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The concept of a quasi-jet of order two was introduced by Pradines, [3]. In the present paper, using the canonical structure properties of the r-sector spaces $T_rM = \underbrace{T\dots T}_{r\text{-times}}M$ and T_rN, see [4], we formulate the definition and the basic properties of quasi-jets of order r. This notion is a generalisation of the one of the jet and it seems to be a useful tool for studying several geometrical objects (for example connections) on T_rM. Our considerations are in the category C^∞.

1. Throughout this paper we will use the short notation (π) for a fibre bundle $\pi\colon Y \to M$, and $p_M\colon TM \to M$ or $TF\colon TM \to TN$ will denote the tangent bundle of a manifold M or the tangent mapping of a differentiable map $f\colon M \to N$, respectively.

We first recall some properties of vector bundles. Those which are generally known we introduce without proof.

Lemma 1. *$Tf\colon TM \to TN$ is a vector bundle morphism (shortly a v.b.m.).*

Lemma 2. *Let $q\colon E \to M$ be a vector bundle. Then p_E is a v.b.m. from (Tq) to (q).*

Lemma 3. *Let $\varphi\colon E_1 \to E_2$ be a v.b.m. from $q_1\colon E_1 \to M_1$ to $q_2\colon E_2 \to M$. Then $T\varphi$ is a v.b.m. from (Tq_1) to (Tq_2).*

Lemma 4. *Let $q_i\colon E_i \to M_i$, $i = 1, 2$, be two vector bundles. If $\psi\colon TE_1 \to TE_2$ is both a v.b.m. from (p_{E_1}) to (p_{E_2}) over the underlying base map $\psi_1\colon E_1 \to E_2$ and a v.b.m. from (Tq_1) to (Tq_2) over the underlying base* map *$\psi_2\colon TM_1 \to TM_2$ then ψ_1 and ψ_2 are v.b. morphisms.*

Proof. Let $u_1, u_2 \in (E_1)_x$. By Lemma 2 there exist $\bar{u}_1, \bar{u}_2$ in the same fibre of (Tq_1) such that $p_{E_1}(\bar{u}_i) = u_i$, $i = 1, 2$, $p_{E_1}(t_1\bar{u}_1 + t_2\bar{u}_2) = t_1u_1 + t_2u_2$. Then $\psi_1(t_1u_1 + t_2u_2) = p_{E_2} . \psi(t_1\bar{u}_1 + t_2\bar{u}_2) = t_1p_{E_2} . \psi(\bar{u}_1) + t_2p_{E_2} . \psi(\bar{u}_2) = t_1\psi_1(u_1) + t_2\,\psi_2(u_2)$ If $v_1, v_2 \in (TM_1)_x$, then by Lemma 1 there exist $\bar{v}_1, \bar{v}_2$ in the same fibre of (p_{E_1}) such that $Tq_1(\bar{v}_i) = v_i$, $i = 1, 2$. Then $\psi_2(t_1v_1 + t_2v_2) = Tq_2 . \psi(t_1\bar{v}_1 + t_2\bar{v}_2) = t_1Tq_2 . \psi(\bar{v}_1) + t_2Tq_2 . \psi(\bar{v}_2) = t_1\,\psi_2(v_1) + t_2\,\psi(v_2)$. Q.E.D.

Let $q: E \to M$ be a vector bundle and let VE_0 denote the set of all tangent vectors v on E such that $p_E(v) = 0 \in E$, $T_q(v) = 0 \in TM$ (vertical vectors at $0 \in E$). Let us recall the canonical identification $E \equiv VE_0 \subset TE$ which gives the canonical embedding $V_0: E \to TE$, $V_0(a) = j_0^1(t \mapsto ta)$, $t \in R$. If (x^i, y^α) or $(x^i, y^\alpha, dx^i, dy^\alpha)$ is a chart on E or TE, respectively, then

$$V_0(x^i, y^\alpha) = (x^i, 0, 0, y^\alpha)\,. \tag{1}$$

This immediately yields:

Lemma 5. *Let $q: E \to M$ be a vector bundle. Then $V_0: E \to TE$ is both a v.b.m. from (q) to (p_E) and a v.b.m. from (q) to (Tq).*

Lemma 6. *Let $\varphi: E_1 \to E_2$ be a v.b.m. of vector bundles $q_1: E_1 \to M_1$ and $q_2: E_2 \to M_2$. Then the diagram*

$$\begin{tikzcd} E_1 \arrow[r, "\varphi"] \arrow[d, "V_0"] & E_2 \arrow[d, "V_0"] \\ TE_1 \arrow[r, "T\varphi"] & TE_2 \end{tikzcd}$$

commutes.

Proof. Let $a \in (E_1)_x$. Then $T\varphi \,.\, V_0(a) = T\varphi(j_0^1(ta)) = j_0^1\, \varphi(ta)$. On the other hand, $V_0\, \varphi(a) = j_0^1(t\, \varphi(a)) = j_0^1\, \varphi(ta)$.

