1, \ldots, X_l) \in F(L)$ and $\eta(\beta; Y_1, \ldots, Y_l) = \xi(\alpha; Y_1, \ldots, Y_l) \in F(L)$. Moreover, we have $a\,\eta(t) = a g_*(\xi(t)) = a\xi(t)$ and $b\eta(t) = b j^k_{a\xi(t)}(g \circ f_t) = g(f_t(a\,\xi(t))) = g(b\,\xi(t)) = (g \circ b \circ \xi)(t)$ for $t \in$ *PointsL*. Hence it follows that $a \circ \eta\colon L \to M$ and $b \circ \eta\colon L \to P$. These relations yield $\eta \in (MP)^{(k)} L$. Therefore $\gamma_1 \circ \xi = \gamma \circ \eta \in F(L)$. Thus, $\gamma_1 \in F(J^k(M, N))$.

Now, let us take a diffeomorphism (3.8) and $\gamma\colon \mathbf{J}^k(P, N) \to \mathbf{R}$ such that $\gamma \circ \eta \in F(L)$ for each d.s. L and any $\eta \in (PN)^{(k)} L$. Set $\gamma_1 = \gamma \circ h^*$. Let $\xi \in (MN)^{(k)} L$, $X_1, \ldots, X_k \in \mathscr{X}(L)$, $Y_1, \ldots, Y_k \in \mathscr{X}(P)$, $\beta \in F(N)$ and $t \in$ *PointsL*. Setting

$$(3.10) \qquad \eta = h^* \circ \xi$$

we get $\eta(t) = j^k(f_t \circ h)(h^{-1}(a\xi(t)))$, where $\xi(t) = j^k_{a\xi(t)} f_t$, $a\eta(t) = h^{-1}(a\xi(t))$, $(h \circ a \circ \eta)(t) = (a \circ \xi)(t)$, $\eta(\beta, X_1, \ldots, X_l)(t) = \partial_{X_1} \ldots \partial_{X_l}(\beta \circ f_t \circ h \circ a \circ \eta)(t) = \partial_{X_1} \ldots \partial_{X_l}(\beta \circ f_t \circ a \circ \xi)(t) = \xi(\beta, X_1, \ldots, X_l)(t)$ and $\eta(\beta;\ Y_1, \ldots, Y_l)(t) = \partial_{Y_1} \ldots \partial_{Y_l}(\beta \circ f_t \circ h)(a\,\eta(t)) = \partial_{Y_1} \ldots \partial_{Y_l}(\beta \circ f_t \circ h)(h^{-1}(a\,\xi(t))) = \partial_{V_1} \ldots \partial_{V_l}(\beta \circ f_t)(a\,\xi(t)) = \xi(\beta;\ V_1, \ldots, V_l)(t)$, where $V_i = h_* \circ Y_i \circ h^{-1} \in \mathcal{X}(M)$, $i = 1, \ldots, k$. Hence it follows that for $l \leqq k$, $\eta(\beta, X_1, \ldots, X_l) = \xi(\beta, X_1, \ldots, X_l) \in F(L)$ and $\eta(\beta;\ Y_1, \ldots, Y_l) = \xi(\beta;\ V_1, \ldots, V_l) \in F(L)$. Moreover, we notice that $b\,h^*(\mu) = b\mu$ for $\mu \in \mathbf{J}^k(M, N)$. So, for η given by (3.10) we have $b \circ \eta = b \circ \xi$. Therefore, $\eta \in (MP)^{(k)} L$. Thus, $\gamma \circ h^* \circ \xi = \gamma \circ \eta \in F(L)$. Hence the mapping (3.9) is smooth. It is easy to check that the mapping $h^{-1*}\colon J^k(P, N) \to J^k(M, N)$ is the inverse mapping to (3.9). Therefore (3.9) is a diffeomorphism. This completes the proof.

4. THE CASE WHEN DIFFERENTIAL SPACES ARE DIFFERENTIABLE MANIFOLDS

For any $h = (h_1, \ldots, h_m)$, where $h_1, \ldots, h_m$ are non-negative integers, we set $|h| = h_1 + \ldots + h_m$ and $h! = h_1! \ldots h_m!$. The set of all systems u of the form

