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ON THE MONOTONICITY OF THE PERIOD FUNCTION OF SOME SECOND ORDER EQUATIONS

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Dedicated to Professor Jaroslav Kurzweil on the occasion of his sixtieth birthday

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SECTION 1. INTRODUCTION

Consider the scalar equation

$$x'' + g(x) = 0\,, \quad \left(x'' = \frac{d^2x}{dt^2}\right) \tag{1.1}$$

where $g(x)$ is smooth for all $x \in \mathbb{R}$. Let

$$G(x) = \int_0^x g(\xi)\, d\xi\,. \tag{1.2}$$

If there exist $a < 0 < b$ such that $G(a) = G(b) = c$, $G(x) < c$ for all $a < x < b$ and $g(a)\, g(b) \neq 0$, then there exists a periodic orbit of (1.1) in the phase plane with energy c, intersecting the x-axis at $(a, 0)$ and $(b, 0)$. Let the least period of this periodic orbit be denoted by $p(c)$, which is called the *period function* in this note. It is well known that $p(c)$ is a smooth function of c. In fact, if g is C^γ, $\gamma \geqq 1$, then p is C^γ. Furthermore, $p(c)$ is given by the following formula

$$p(c) = \sqrt{(2)} \int_a^b \frac{dx}{\sqrt{(c - G(x))}}\,. \tag{1.3}$$

In this note, we will discuss the monotonicity of $p(c)$. This problem has been studied by many authors, e.g., Loud [5], Opial [7], Obi [6] and Schaaf [8]. When $g(x)$ is a polynomial of degree n, the above problem is a special case of the weakened 16-th Hilbert problem proposed by V. I. Arnold ([1], p. 303).

In this note, we will derive some formulae for $p'(c)$ and $p''(c)$ which are useful for determining the monotonicity of the period function. In Section 3, we study the period functions of (1.1) for different g's. We will prove the monotonicity of the period function of equation

$$x'' + e^x - 1 = 0\,. \tag{1.4}$$

This will complement the results of Wang [9] and will be useful for bifurcation problems [3].

Let $g(x)$, $G(x)$ and $p(c)$ be as in Section 1.

Since we are interested mainly in either the monotonicity or the number of critical points of $p(c)$, i.e., the points c at which $p'(c) = 0$, we may assume that $g(x)$ has been scaled by $g(x) \to k\, g(\alpha x + \beta)$, where $k\alpha > 0$. Hence, we will assume $g(0) = 0$.

We consider now the periodic orbits which contain only one critical point in their interiors. Consider the hypothesis:

(H1) *There exist* $-\infty \leqq a^* < 0 < b^* \leqq +\infty$, *an integer* $N \geqq 0$ *and a positive smooth function* $h(x)$ *such that*

$$g(x) = x^{2N+1} h(x), \quad a^* < x < b^*, \tag{2.1}$$

and

$$0 < G(a^*) = G(b^*) = c^* \leqq +\infty .$$

Note that under the above hypothesis, the graph of $y = G(x)$ and the corresponding phase portrait of (1.1) are shown in Fig. 2.1. Furthermore, $p(c)$ is defined for every $0 < c < c^*$.