Lemma 7. *Let $q_i: E_i \to M_i$, $i = 1, 2$, be two vector bundles. Let $\psi: TE_1 \to TE_2$ be both a v.b.m. from (p_{E_1}) to (p_{E_2}) and a v.b.m. from (Tq_1) to (Tq_2). Then $\psi(V_0(E_1)) \subset V_0(E_2)$.*

Proof. We need to show $Tq_2 \,.\, \psi \,.\, V_0(E_1) = 0 = p_{E_2} \,.\, \psi \,.\, V_0(E_1)$. By Lemma 4, the underlying basic maps of ψ, both $\psi_1: E_1 \to E_2$ and $\psi_2: TM_1 \to TM_2$, are v.b. morphisms. If $h \in V_0(E_1)$ then $p_{E_1}(h) = 0$, $Tq_1(h) = 0$. Consequently $Tq_2 \,.\, \psi \,.\, . V_0(h) = \psi_2 \,.\, Tq_1(h) = 0$, $p_{E_2} \,.\, \psi \,.\, V_0(h) = \psi_1 \,.\, p_{E_1}(h) = 0$.

Lemma 8. *Let $q_1: E \to Y$, $q_2: E \to Y$; $q: Y \to M$ be vector bundles. Let q_2 be a v.b.m. from (q_1) to (q). Then the canonical embedding $V_0^1: E \to TE$, determined by q_1, preserves the fibres from (q_2) into (Tq_2).*

Proof. Let $a, b \in E$, $q_2(a) = q_2(b)$. Then $Tq_2 V_0^1(a) = Tq_2(j_0^1 ta) = j_0^1 q_2(ta) = = j(_0^1 tq_2(a)) = j_0^1(tq_2 b) = j_0^1 q_2(tb) = Tq_2(j_0^1 tb) = Tq_2 V_0^1(b)$.

Remark 1. In general, V_0^1 is not a v.b.m. from (q_2) to (Tq_2).

2. Quasijets. Let T be the tangent functor. By iteration we get $T_k M := \underbrace{T \dots T}_{k\text{-times}} M$ and $T_k f := T(\dots Tf): T_k M \to T_k N$. Denote by $p_j^M: T(T_{j-1}M) \to T_{j-1}M$ the tangent bundle projection $p_{T_{j-1}M}$. Then $T_{r-j}p_j^M: T_rM \to T_{r-1}M$ is a vector bundle. Hence on T_rM the following canonical vector bundle structures exist: $(T_{r-1}p_1^M), \ldots, (Tp_{r-1}^M)$, (p_r^M), see [4].

Lemma 9. *The tangent bundle projection* $p_i^M: T(T_{i-1}M) \to T_{i-1}M$ *is a v.b.m. from* $(T_{i-k}p_k^M)$ *to* $(T_{i-k-1}p_k^M)$, $k = 1, \ldots, i-1$.

Proof. Because $(T_{i-k-1}p_k^M)$, $k = 1, \ldots, i-1$, is a vector bundle structure on $T_{i-1}M$, therefore, by Lemma 2, p_i^M is a v.b.m. from $T(T_{i-k-1}p_k^M) \equiv T_{i-k}p_k^M$ to $(T_{i-k-1}p_k^M)$.

Corollary 1. *As* p_i *is a v.b.m. from* $(T_{i-k}p_k^M)$ *to* $(T_{i-k-1}p_k^M)$ *then, by Lemma* 3, $T_{r-i}p_i^M$ *is a v.b.m. from* $(T_{r-1}p_k^M)$ *to* $(T_{r-k-1}p_k^M)$, *where* $k = 1, \ldots, i-1$, $i \geqq 2$.

Corollary 2. *For* $s > i$, $T_{s-i-1}p_i: T_{s-1}M \to T_{s-2}M$ *is smooth. Then, by Lemma* 3, $T(T_{s-i-1})\, p_i \equiv T_{s-i}p_i$ *is a v.b.m. from* (p_s^M) *to* (p_{s-1}^M). *Therefore, by Lemma* 3, $T_{r-i}p_i$ *is a v.b.m. from* $(T_{r-s}p_s^M)$ *into* $(T_{r-s}p_{s-1})$, $i < s \leqq r$.

Definition 1. *Let* M, N *be smooth manifolds. A quasijet of order* r *with the source* $x \in M$ *and the target* $y \in N$ *is a map* $\varphi: (T_rM)_x \to (T_rN)_y$ *which is a v.b.m. from* $(T_{r-k}p_k^M)_x$ *to* $(T_{r-k}p_k^N)_y$ *for every* $k = 1, \ldots, r$. *The set of all quasijets of order* r *with the source* $x \in M$ *and the target* $y \in N$ *will be denoted by* $QJ_x^r(M, N)_y$. *Then* $QJ^r(M, N)$ *will mean the set of all quasijets from* M *to* N.