$$(4.1) \qquad ((u^1, \ldots, u^m),\ (u^j_h;\ |h| \leqq k,\ j \leqq n)),$$

where u^i, u^j_h are reals, will be denoted by $E^k_{m,n}$. The set $E^k_{m,n}$ is in a natural way an $\left(m + n\binom{k+m}{m}\right)$-dimensional Euclidean space. Let M and N be differential spaces. We will examine in this section the d.s. M and N under the hypothesis that they are differentiable manifolds of dimensions m and n, respectively. Let x and y be any charts of M and N, respectively. For any θ in $J^k(M, N)$ such that $a\theta$ and $b\theta$ belong to the domains D_x and D_y of charts x and y, respectively, we set

$$(4.2) \qquad y\theta x = (x(a\theta),\ (\partial^{h_1}_{x_1} \ldots \partial^{h_m}_{x_m}(y^j \circ f)(a\theta);\ |h| \leqq k,\ j \leqq n)),$$

where $\theta = j^k f(a\theta)$, $f\colon U \to N$, U being an open differential subspace of M around the point $a\theta$. Here $x_i(p)(\alpha) = \partial_i(\alpha \circ x^{-1})(x(p))$ for $\alpha \in F(M_A)$, A is an open neighbourhood of p contained in D_x, $\partial^{h_i}_{x_i}$ stands for the h_i-times repeated operation ∂_{x_i} which corresponds to the vector field x_i when $h_i > 0$, and $\partial^{h_i}_{x_i} = \mathrm{id}$ when $h_i = 0$. The formula (4.2) defines then the mapping $y\,.\,x$ of the form

$$(4.3) \qquad a^{-1}[D_x] \cap b^{-1}[D_y] \ni \theta \mapsto y\theta x\,.$$

The mapping (4.3) is a chart of the manifold of all Ehresmann's k-jets from M into N (cf. [1]). Now, we evaluate the value $(y\,.\,x)^{-1}(u)$ of the inverse mapping to (4.2) at a point u of the form (4.1).

Let $\theta = (y\,.\,x)^{-1}(u)$. So, we have $y\theta x = u$. Hence we get

$$(4.4) \qquad x(a\theta) = (u^1, \ldots, u^m) \text{ and } \partial^{h_1}_{x_1} \ldots \partial^{h_m}_{x_m}(y^j \circ f)(a\theta) = u^j_{h_1 \ldots h_m}\,.$$

Thus, $\partial_1^{h_1} \dots \partial_m^{h_m}(y^j \circ f \circ x^{-1})(a\theta) = u^j_{h_1 \dots h_m}$, where ∂_i denotes the partial differentiation with respect to the i-th variable. Let us set

$$x^j(p, u) = \sum_{|h| \leqq k} \frac{1}{h!} u^j_h (x^1(p) - u^1)^{h_1} \dots (x^m(p) - u^m)^{h_m} \tag{4.5}$$

and

$$x(p, u) = (x^1(p, u), \dots, x^m(p, u)) \quad \text{for} \quad p \in D_x \quad \text{and} \quad u \in E^k_{m,n}. \tag{4.6}$$

From (4 4), (4 5) and (4.6) we get

$$(y . x)^{-1}(u) = j^k(y^{-1} \circ x(\cdot, u))(x^{-1}(u^1, \dots, u^m)). \tag{4.7}$$

In Ehresmann's theory, the differentiable manifold of all jets of order k of mappings from M into N has the atlas generated by all maps of the form $y . x$, where x is any chart of M and y is any chart of N. We will prove the basic theorem about compatibility.

Theorem. *If d.s. M and N are differentiable manifolds, then the d.s. $J^k(M, N)$ coincides with the differentiable manifold of all Ehresmann's jets of order k of mappings from M into N.*

Before proving the above theorem we prove three lemmas.

Lemma 1. *If M and N are differentiable manifolds of dimensions m and n, respectively, then the set of all Ehresmann's jets of order k from M into N is equal to* $\mathbf{J}^k(M, N)$.

Proof. It suffices to prove that, under the assumption of (i), the condition (ii) is equivalent to

(ii′) for any chart x of M around p and any chart y of N around $f(p)$, if $h_1 + \dots + h_m \leqq k$, then for $j \leqq n$

$$\partial_1^{h_1} \dots \partial_m^{h_m}(y^j \circ f \circ x^{-1})(x(p)) = \partial_1^{h_1} \dots \partial_m^{h_m}(y^j \circ f_1 \circ x^{-1})(x(p)). \tag{4.8}$$