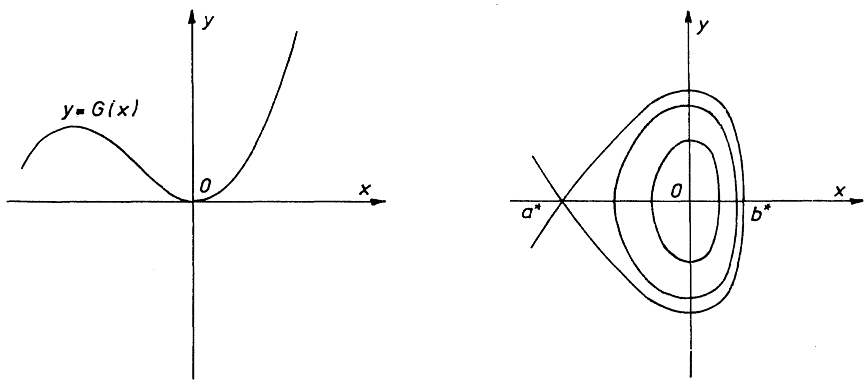

Fig. 2.1

For simplicity, let

$$\gamma(x, c) = 2(c - G(x)) . \tag{2.2}$$

Note that

$$\frac{\partial \gamma}{\partial x} = -2g(x), \tag{2.3}$$

$$p(c) = 2\int_a^b \frac{dx}{\sqrt{\gamma}}, \tag{2.4}$$

where $a^* < a < 0 < b < b^*$, $\gamma(a, c) = \gamma(b, c) = 0$, $\gamma(x, c) > 0$ if $a < x < b$.

Theorem 2.1. *Assume that* (H1) *holds. Then for any* $0 < c < c^*$,

$$c\,p'(c) = \int_a^b \frac{R(x)}{\sqrt{(\gamma)}\, g^2(x)}\,\mathrm{d}x\,, \tag{2.5}$$

where $a^* < a < 0 < b < b^*$, $G(a) = G(b) = c$, *and*

$$R(x) = g^2(x) - 2\,G(x)\,g'(x)\,. \tag{2.6}$$

Proof. Let

$$I = \int_a^b \sqrt{(\gamma)}\,\mathrm{d}x\,,$$

and

$$J = \int_a^b (\gamma - 2c)\sqrt{(\gamma)}\,\mathrm{d}x\,. \tag{2.7}$$

Then

$$I' = \int_a^b \frac{1}{\sqrt{\gamma}}\,\mathrm{d}x\,, \tag{2.8}$$

$$J' = \int_a^b \frac{\gamma - 2c}{\sqrt{\gamma}}\,\mathrm{d}x = I - 2cI'\,.$$

Hence

$$J'' = -I' - 2cI''\,. \tag{2.9}$$

On the other hand, integration by parts for (2.7) yields

$$J = \frac{2}{3}\int_a^b \frac{\gamma - 2c}{\left(\dfrac{\partial \gamma}{\partial x}\right)}\,\mathrm{d}\gamma^{3/2} = -\frac{2}{3}\int_a^b \gamma^{3/2}\,\mathrm{d}\left[\frac{\gamma - 2c}{\left(\dfrac{\partial \gamma}{\partial x}\right)}\right] =$$

$$= -\frac{2}{3}\int_a^b \frac{\gamma^{3/2}(g^2(x) - G(x)\,g'(x))}{g^2(x)}\,\mathrm{d}x\,.$$

Differentiating the above equality with respect to c twice, we have

$$J'' = -2\int_a^b \frac{g^2(x) - G(x)\,g'(x)}{\sqrt{(\gamma)}\,g^2(x)}\,\mathrm{d}x\,. \tag{2.10}$$

Then from (2.8), (2.9) and (2.10), we have

$$2cI'' = 2\int_a^b \frac{g^2(x) - G(x)\,g'(x)}{\sqrt{(\gamma)}\,g^2(x)}\,\mathrm{d}x - I' = \tag{2.11}$$

$$= \int_a^b \frac{R(x)}{\sqrt{(\gamma)}\,g^2(x)}\,\mathrm{d}x\,.$$

Note that $p(c) = 2I'$. Therefore (2.11) gives the desired result.

Remark 2.2. Hypothesis (H1) guarantees that all the integrations in the proof of Theorem 2.1 make sense.

Corollary 2.3. *If* (H1) *holds and*

$$x\, g''(x) < 0 \ (or > 0), \quad x \neq 0, \quad a^* < x < b^*,$$

then

$$p'(c) > 0 \ (or < 0), \quad 0 < c < c^*.$$

Proof. Since $R'(x) = -2G(x)\, g''(x)$, $R(0) = 0$, then $R(x) > 0$ (or <0), $x \neq 0$, $a^* < x < b^*$.