Proposition 1. *Let* $\varphi \in QJ_x^r(M, N)_y$. *Then the basic underlying map* $\varphi_i: (T_{r-1}M)_x \to$ $\to (T_{r-1}N)_y$ *of the v.b. morphism* φ *from* $(T_{r-i}p_i^M)_x$ *into* $(T_{r-i}p_i^N)_y$ *is a quasijet of order* $r-1$, *i.e.* $\varphi_i \in QJ^{r-1}{}_x(M, N)_y$.

Proof. It is necessary to prove that φ_i is a v.b.m. from $(T_{r-1-k}p_k^M)_x$ to $(T_{r-1-k}p_k^N)_y$ for $k = 1, \ldots, r-1$. By Corollary 1, $T_{r-i}p_i$ is a v.b.m. from $(T_{r-k}p_k^M)$ into $(T_{r-k-1}p_k^M)$ for $k = 1, \ldots, i-1$ and by Corollary 2, $T_{r-i}p_i$ is a v.b.m. from $(T_{r-k}p_k^M)$ to $(T_{r-k}p_{k-1}^M)$ for $i < k \leqq r$. Let u_1, u_2 be from the same fibre of $(T_{r-j-1}p_k^M)$, $k \leqq i-1$, or of $(T_{r-k}p_{k-1}^M)$, $i \leqq k-1 \leqq r-1$. Then there exist $\bar{u}_1, \bar{u}_2 \in (T_{r-k}p_k^M)$ such that $T_{r-i}p_i(\bar{u}_j) = u_j$, $j = 1, 2$. Then $\varphi_i(t_1u_1 + t_2u_2) = T_{r-i}p_i^N \cdot \varphi(t_1\bar{u}_1 + t_2\bar{u}_2) =$ $= t_1T_{r-i}p_i^N \cdot \varphi(\bar{u}_1) + t_2T_{r-i}p_i^N \cdot \varphi(\bar{u}_2) = t_1\, \varphi_i(u_1) + t_2\, \varphi_i(u_2)$. Q.E.D.

The map $\varkappa_i: QJ^r(M, N) \to QJ^{r-1}(M, N)$, where $\varkappa_i(\varphi) = \varphi_i$ is the underlying basic map of the v.b.m. $\varphi: (T_{r-i}p_i^M) \to (T_{r-i}p_i^N)_y$, will be called the i-basic projection.

Coordinates on $QJ^r(M, N)$. Let (x^i) or (y^α) be a chart on M or on N, respectively. Then $(x^i_{\varepsilon_1\ldots\varepsilon_r})$ or $(y_{\varepsilon_1\ldots\varepsilon_r})$, where $\varepsilon_i \in \{0, 1\}$, is the induced chart on T_rM or on T_rN, respectively. For example, $(x^i_{00}, x^i_{10}, x^i_{01}, x^i_{11})$ is a chart on T_2M. Since a quasijet of the first order is a linear map from T_xM into T_yN therefore $(x^i, {}^1a^\alpha_i, y^\alpha)$ is a local chart on $QJ^1(M,N)$. Let us suppose that we know the coordinate formulas for quasijets of order $r - 1$. Then by Proposition 1, for $\varphi \in QJ^r(M, N)$ it is sufficient to determine the form of the function $y^\alpha_{1\ldots1} = \varphi^\alpha_{1\ldots1}(x^i_{\varepsilon_1\ldots\varepsilon_r})$. However, by Definition 1, $\varphi^\alpha_{1\ldots1}(x^i_{\varepsilon_1\ldots\varepsilon_r})$ is a sector r-form and, by Whitte [4], its coordinate form is

$$(2) \qquad y^\alpha_{1\ldots1} = \sum_{(\gamma_1\ldots\gamma_k)\in S} a^{\alpha\gamma_1\ldots\gamma_k}_{i_1\ldots i_k} x^{i_1}_{\gamma_1} \ldots x^{i_k}_{\gamma_k},$$

where S denotes the set of all admissible decompositions of $e = (1, \ldots, 1)$ on $\gamma_1 = (e^1_1, \ldots, e^1_r), \ldots, \gamma_k = (e^k_1, \ldots, e^k_r)$, $e^u_s \in \{0, 1\}$ such that

(i) $\gamma_1 + \ldots + \gamma_k = (e^1_1, + \ldots + e^k_1, \ldots, e^1_r + \ldots + e^k_r) = e$,

(ii) if $i < j$ then $\deg \gamma_i < \deg \gamma_j$, where the number $\deg(\varepsilon_1, \ldots, \varepsilon_r)$ is defined in the following way: if $\varepsilon_1 = \ldots = \varepsilon_{s-1} = 0$ and $\varepsilon_s = 1$ then $\deg(\varepsilon_1, \ldots, \varepsilon_r) = s$.

Consequently, $\varphi \in QJ^r(M, N)$ has the coordinate form

$$(3) \qquad y^\alpha_{\bar\varepsilon_1\ldots\bar\varepsilon_r} = \sum_{(\gamma_1,\ldots,\gamma_k)} a^{\alpha\gamma_1\ldots\gamma_k}_{i_1\ldots i_k} x^{i_1}_{\gamma_1} \ldots x^{i_k}_{\gamma_k}$$

where $\gamma_1 + \ldots + \gamma_k = (\bar\varepsilon_1, \ldots, \bar\varepsilon_r)$ and $\deg \gamma_i < \deg \gamma_j$ if $i < j$.