Assuming (ii) let us take charts x and y as in (ii′), M_{D_x} as L and id as φ in (ii). Diminishing, if necessary, the domain D_y of functions y^j to a neighbourhood of $f(p)$ we can take some functions $\beta^j \in F(N)$ such that β^j is equal to y^j in a neighbourhood of $f(p)$. We have then by (ii)

$$\begin{aligned}\partial_1^{h_1} \dots \partial_m^{h_m}(y^j \circ f \circ x^{-1})(x(p)) &= \partial_{x_1}^{h_1} \dots \partial_{x_m}^{h_m}(y^j \circ f)(p) = \\ &= \partial_{x_1}^{h_1} \dots \partial_{x_m}^{h_m}(\beta^j \circ f \circ \varphi)(p) = \partial_{x_1}^{h_1} \dots \partial_{x_m}^{h_m}(\beta^j \circ f_1 \circ \varphi)(p) = \\ &= \partial_{x_1}^{h_1} \dots \partial_{x_m}^{h_m}(y^j \circ f_1)(p) = \partial_1^{h_1} \dots \partial_m^{h_m}(y^j \circ f_1 \circ x^{-1})(p).\end{aligned}$$

Let us assume (ii′). For any $\varphi \colon L \to M$, $X \in \mathscr{X}(M)$, $t \in PointsL$ and $\alpha \in F(M, \varphi(t))$ we have

$$\partial_X(\alpha \circ \varphi)(t) = X(t)(\alpha \circ \varphi) = \varphi_*(X(t))(\alpha) = \varphi_*(X(t))(x^i)\, x_i(\varphi(t))(\alpha) =$$
$$= X(t)(x^i \circ \varphi)(\partial_{x_i}\alpha)(\varphi(t)) = \partial_X(x^i \circ \varphi)(t)((\partial_{x_i}\alpha) \circ \varphi)(t)\,.$$

Thus, assuming without loss of generality that $\varphi[PointsL] \subset D_x$ we get

(4.9) $\quad \partial_X(\alpha \circ \varphi) = \partial_X(x^i \circ \varphi)(\partial_{x_i}\alpha) \circ \varphi \quad$ for $\quad \alpha \in F(M_A)$, where A is open in M.

Applying (4.9) l-times, $l \leqq k$, we obtain the equality

$$\partial_{X_1} \ldots \partial_{X_l}(\alpha \circ \varphi) = \sum_{r=1}^{l} \gamma_l^{i_1 \ldots i_r}(\partial_{x_{i_1}} \ldots \partial_{x_{i_r}}\alpha) \circ \varphi\,,$$

where $\gamma_l^{i_1 \ldots i_r}$ is a smooth function. Assume $(p, f) \equiv_k (p_1, f_1)$. Let $\beta \in F(N)$ and let t be a point of L such that $\varphi(t) = p$. By (4.8), we then have

$$\partial_{X_1} \ldots \partial_{X_l}(\beta \circ f \circ \varphi)(t) = \sum_{r=1}^{l} \gamma_l^{i_1 \ldots i_r}(t)\, \partial_{x_{i_1}} \ldots \partial_{x_{i_r}}(\beta \circ f)(p) =$$
$$= \sum_{r=1}^{l} \gamma_l^{i_1 \ldots i_r}(t)\, \partial_{i_1} \ldots \partial_{i_r}(\beta \circ f \circ x^{-1})(x(p)) =$$
$$= \sum_{r=1}^{l} \gamma_l^{i_1 \ldots i_r}(t)\, \partial_{i_1} \ldots \partial_{i_r}(\beta \circ f_1 \circ x^{-1})(x(p)) = \partial_{X_1} \ldots \partial_{X_l}(\beta \circ f_1 \circ \varphi)(t)\,.$$

So, the condition (ii) is satisfied. This completes the proof of Lemma.

Lemma 2. *If x and y are charts on differentiable manifolds M and N, respectively, L is a d.s., $\xi \in (MN)^{(k)} L$, and for any $t \in PointsL$ (3.3) holds, then the set $\xi^{-1}[D_{y.x}] \in$ $\in topL$ and for $j \leqq n$ and any $h_1, \ldots, h_m$ such that $h_1 + \ldots + h_m \leqq k$, $m = dimM$, the function*

(4.10) $$t \mapsto \partial_{x_1}^{h_1} \ldots \partial_{x_m}^{h_m}(y^j \circ f_t)(a\,\xi(t))$$

belongs to $F(L)_{\xi^{-1}[Dy,x]}$.