Corollary 2.4. *If* (H1) *holds and*

$$\frac{R(x)}{g^3(x)} - \frac{R(A(x))}{g^3(A(x))} < 0 \ (or > 0), \quad a^* < x < 0,$$

where $R(x) = g^2(x) - 2G(x)\, g'(x)$ *and* $A(x)$ *is defined by*

$$G(A(x)) = G(x), \quad a^* < x < 0, \quad 0 < A(x) < b^*, \tag{2.12}$$

then

$$p'(c) > 0 \ (or < 0), \quad 0 < c < c^*.$$

Proof. By Implicit Function Theorem, $A(x) \in C^1\, (a^*, 0)$ and

$$A'(x) = \frac{g(x)}{g(A(x))}, \quad a^* < x < 0. \tag{2.13}$$

In the integration

$$\int_0^b \frac{R(x)}{\sqrt{(\gamma)}\, g^2(x)}\, dx,$$

we change variables by $x = A(y)$, then

$$\int_0^b \frac{R(x)}{\sqrt{(\gamma)}\, g^2(x)}\, dx = \int_0^a \frac{R(A(x))}{\sqrt{(\gamma)}\, g^2(A(x))}\, A'(x)\, dx. \tag{2.14}$$

From Theorem 2.1 and (2.13), (2.14) we have

$$c\, p'(c) = \int_a^0 \frac{g(x)}{\sqrt{\gamma}} \left[\frac{R(x)}{g^3(x)} - \frac{R(A(x))}{g^3(A(x))}\right] dx.$$

Note that $g(x) < 0$, $a^* < x < 0$. The conclusion follows.

Corollary 2.5. *Suppose* (H1) *holds. If* $g'(0) > 0$ *and*

$$H(x) = g^2(x) + \frac{g''(0)}{3(g'(0))^2}\, g^3(x) - 2G(x)\, g'(x) > 0 \ (or < 0) \tag{2.15}$$

$$x \neq 0\,, \quad a^* < x < b^*\,,$$

then

$$p'(c) > 0 \ (\mathit{or}\ <0)\,, \quad 0 < c < c^*\,.$$

Proof. By L'Hospital's rule,

$$\lim_{x\to 0} \frac{R(x)}{g^3(x)} = -\frac{1}{3}\frac{g''(0)}{(g'(0))^2}\,.$$

Then $H(x) > 0$ (or <0) implies

$$\frac{R(x)}{g^3(x)} < -\frac{1}{3}\frac{g''(0)}{(g'(0))^2} < \frac{R(A(x))}{g^3(A(x))}\,, \quad a^* < x < 0\,,$$

$$\left(\text{or } \frac{R(x)}{g^3(x)} > -\frac{1}{3}\frac{g''(0)}{(g'(0))^2} > \frac{R(A(x))}{g^3(A(x))}\,, \quad a^* < x < 0\right),$$

since $A(x) > 0$, $x \in (a^*, 0)$ and $x\,g(x) > 0$ for $x \neq 0$, $x \in (a^*, b^*)$. By Corollary 2.4,

$$p'(c) > 0 \ (\text{or } <0)\,, \quad 0 < c < c^*\,.$$

Corollary 2.6. *Suppose* (H1) *holds. If* $g'(0) > 0$ *and*

(2.16) $$\nabla = 5(g''(0))^2 - 3\,g'(0)\,g'''(0) > 0 \ (\mathit{or}\ <0)\,,$$

then there exists $\delta > 0$ *such that*

$$p'(c) > 0 \ (\mathit{or}\ <0)\,, \quad 0 < c < \delta\,.$$

Proof. By Taylor's expansion technique,

$$H(x) = \tfrac{1}{12}x^4\,\nabla + O(|x|^5) \quad \text{as} \quad |x| \to 0\,.$$

The conclusion follows from Corollary 2.5.

Theorem 2.7. *Suppose* (H1) *holds. Then for any* $0 < c < c^*$,

$$2c^2\,p''(c) = \int_a^b \frac{S(x)}{\sqrt{(\gamma)}\,g^4(x)}\,\mathrm{d}x\,,$$

where $a^* < a < 0 < b < b^*$, $G(a) = G(b) = c$, *and*

(2.17) $$S(x) = -g^4(x) - 4\,G(x)\,g^2(x)\,g'(x) - 4\,G^2(x)\,g(x)\,g''(x) + {} + 12\,G^2(x)\,(g'(x))^2\,.$$