For example, in the case when $\varphi \in QJ^2(M, N)$ we have:

$$y^\alpha_{10} = a^{\alpha 10}_i x^i_{10}, \quad y^\alpha_{01} = a^{\alpha 01}_i x^i_{01}, \quad y_{11} = a^{\alpha(10)(01)}_{ij} x^i_{10} x^j_{01} + a^{\alpha 11}_i x^i_{11}.$$

It implies a chart $(x^i, a^{\alpha 10}_i, a^{\alpha 01}_i, a^{\alpha 11}_i, a^{\alpha(10)(01)}_{ij})$ on $QJ^2(M, N)$.

By the standard procedure, one can show that $QJ^r(M, N)$ is a differentiable manifold and $\varkappa_i: QJ^r(M, N) \to QJ^{r-1}(M, N)$, $i = 1, \ldots, r$, is a fibre bundle structure. In coordinates, if $\varphi = (a^{\alpha\gamma_1\ldots\gamma_k}_{i_1\ldots i_k})$ then just the coordinates of φ for which $\gamma_j = (\gamma^j_1, \ldots, \gamma^j_i = 0, \ldots, \gamma^j_r)$ for all j, are also the coordinates of $\varkappa_i\varphi$.

Now we describe the other canonical submersion from $QJ^r(M, N)$ onto $QJ^{r-1}(M, N)$. On T_kM there exist k vector bundle structures: $(T_{k-i}p^M_i)$, $i = 1, \ldots, k$. Let $V^{iM}_{0k}: T_kM \to T(T_kM)$ denote the embedding determined by the vector bundle structure $(T_{k-i}p^M_i)$.

Lemma 10. *V^{iM}_{0K} is a v.b.m. from $(T_{k-j}p^M_j)$ to $(T_{k-j+1}p^M_j)$, $j = 1, \ldots, k$.*

Proof. Let $(x^i_{\varepsilon_1\ldots\varepsilon_k})$ be the induced chart on T_kM. Then the induced chart on $T_{k+1}M$ can be written in the form $(x^i_{\varepsilon_1\ldots\varepsilon_k 0}, x^i_{\varepsilon_1\ldots\varepsilon_k 1})$, i.e. the fibres of $T(T_kM)$ are determined by $x^i_{\varepsilon_1\ldots\varepsilon_k 0} = \text{const}$, i.e. $x^i_{\varepsilon_1\ldots\varepsilon_k 1}$ are the variables on fibres of $T(T_kM)$. In general, the coordinates $x^i_{\varepsilon_1\ldots\varepsilon_{k+1}}$, for which $\varepsilon_j = 1$, are variables on fibres of $(T_{k-j}p_j)$. By (1), the equations of V^{iM}_{0k} can be written in the form:

$$(4) \qquad \begin{aligned} &\bar x_{\varepsilon_1\ldots\varepsilon_i=1\ldots\varepsilon_k 0} = 0, \quad \bar x_{\varepsilon_1\ldots\varepsilon_i=0\ldots\varepsilon_k 1} = 0, \\ &\bar x_{\varepsilon_1\ldots\varepsilon_i=1\ldots\varepsilon_k 1} = x_{\varepsilon_1\ldots\varepsilon_i=1\ldots\varepsilon_k}, \\ &\bar x_{\varepsilon_1\ldots\varepsilon_i=0\ldots\varepsilon_k 0} = x_{\varepsilon_1\ldots\varepsilon_i=0\ldots\varepsilon_k}. \end{aligned}$$

It is clear that if $x_{\varepsilon_1\ldots\varepsilon_j=0\ldots\varepsilon_k} = \text{const}$ then $\bar x_{\varepsilon_1\ldots\varepsilon_j=0\ldots\varepsilon_{k+1}} = \text{const}$, i.e. V^i_{0k} preserves fibres from $(T_{k-j}p_j)$ to $(T_{k-j+1}p_j)$. The linearity of $V^i_{0K}: (T_{k-j}p^M_j) \to (T_{k-j+1}p_j)$ follows from (4) for $\varepsilon_j = 1$.

For every $k \leqq r - 1$ we have the embeddings $T_{r-k-1}V^i_{0k}\colon T_{r-1}M \to T_rM$, $i = 1, \dots, k$. Hence, in this way, we get $1 + \dots + r - 1$ vertical embeddings from T_{r-1} to T_rM. Denote $(T_{r-k-1}V^{iM}_{0K})^{-1}\colon T_{r-k-1}V^{iM}_{0k}(T_{r-1}M) \to T_{r-1}$ the map inverse to $T_{r-k-1}V^{iM}_{0k}$.