Proof. Let us set

$$\eta(t) = \partial_{x_1}^{h_1} \ldots \partial_{x_m}^{h_m}(y^j \circ f_t)(a\xi(t))\,, \quad \text{where} \quad h_1 + \ldots + h_m \leqq k\,,$$

$\xi(t) = j^k f_t(a(t))$, where $f_t \colon U_t \to N$, U_t is a d.s. open in M such that $a\xi(t) \in Points\,U_t$ for $t \in \xi^{-1}[D_{y.x}]$. We have $D_x \in topM$ and $D_y \in topN$. So, by (3.2) we have $(a \circ \xi)^{-1}[D_x]$, $(b \circ \xi)^{-1}[D_y] \in topL$. Hence, by the equality $D_{y.x} = a^{-1}[D_x] \cap$ $\cap\, b^{-1}[D_y]$ we get $\xi^{-1}[D_{y.x}] = (a \circ \xi)^{-1}[D_x] \cap (b \circ \xi)^{-1}[D_y] \in topL$. Take any $s \in A$, $A = \xi^{-1}[D_{y.x}]$. Then there exist $B_0, B_1 \in topM$, $C_0, C_1 \in topN$, $\varphi \in F(M)$ and $\psi \in F(N)$ such that $a\xi(s) \in B_1$, $b\xi(s) \in C_1$, $D_x \cup B_0 = PointsM$, $D_y \cup C_0 =$ $= PointsN$, $\varphi(p) = 1$ for $p \in B_1$, $\varphi(p) = 0$ for $p \in B_0$, $\psi(q) = 1$ for $q \in C_1$ and $\psi(q) = 0$ for $q \in C_0$. Now, let us set

$$\beta^j(q) = \begin{cases} \psi(q)\, y^j(q) & \text{for} \quad q \in D_y\,, \\ 0 \quad \text{for} & q \in PointsN - D_y\,, \quad \text{and} \end{cases}$$

$$Y_i(p) = \begin{cases} \varphi(p)\, x_i(p) & \text{for} \quad p \in D_x\,, \\ 0 \quad \text{for} & p \in PointsM - D_x\,. \end{cases}$$

Thus, $\beta^j \in F(N)$, $Y_i \in \mathscr{X}(M)$, $\beta^j(q) = y^j(q)$ for $q \in C_1$ and $Y_i(p) = x_i(p)$ for $p \in B_1$. We set $A_1 = (a \circ \xi)^{-1}[B_1] \cap (b \circ \xi)^{-1}[C_1]$. We then have $s \in A_1 \in topL$ and $A_1 \subset A$. For any $p \in B_1$ and any $\alpha \in F(M, p)$ we have

$$(\partial^{h_1}_{Y_1} \dots \partial^{h_m}_{Y_m} \alpha)(p) = (\partial^{h_1}_{x_1} \dots \partial^{h_m}_{x_m} \alpha)(p)\,.$$

In particular,

$$(\partial^{h_1}_{Y_1} \dots \partial^{h_m}_{Y_m}(\beta^j \circ f_t))(a\, \xi(t)) = (\partial^{h_1}_{x_1} \dots \partial^{h_m}_{x_m}(y^j \circ f_t))(a\, \xi(t)) \quad \text{for} \quad t \in A_1\,.$$

Thus,

$$\xi(\beta^j; Y_1, \dots, Y_1, \dots, Y_m, \dots, Y_m)(t) = (\partial^{h_1}_{Y_1} \dots \partial^{h_m}_{Y_m}(\beta^j \circ f_t))(a\, \xi(t)) =$$
$$= (\partial^{h_1}_{x_1} \dots \partial^{h_m}_{x_m}(y^j \circ f_t))(a\, \xi(t)) \quad \text{for} \quad t \in A_1\,.$$

From the hypothesis $\xi \in (MN)^{(k)} L$ we get $\xi(\beta^j; Y_1, \dots, Y_1, \dots, Y_m, \dots, Y_m) \in F(L)$. Hence it follows that the function (4.10) belongs to $F(L)_A$. This completes the proof of Lemma.

Lemma 3. *If x and y are charts on differentiable manifolds M and N, respectively, then $(y\,.\,x)[D_{y.x}]$ is open in $E^k_{m,n}$ and $(y\,.\,x)^{-1}$ belongs to $(MN)^{(k)} L$, where L denotes the natural d.s. of the set $(y\,.\,x)[D_{y.x}]$.*