Proof. Let

(2.18) $$K = \int_a^b \frac{R(x)\sqrt{\gamma}}{g^2(x)}\,\mathrm{d}x$$

and

(2.19) $$L = \int_a^b \frac{2\, R(x)\, G(x) \sqrt{\gamma}}{g^2(x)}\, \mathrm{d}x\,.$$

Differentiating (2.18) and (2.19) with respect to c we obtain

(2.20) $$K' = \int_a^b \frac{R(x)}{\sqrt{(\gamma)}\, g^2(x)}\, \mathrm{d}x\,,$$

$$L' = \int_a^b \frac{2\, R(x)\, G(x)}{\sqrt{(\gamma)}\, g^2(x)}\, \mathrm{d}x\,.$$

By Theorem 2.1, $K' = cp'(c)$. Since $\gamma - 2c = -2\, G(x)$, $-L' = K - 2cK'$, we have

(2.21) $$\begin{cases} p'(c) + c\, p''(c) = K''\,, \\ K' + 2cK'' = L''\,. \end{cases}$$

On the other hand, integration by parts for (2.18) yields

(2.22) $$L = -\frac{2}{3}\int_a^b \frac{R(x)\, G(x)}{g^3(x)}\, \mathrm{d}\gamma^{3/2} =$$

$$= \frac{2}{3}\int_a^b \gamma^{3/2}\, \mathrm{d}\left[\frac{R(x)\, G(x)}{g^3(x)}\right] =$$

$$= \frac{2}{3}\int_a^b \frac{\gamma^{3/2}\, S_1(x)}{g^4(x)}\, \mathrm{d}x\,,$$

where

(2.23) $S_1(x) = g^4(x) - 5\, G(x)\, g^2(x)\, g'(x) - 2\, G^2(x)\, g(x)\, g''(x) + 6\, G^2(x)\, (g'(x))^2\,.$

Differentiating (2.22) with respect to c twice we obtain

(2.24) $$L'' = 2\int_a^b \frac{S_1(x)}{\sqrt{(\gamma)}\, g^4(x)}\, \mathrm{d}x\,.$$

From (2.20), (2.21) and (2.24), and by Theorem 2.1,

$$2c^2\, p''(c) = L'' - 3cp'(c) =$$

$$= 2\int_a^b \frac{S_1(x)}{\sqrt{(\gamma)}\, g^4(x)}\, \mathrm{d}x - 3\int_a^b \frac{R(x)}{\sqrt{(\gamma)}\, g^2(x)}\, \mathrm{d}x =$$

$$= \int_a^b \frac{2\, S_1(x) - 3\, R(x)\, g^2(x)}{\sqrt{(\gamma)}\, g^4(x)}\, \mathrm{d}x\,.$$

The desired result follows from (2.6) and (2.23).

Remark 2.8. Hypothesis (H1) guarantees that all the integrations in the above proof make sense.

We will now extend the previous results to periodic orbits whose interiors may contain more than one critical points.

Note that we can also define $G(x)$ as follows:

$$G(x) = \int_0^x g(\xi)\,d\xi + c_0 ,$$

where c_0 can be any real number.

We need the following hypothesis:

(H2) *There exist* $-\infty \leqq a^* < \alpha \leqq 0 \leqq \beta < b^* \leqq +\infty$, *integers* $M \geqq 0$, $N \geqq 0$ *and a nonnegative smooth function* $h(x)$ *such that*

$$x\,g(x) > 0 , \quad a^* < x < \alpha , \quad \beta < x < b^* ,$$
$$0 < G(a^*) = G(b^*) = c^* \leqq +\infty ,$$

and

(2.25) $$G(x) = (x - \alpha)^{2M+1} (x - \beta)^{2N+1} h(x) , \quad a^* < x < b^* .$$

The graph of $y = G(x)$ and the corresponding phase portrait of (1.1) are shown in Fig. 2.2.