Proposition 2. *Let $\varphi \in QJ^r_x(M, N)_y$. Let $i \leqq k < r$. Then the map $\varphi^i_k = (T_{r-k-1}V^{iN}_{0k})^{-1} \cdot \varphi \cdot T_{r-k-1}V^{iM}_{0k}\colon (T_{r-1}M)_x \to (T_{r-1}N)_y$ is a quasijet of order $r - 1$.*

Proof. To prove it we must show that φ^i_k is a v.b.m. from $(V_{r-j-1}p^M_j)$ to $(T_{r-j-1}p^N_j)$, $j = 1, \dots, r-1$. Let $j \leqq k$. By Lemma 10, V^i_{0k} is a v.b.m. from $(T_{k-j}p_j)$ to $(T_{k-j+1}p_j)$, $j = 1, \dots, k$. Then by Lemma 3, $T_{r-k-1}V^i_{0k}$ is a v.b.m. from $(T_{r-j-1}p_j)$ to $(T_{r-j}p_j)$. By Lemma 7, $\varphi \cdot T_{r-k-1}V^{iM}_{0k}(T_{r-j-1}p^M_j) \subset T_{r-k-1}V^{iN}_{0k}(T_{r-j-1}p^N_j)$. Therefore φ^i_k is a v.b.m. from $(T_{r-j-1}p^M_j)$ into $(T_{r-j-1}p^N_j)$. Let $r > j > k$. Since $T_{j-k-1}V^{iM}_{0k}\colon T_{j-1}M \to T_jM$ is smooth, then $T(T_{j-k-1}V^{iM}_{0k})$ is a v.b.m. from (p^M_j) to (p^M_{j+1}). Hence $T_{r-k-1}V^{iM}_{0k}$ is a v.b.m. from $(T_{r-j-1}p^M_j)$ to $(T_{r-j-1}p^M_{j+1})$ and thus φ^i_k is a v.b.m. from $(T_{r-j-1}p^M_j)$ to $(T_{r-1-j}p^N_j)$.

By the equations (4), it is clear that the induced coordinates of φ^i_k are just the coordinates of $\varphi = (a^{\alpha\gamma_1\dots\gamma_s}_{i_1\dots i_s})$ for which $\gamma_j = (\varepsilon^j_1 \dots \varepsilon^j_i = 0 \dots \varepsilon^j_{k+1} = 0 \dots \varepsilon^j_r)$ or $\gamma_j = (\varepsilon^j_1 \dots \varepsilon^j_i = 1 \dots \varepsilon^j_{k+1} = 1 \dots \varepsilon^j_r)$ for every j. Therefore the map $\varkappa^i_k\colon QJ^r(M, N) \to QJ^{r-1}(M, N)$, $\varkappa^i_k(\varphi) = \varphi^i_k$, is a submersion. For instance, in the case $r = 2$, $\varkappa^1_1(x^i, a^{\alpha10}_i, a^{\alpha01}_i, a^{\alpha11}_i, a^{\alpha(10)(01)}_{ij}, y^\alpha) = (x^i, a^{\alpha11}_i, y^\alpha)$.

Let $a\colon QJ^r(M, N) \to M$ be the source projection. Then a local cross-section of (a) $\psi\colon M \supset U \to QJ^r(M, N)$ determines a map $\bar{\psi}\colon (T_rU) \to T_rN$, $\bar{\psi}(u) = \psi(a(u))(u)$ Let $x \in U$. Denote $T\bar{\psi}(x) := T\bar{\psi}|_{(T_{r+1}M)_x}$.

Lemma 11. *Let $\psi\colon M \supset U \to (a)$ be a local cross-section. Then for every $x \in U$, $T\bar{\psi}(x) \in QJ^{r+1}_x(M, N)$ and $\varkappa^i_r \cdot T\bar{\psi}(x) = \psi(x)$, $i = 1, \dots, r$.*

Proof. By Lemma 1, $T\bar{\psi}$ is a v.b.m. from (p^M_{r+1}) to (p^N_{r+1}). As $\bar{\psi}$ is a v.b.m. from $(T_{r-k}p^U_k)$ to $(T_{r-k}p^N_k)$ for $k = 1, \dots, r$, therefore, by Lemma 3, $T\bar{\psi}$ is a v.b.m. from $(T_{r-k+1}p^U_k)$ to $(T_{r-k+1}p^N_k)$. It means that $T\bar{\psi}(x) \in QJ^{r+1}_x(M, N)$. By Lemma 6, the diagram

$$\begin{array}{ccc} (T_{r-i}p^U_i) & \xrightarrow{\ \bar{\psi}\ } & (T_{r-i}p^N_i) \\ \big\downarrow{\scriptstyle V^{iU}_{or}} & & \big\downarrow{\scriptstyle V^i_{or}} \\ T(T_{r-i}p^U_i) & \xrightarrow{\ T\bar{\psi}\ } & T(T_{r-i}p^N_i) \end{array}$$

is commutative. It implies $\varkappa^i_r(T, \bar{\psi}) = \bar{\psi}$, $i = 1, \dots, r$.