Proof. Let us set $\xi(u) = (y\,.\,x)^{-1}(u)$ and $\varphi(u) = a\, \xi(u)$ for u in L. We will check that $\xi \in (MN)^{(k)} L$. From (4.2) it follows that $(y\,.\,x)[D_{y.x}]$ is the set of all points u of the form (4.1) such that $(u^1, \dots, u^m) \in x[D_x]$, $(u^1_{0\dots0}, \dots, u^n_{0\dots0}) \in y[D_y]$ and u^j_h are any reals, when $0 < |h| \leqq k$ and $j \leqq n$. Thus, $(y\,.\,x)[D_{y.x}]$ is open in $E^k_{m,n}$. Now, let us take $\beta \in F(N)$, $X_1, \dots, X_k \in \mathscr{X}(L)$ and $Y_1, \dots, Y_k \in \mathscr{X}(M)$. By (4.7) we have $a\, \xi(u) = x^{-1}(u^1, \dots, u^m) = (x^{-1} \circ pr)(u)$, where $pr(u) = (u^1, \dots, u^m)$ for any u of he form (4.1). Further,

$$\xi(\beta, X_1, \dots, X_l)(u) = \partial_{X_1} \dots \partial_{X_l}(\beta \circ y^{-1} \circ x(\cdot, u) \circ x^{-1} \circ pr)(u)$$

and

$$\xi(\beta; Y_1, \dots, Y_l)(u) = \partial_{Y_1} \dots \partial_{Y_l}(\beta \circ y^{-1} \circ x(\cdot, u))(x^{-1}(u^1, \dots, u^m))$$

for any point u in L. Hence it follows that $\xi(\beta, X_1, \dots, X_l)$ and $\xi(\beta; Y_1, \dots, Y_l)$ belong to $F(L)$ for $l \leqq k$. This completes the proof of Lemma.

Proof of Theorem. According to Lemma 1 the set of all Ehresmann's jets from the manifold M into the manifold N coincides with the set of all jets from M into N treated as d.s. Let γ be any real function on $\mathbf{J}^k(M, N)$ fulfilling the following condition: $\gamma \circ \xi \in F(L)$ for $\xi \in (MN)^{(k)} L$ and any d.s. L. By Lemma 3 we have that $\gamma \circ (y \,.\, x)^{-1}$ is of class C^∞ on $(y \,.\, x)[D_{y.x}]$ for all charts x and y of the manifolds M and N, respectively. Thus, every $\gamma \in F(J^k(M, N))$ is smooth on Ehresmann's manifold of all k-jets from the manifold M into the manifold N. To complete the proof we take any smooth function γ on Ehresmann's manifold of all k-jets from M into N. Then $\gamma \circ (y \,.\, x)^{-1}$ is of class C^∞ for any charts x and y of the manifolds M and N, respectively. Taking any d.s. L and any $\xi \in (MN)^{(k)} L$ we have

$$y\,\xi(t)\,x = \left(x(a\,\xi(t)),\ (\partial_{x_1}^{h_1} \ldots \partial_{x_m}^{h_m}(y^j \circ f_t)\,(a\xi(t));\ |h| \leqq k,\ j \leqq n)\right),$$

for $t \in (y \,.\, x)[D_{y,x}]$, where $\xi(t) = j^k f_t(a\xi(t))$, $(a\xi(t), f_t) \in (MN)$. Thus, by Lemma 2 we have a smooth mapping

$$t \mapsto y\,\xi(t)\,x : (\xi^{-1}[D_{y,x}],\ F(L)_{\xi^{-1}[D_{y,x}]}) \to E^k_{m,n}\,.$$

Hence it follows that $\gamma \circ \xi \mid \xi^{-1}[D_{y,x}] = \gamma \circ (y \,.\, x)^{-1} \circ (y \,.\, x) \circ \xi \mid \xi^{-1}[D_{y,x}]$ belongs to $F(L)_{\xi^{-1}[D_{y,x}]}$. So, $\gamma \circ \xi \in F(L)$. This completes the proof of Theorem.

References

[1] *C. Ehresmann:* Le prolongements d'une variété différentiable I. Calcul des jets, prolongement principal. Comptes rendus hebdomadaires des Séances de l'Académie des Sciences, *233* (1951), p. 598—600.

[2] *I. Kolář:* On the jet prolongations of smooth categories. Bulletin de l'Académie Polonaise des Sciences, Série des sciences math., astr. et phys. *24*, 10 (1978), p. 883—887.

[3] *S. Mac Lane:* Differentiable spaces. Notes for Geometrical Mechanics, Winter 1970, p. 1—9 (unpublished).

[4] *H. Matuszczyk:* On the formula of Slebodziński for Lie derivative of tensor fields in a differential space. Colloquium Mathematicum, *46*, 2 (1982), p. 233—241.

[5] *R. Sikorski:* Abstract covariant derivative. Colloquium Mathematicum *18* (1967), p. 251—272.

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