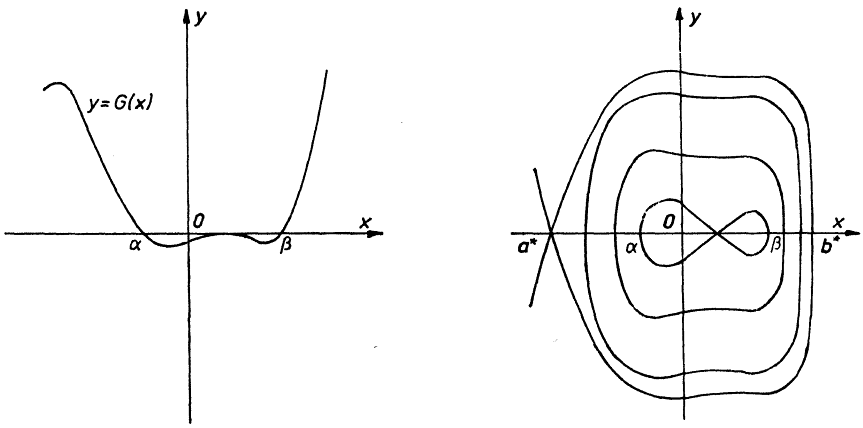

Fig. 2.2

Theorem 2.9. *Suppose* (H2) *holds. Then for* $0 < c < c^*$, *we have*

(2.26) $$c\,p'(c) = \left[\int_a^\alpha + \int_\beta^b\right] \frac{R(x)}{\sqrt{(\gamma)}\,g^2(x)}\,dx - 2c\int_\alpha^\beta \frac{dx}{\gamma^{3/2}} ,$$

where $a^* < a < \alpha$, $\beta < b < b^*$, $G(a) = G(b) = c$, $R(x)$, $\gamma(x, c)$ *are the same as those in Theorem* 2.1.

Proof. Define

$$J = \int_a^b (\gamma - 2c)\sqrt{(\gamma)}\,dx .$$

Note that $G(\alpha) = G(\beta) = 0$. Hence

$$J = \frac{2}{3}\left[\int_a^\alpha + \int_\beta^b\right] \frac{\gamma - 2c}{\left(\dfrac{\partial \gamma}{\partial x}\right)} \mathrm{d}\gamma^{3/2} + \int_\alpha^\beta (\gamma - 2c)\sqrt{(\gamma)}\,\mathrm{d}x =$$

$$= -\frac{2}{3}\left[\int_a^\alpha + \int_\beta^b\right] \frac{\gamma^{3/2}(g^2(x) - G(x)\,g'(x))}{g^2(x)}\,\mathrm{d}x +$$

$$+ \int_\alpha^\beta (\gamma - 2c)\sqrt{(\gamma)}\,\mathrm{d}x\,.$$

The rest of the proof is similar to that of Theorem 2.1.

Remark 2.10. Hypothesis (H2) guarantees that all the integrations in the above proof make sense.

Corollary 2.11. *Suppose* (H2) *holds. If*

(i) $g(x)$ *is odd,*

(ii) $g(\alpha) = 0$,

(iii) $g''(x) \leqq 0,\ a^* < x < \alpha$,

then

$$p'(c) < 0\,, \quad 0 < c < c^*\,.$$

Proof. From oddness, (2.26) becomes

$$c\,p'(c) = \int_a^\alpha \frac{2\,R(x)}{\sqrt{(\gamma)}\,g^2(x)}\,\mathrm{d}x - 2c\int_\alpha^{-\alpha} \frac{\mathrm{d}x}{\gamma^{3/2}}\,. \tag{2.27}$$

Because $g''(x) \leqq 0$, so $R'(x) \geqq 0$, $a^* < x < \alpha$. Therefore

$$R(x) \leqq R(\alpha) = g^2(\alpha) - 2\,G(\alpha)\,g'(\alpha) = 0\,, \quad a^* < x < \alpha\,.$$

From (2.27), the conclusion is obvious.

Theorem 2.12. *Suppose* (H2) *holds. Then for any* $0 < c < c^*$, *we have*

$$2c^2\,p''(c) = \left[\int_a^\alpha + \int_\beta^b\right] \frac{S(x)}{\sqrt{(\gamma)}\,g^4(x)}\,\mathrm{d}x + 12c^2\int_\alpha^\beta \frac{\mathrm{d}x}{\gamma^{5/2}}\,,$$

where $S(x)$, $\gamma(x, c)$, a, b, c *are the same as those in Theorem* 2.7.