Using (3) it is easy to verify the converse of Lemma 11:

Lemma 12. *If $A \in QJ^{r+1}_x(M, N)_y$ and $\varkappa^1_r A = \dots = \varkappa^r_r A = \varkappa_{r+1} A$, then there exists a local cross-section ψ of* (a) *such that $A = T\bar{\psi}(x)$.*

Relations to the theory of jets. The basic ideas of jets were introduced by Ehresmann, [1]. They can be reformulated to a form suitable for our purpose, see [3]. Let U be a neighbourhood of $x \in M$. Let $f: U \to N$ be a smooth map. Then $Tf(x)$ is a 1-jet with the source x and the target $y = f(x)$. Generally, $T_r f(x) = T(\dots Tf)_{(T_r M)_x}$ is a holonomic r-jet with the source x and the target y. We use the notation $j^r_x f$ for $T_r f(x)$ and $J^r(M, N)$ for the set of all holonomic r-jets. Let $a: J^1(M, N) \to M$ or $b: J^1(M, N) \to N$ be the source or the target projection. A local cross-section $\psi: U \to (a)$ gives a map $\bar{\psi}: TU \to TN$, $\bar{\psi}(h) = \psi(a(u))(u)$. Then $T\bar{\psi}(x) = T\bar{\psi}|_{(T_2M)_x}$: $(T_2M)_x \to T_2N$ is a non-holonomic 2-jet denoted by $j^1_x\psi$. IF $\psi(x) = T_x(b\psi)$ then $j^1_x\psi$ is said to be semiholonomic. Let $\tilde{J}^2(M, N)$ or $\bar{J}^2(M, N)$ be respectively the set of all non-holonomic or semiholonomic 2-jets from M to N. Denote by $\pi^2_1: \tilde{J}^2(M, N) \to$ $\to J^1(M, N)$ the canonical submersion $\pi^2_1(j^1_x\psi) = \psi(x)$. By induction, let $\tilde{J}^{r-1}(M, N)$ or $\bar{J}^{r-1}(M, N)$ be respectively the set of all non-holonomic or semiholonomic $(r - 1)$-jets and let $\pi^{r-1}_{r-2}: \tilde{J}^{r-1}(M, N) \to \tilde{J}^{r-2}(M, N)$ be the canonical projection. Let $\psi: U \to$ $\to \tilde{J}^{r-1}(M, N)$ be a cross-section of the source submersion $a: \tilde{J}^{r-1}(M, N) \to M$. Then $T\bar{\psi}(x) = T\bar{\psi}|_{(T_rM)_x}: (T_rM)_x \to T_rN$, where $\bar{\psi}(h) = \psi(a(h))(h)$, is a non-holonomic r-jet. If the values of ψ are semiholonomic $(r - 1)$-jets and $\psi(x) =$ $= T(\overline{\pi^{r-1}_{r-2}\psi})|_{(T_{r-1}M)_x}$ then $j^1_x\psi$ is a semiholonomic $(r - 1)$-jet. It is obvious that r-jets are quasijets of order r and the canonical projection π^r_{r-1} is identical with the above defined submersion $\varkappa_r$.

Using Lemmas 11 and 12 we get:

Proposition 3. *Let $\varphi \in QJ^r(M, N)$. Then φ is a non-holonomic r-jet iff* $\varkappa^1_1\varphi = \varkappa_2\varphi$, $\varkappa^1_2\varphi = \varkappa^2_2\varphi = \varkappa_3\varphi, \dots, \varkappa^1_{r-1}\varphi = \dots = \varkappa^{r-1}_{r-1}\varphi = \varkappa_r$.

Proposition 4. *A quasijet φ of order r is a semiholonomic r-jet iff* $\varkappa_1\varphi = \dots$ $\dots = \varkappa_r\varphi = \varkappa^1_1\varphi = \varkappa^1_2\varphi = \dots = \varkappa^{r-1}_{r-1}\varphi$.

Let us emphasize that $\varphi \in QJ^r_x(M, N)_y$ is a v.b.m. from $(T_{r-i}p^M_i)\,x$ to $(T_{r-i}p^N_i)_y$ for every $i = 1, \dots, r$. One can study v.b. morphisms from $(T_{r-j}p^M_j)$ to $(T_{r-k}p^N_k)$, $j \neq k$. As an example of such maps we introduce the canonical involutions on T_rM. Let us recall, see [2], that the canonical involution $i_2: T_2M \to T_2M$ has the following coordinate form: $i_2(x^i_{00}, x^i_{10}, x^i_{01}, x^i_{11}) = (x^i_{00}, x^i_{01}, x^i_{10}, x^i_{11})$ Denote by $i_3, \dots, i_r$ the canonical involutions on $T_2(TM) \equiv T_3M, \dots, T_2(T_{r-2}M)$. Then $T_{r-2}i_2, \dots$ $\dots, Ti_{r-1}, i_r$ are involutions on T_rM. It is easy to see that $T_{r-j}i_j$ is a v.b.m. from $(T_{r-j}p^M_j)$ to $(T_{r-j+1}p^M_{j-1})$. Denote by I_r the group of diffeomorphisms on T_rM which is generated by $T_{r-2}i_2, \dots, i_r$. I_r is isomorphic with the group of all permutations of the set $\{1, \dots, n\}$. Obviously $g \in I_r$ is a v.b.m. from $(T_{r-k}p^M_k)$ to $(T_{r-g(k)}p^M_{g(k)})$. It means that $\varphi . g$, id $\neq g \in I_r$, $\varphi \in QJ^r(M, N)$, is not a quasijet, but $g^{-1} . \varphi . g$ is. Let us remark that if A is a semiholonomic r-jet then the quasijet $B = T_{r-k}i_k . A .$ $. T_{r-k}i_k$ need not be semiholonomic. For instance, if $A = (x^i, a^\alpha_i, a^\alpha_{ij}, a^\alpha_{ijk}, y^\alpha) \in$ $\in \bar{J}^3(M, N)$ then $\varkappa_1(Tp_2 . A . Tp_2) = (x^i, a^\alpha_i, b^\alpha_{ij} = a^\alpha_{ji}, y^\alpha)$ and $\varkappa_3(Tp_2 . A . Tp_2) =$ $= (x^i, a^\alpha_i, c^\alpha_{ij} = a^\alpha_{ij}, y^\alpha)$, i.e. if $a_{ij} \neq a_{ji}$ then $\varkappa_3B \neq \varkappa_2B$, $B = Tp_2 . A . Tp_2$.

Lemma 13. *Let A be a semiholonomic r-jet. If $T_{r-j}i_j \, . \, A \, . \, T_{r-j}i_j = A, j \leqq r - 1$, then $T_{r-j-1}i_j \, . \, \pi^r_{r-1}A \, . \, T_{r-j-1}i_j = \pi^r_{r-1}A$.*

Proof $A = j^1_x\psi$, $A = T\bar{\psi}|_{(T_rM)_x}$, $\psi\colon M \to \bar{J}^{r-1}(M, N)$, $\psi(x) = \pi^r_{r-1}A$. As $T_{r-j}i_j \, . \, A \, . \, T_{r-j}i_j = (T_{r-j}i_j \, . \, T\bar{\psi} \, . \, T_{r-j}i_j)|_{(T_rM)_x} = T(T_{r-j-1}i_j \, . \, \bar{\psi} \, . \, T_{r-j-1}i_j)|_{(T_rM)_x}$, therefore if $T_{r-j}i_j \, . \, A \, . \, T_{r-j}i_j = A \equiv T\bar{\psi}|_{(T_rM)_x}$ then $\psi(x) = T_{r-j-1}i_j \, . \, \psi(x) \, . \, T_{r-j-1}i_j$. This completes our proof.

Let us recall that a semiholonomic jet A is holonomic iff all its coordinates $a_{i_1 \dots i_s}$ are symmetric in all subscripts.

Proposition 5. *If A is a holonomic r-jet then $T_{r-k}i_k \, . \, A \, . \, T_{r-k}i_k = A$ for every $k = 1, \dots, r$.*

Proof by induction. Because $a^\alpha_{ij} = a^\alpha_{ji}$ then Proposition 5 is true in the case $r = 2$. Let it be true for $r - 1$. Let $A = j^r_xf \equiv T_rf|_{(T_rM)_x}$, $F\colon M \to N$. Then $A = T(T_{r-1}f)|_{(T_rM)_x}$. By the induction assumption $T_{r-j-1}i_j \, . \, T_{r-1}f \, . \, T_{r-j-1}i_j = T_{r-1}f$ for $j = 1, \dots, r - 1$. Consequently, $T_{r-j}i_j \, . \, A \, . \, T_{r-j}i_j = T(T_{r-j-1}i_j \, . \, T_{r-1}f \, . \, T_{r-j-1}i_j)|_{T_rM_x} = T(T_{r-1}f)|_{(T_rM)_x} = A$, $j = 1, \dots, r - 1$. It is necessary to prove that $i_r \, . \, A \, . \, i_r = A$. Since $T_rf|_{(T_rM)_x} = T_2(T_{r-2}f)|_{(T_rM)_x}$, i_r is the canonical involution on $T_2(T_{r-2}M)$ and Proposition 5 is true for $r = 2$ then $i_r \, . \, A \, . \, i_r = i_r \, . \, T_2(T_{r-2}f) \, . \, i_r|_{(T_rM)_x} = T_2(T_{r-2}f)|_{(T_rM)_x} = A$.