Remark 2.13. Theorems 2.1 and 2.7 are special cases of Theorems 2.9 and 2.12, respectively.

SECTION 3. APPLICATIONS

In this section, the results of Section 2 will be applied to several examples to show the monotonicity of the period function $p(c)$. The following simple proposition is useful in applications.

Proposition 3.1. *Suppose* (H1) *holds. If* $g'(0) > 0$, $g''(0) \geqq 0$, *then each of the following conditions implies* $H(x) > 0$ (see (2.15)) *for* $x \neq 0$, $x \in (a_1, b_1)$.

(i) $g''(x) > 0$

and

$$\Delta(x) = x(g''(0)\, g'(x) - g'(0)\, g''(x)) \geqq 0\,, \quad x \in (a_1, b_1)\,,$$

where $a^* \leqq a_1 \leqq 0 \leqq b_1 \leqq b^*$;

(ii) $g''(x) > 0$, $g'''(x) \leqq 0$, $x \in (a_1, b_1)$,

where $a^* \leqq a_1 \leqq 0 \leqq b_1 \leqq b^*$;

(iii) $g''(x) < 0$, $g'(x) \geqq 0$, $0 \leqq a_1 < x < b_1 \leqq b^*$ *and* $H(a_1) \geqq 0$;

(iv) $g'(x) \leqq 0$, $0 < a_1 < x < b_1 \leqq b^*$;

(v) $g''(x) < 0$, $g'''(x) \geqq 0$, $a^* \leqq a_1 < x < b_1 < 0$ *and* $H(a_1) \geqq 0$, $H(b_1) \geqq 0$.

Example 1. Let

$$g(x) = \mathrm{e}^x - 1\,, \quad -\infty < x < +\infty\,.$$

Since $g'(x) = g''(x) = \mathrm{e}^x > 0$, $-\infty < x < +\infty$, and $\Delta(x) = x(g''(0)\, g'(x) - g'(0)\, g''(x)) \equiv 0$, by Proposition 3.1 (i) and Corollary 2.5 we have

$$p'(c) > 0\,, \quad 0 < c < +\infty\,.$$

From the results of Opial ([7]),

$$\lim_{c \to 0+} P(c) = 2\pi\,, \quad \lim_{c \to +\infty} p(c) = +\infty\,.$$

Remark 3.2. It seems that the above result does not follow from the monotonicity results in [5], [6], [7] and [8].

Example 2. Let $g(x)$ be a quadratic polynomial. We can consider the normal form ([4]):

$$g(x) = x(x + 1)\,, \quad -1 < x < +\infty\,.$$

Since $g''(x) = 2$, $g'''(x) \equiv 0$, by Proposition 3.1 (ii) and Corollary 2.5 we have

$$p'(c) > 0\,, \quad 0 < c < c^* = \tfrac{1}{6}\,.$$

Because c^* corresponds to a homoclinic orbit, so $\lim_{c \to c^*-} p(c) = +\infty$. By the result of Opial [7], $\lim_{c \to 0^+} p(c) = 2\pi$.

Example 3. Let $g(x)$ be a cubic polynomial. For periodic orbits with only one critical point in their interiors, we may consider the following normal forms:

(3.a) $\quad g(x) = -(x + a)\,x(x - 1)\,, \quad 0 < a \leqq 1\,, \quad -a < x < 1\,;$

(3.b) $\quad g(x) = x(x + a)\,(x + 1)\,, \quad 0 < a \leqq 1\,, \quad -a < x < +\infty\,;$

(3.c) $\quad g(x) = x(x^2 + bx + 1)\,, \quad 0 \leqq b < 2\,, \quad -\infty < x < +\infty\,;$