Corollary 3. *If A is a holonomic r-jet then $g^{-1}Ag = A$ for every $g \in I_r$.*

Proposition 6. *If A is a semiholonomic r-jet and $T_{r-k}i_k \, . \, A \, . \, T_{r-k}i_k = A$ for every $k = 1, \dots, r$, then A is holonomic.*

Proof. In local coordinates it is easy to see that Proposition 5 is true for $r = 2$. Let it be true for $r - 1$. By Lemma 13, if $T_{r-j}i_j \, . \, A \, . \, T_{r-j}i_j = A, j = 1, \dots, r - 1$, then $T_{r-j-1}i_j \, . \, \pi^r_{r-1} \, . \, A \, . \, T_{r-j-1}i_j = \pi^r_{r-1}A$. Then, by the induction assumption, $\pi^r_{r-1}A$ is holonomic. As A is semiholonomic, it is sufficient to prove that its coordinates $a^\alpha_{i_1 \dots i_r}$ are symmetric in all subscripts. By (2), if $T_{r-k}i_k \, . \, A \, . \, T_{r-k}i_k = A$ then

$$a^\alpha_{i_1 \dots i_{k-1} i_k \dots i_r} x^{i_1}_{10\dots0} \cdots x^{i_{k-1}}_{0\dots\varepsilon_{k-1}=10\dots0} x^{i_k}_{0\dots\varepsilon_k=1,0\dots0} \cdots x^{i_r}_{0\dots01} = a_{i_1 \dots i_{k-1} i_k \dots i_r} x^{i_1}_{10\dots0} \cdots$$
$$\cdots x^{i_{k-1}}_{0\dots0\varepsilon_k=10\dots0} \, . \, x^{i_k}_{0\dots0\varepsilon_{k-1}=10\dots0} \cdots x^{i_r}_{0\dots01} \quad \text{for every} \quad x^i_{\varepsilon_1 \dots \varepsilon_r} \in (T_rM)_x \, .$$

Then $a_{i_1 \dots i_{k-1} i_k \dots i_r} = = a_{i_1 i_k j_{k-1} \dots i_r}$.

Remark 2. White, [4], defined a sector r-form on M as a real function $f\colon (T_rM)_x \to R$ linear on $(T_{r-k}p^M_k)$ for every $k = 1, \dots, r$. Let $\tau^rM \to M$ be the vector bundle of all sector r-forms on M. In the proof of Proposition 2, it was shown that $T_{r-k-1}V^i_{0k}$ is both a v.b.m. from $(T_{r-j-1}p_j)$ to $(T_{r-j}p_j)$ for $j \leqq k$ and a v.b.m. from $(T_{r-j-1}p_j)$ into $(T_{r-j-1}p_{j+1})$ if $r > j > k$. Hence if $f \in \tau^r_xM$ then $\varkappa^i_kf := f \, . \, T_{r-k-1}V^i_{0k} \in \tau^{r-1}_xM$. One can prove that $\varkappa^i_k\colon \tau^rM \to \tau^{r-1}M$, $f \mapsto \varkappa^i_kf$, is a submersion. Let $(\tau^rM)_0 :=$

$:= \{f \in \tau^r M; \varkappa_k^i f = 0$ for all i, k where $1 \leqq i \leqq k < r\}$. By (4), if $f = \sum_{(\gamma_1 \dots \gamma_j) \in S} a_{i_1 \dots i_j}^{\gamma_1 \dots \gamma_j} \,.$ $. \, x_{\gamma_1}^{i_1} \dots x_{\gamma_j}^{i_j}$, where S is the set of all admissible decompositions of $e = (1, \dots, 1)$, then $f \in (\tau^r M)_0$ iff all coordinates of f vanish with the exception of $a_{i_1 \dots i_r}^{(10\dots0)\dots(0\dots01)} \equiv$ $\equiv a_{i_1 \dots i_r}$. Let $X_1, \dots, X_r \in T_x M$. Then there exist $X \in (T_r M)_x$ such that $p_1 \dots$ $\dots p_{i-1} p_{i+1} \dots p_{r-1} \, p_r(X) = X_i$. Let $f \in (\tau_x^r M)_0$. Set $\bar{f}(X_1, \dots X_r) := f(X) =$ $a_{i_1 \dots i_r} x_1^{i_1} \dots x_r^{i_r}$. It implies the identification $(\tau^r M)_0 \equiv \otimes^r T^* M$.

Remark 3. All notions of the theory of jets can be extended to quasijets. For example, the composition of quasijets is immediately given by the composition of maps; a quasijet $A \in QJ_x^r(M, N)_y$ is called invertible if there exist $B \in QJ_y^r(N, M)_x$ such that $B \,.\, A = \mathrm{id}|_{(T_r M)_x}$. It is suitable to use for quasijets the notation from the theory of jets with the capital letter Q in front of the corresponding symbol. For instance, $QT_k^r M \equiv QJ_0^r(R^k, M)$; $QL_m^r \equiv \mathrm{Inv}\, QJ_0^r(R^m, R^m)_0$ is the set of all invertible quasijets from R^m to R^m with the source and target $0 \in R^m$; $QH^r M \equiv \mathrm{Inv}\, QJ_0^r(R^m, M)$, $\dim M = m$. One can show that $QH^r M$ is a principal fibre bundle with the structure group QL_m^r. Let $q \colon Y \to X$ be a fibre bundle. Then $QJ^r Y = \{u \in QJ^r(X, Y); T_r q \,.\, u =$ $= \mathrm{id}|_{(T^r X)_{\alpha(u)}}\}$ can be called the r-th quasijet prolongation of Y.

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