(3.d) $\quad g(x) = x^3\,, \quad -\infty < x < +\infty\,.$

For (3.a), $g(x) = -x^3 + (1 - a)\,x^2 + ax$. Then

$$g''(x) = -6x + 2(1 - a)\,,$$
$$g'''(x) = -6 < 0\,.$$

By Proposition 3.1 (ii), (iii), (iv) and Corollary 2.5,

$$p'(c) > 0\,, \quad 0 < c < c^* = G(-a)\,.$$

For (3.b), $g(x) = x^3 + (1 + a)\,x^2 + ax$. Then

$$g''(x) = 6x + 2(1 + a)\,,$$
$$g'''(x) = 6 > 0\,.$$

Hence $g''(x) > 0$ if and only if $x > -\frac{1}{3}(1 + a)$. Further,

$$\begin{aligned}\Delta(x) &= x^2[6(1 + a)\,x + 4(1 + a)^2 - 6a] \geqq \\ &\geqq x^2[6(1 + a)\,(-\tfrac{1}{3}(1 + a)) + 4(1 + a)^2 - 6a] = \\ &= x^2[2(1 + a)^2 - 6a] \geqq 0\,, \quad x > -\tfrac{1}{3}(1 + a)\,.\end{aligned}$$

By Proposition 3.1 (i), (v) and Corollary 2.5, we conclude

$$p'(c) > 0\,, \quad 0 < c < c^* = G(-a)\,.$$

For (3.c), if $b = 0$, then $g(x) = x^3 + x$, $g''(x) = 6x$. Then by Corollary 2.3,

$$p'(c) < 0\,, \quad 0 < c < +\infty\,.$$

If $b > \sqrt{(9/10)}$, then

$$\nabla = 5(g''(0))^2 - 3\,g'(0)\,g'''(0) = 20b^2 - 18 > 0\,.$$

By Corollary 2.6, there exists $\delta > 0$ such that $p'(c) > 0$, $0 < c < \delta$. On the other hand, by a result of Opial ([7]), $p(c) \to 0$ as $c \to +\infty$. This implies that $p(c)$ is not monotone.

For (3.d), $g''(x) = 6x$. Then by Corollary 2.3,

$$p'(c) < 0\,, \quad 0 < c < +\infty\,.$$

Remark 3.3. In [4], Chow and Sanders proved that there are at most 3 critical points of the period function when $g(x)$ is a polynomial of degree 3.

Example 4. Let $g(x) = -x^4 + x^3$, $-\infty < x < +1$. A direct calculation shows that

$$S(x) = \tfrac{4}{25}(156x^{16} - 624x^{15} + 896x^{14} - 550x^{13} + 125x^{12}).$$

Hence $S(x) > 0$ for $x < 0$. Furthermore,

$$S(x) = \tfrac{4}{25}x^{12}[156(x - 0{\cdot}65)^4 + 218{\cdot}4(x - 0{\cdot}65)^2 (1 - x) + \\ + 4{\cdot}879025(1 - x) + 2{\cdot}439025(1 - x)\, x + 0{\cdot}659025x^2] > 0, \quad 0 < x < 1.$$

By Theorem 2.7,

$$p''(c) > 0, \quad 0 < c < c^* = G(1) = \tfrac{1}{20}.$$

Since

$$\lim_{c \to 0^+} p(c) = \lim_{c \to c^* - 0} p(c) = +\infty,$$

$p(c)$ has exactly one critical point.

Example 5. Let $g(x) = x(x^2 - 1)^2$ and

$$G(x) = \int_0^x g(\xi)\, d\xi - \tfrac{1}{6}.$$

Then

$$G(x) = \tfrac{1}{6}(x + 1)^3 (x - 1)^3.$$

Since $g(x)$ is odd, $g(-1) = G(-1) = 0$, and $g''(x) = 20x^3 - 12x < 0$, $x < -1$, by Corollary 2.11, the period function of the periodic orbits with 3 critical points in their interiors is decreasing for $c \in (0, +\infty)$.

If we let

$$G(x) = \int_0^x g(\xi)\, d\xi,$$

then by Proposition 3.1 (i), (iii), (iv), (v), we have $H(x) > 0$ for $x \neq 0$, $x \in (-1, 1)$. Therefore

$$p'(c) > 0, \quad 0 < c < c^* = \tfrac{1}{6}.$$

We conclude that there are no critical points of the period function of the equation

$$x'' + x(x^2 - 1)^2 = 0.$$

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