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ON GOOD CHARACTERIZATIONS FOR DIGRAPHS

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Suppose a class $\mathscr{G}$ of digraphs is characterized in two ways: by the nonexistence of a homomorphism *from* some fixed digraph A (this fact is denoted by $\mathscr{G} = A \nrightarrow$) and by the existence of a homomorphism *to* some fixed digraph B (this is denoted by $\mathscr{G} = \rightarrow B$). The equality $A \nrightarrow \; = \; \rightarrow B$ will be then called a *good characterization*. The importance of good characterizations consists in the fact that every theorem of this type makes the efficient solution of the appertaining problem possible.

The aim of this paper is to show some new good characterizations. Further, we shall prove one necessary condition for the digraphs A which appear in good characterizations of this type.

1. INTRODUCTION

In the beginning we shall present some necessary definitions and facts. Denote the set of all digraphs (without loops and multiple edges) as $\mathscr{D}$. A *homomorphism h* in $\mathscr{D}$ from a digraph $G = (V(G), E(G))$ to a digraph $H = (V(H), E(H))$ is a mapping $h: V(G) \rightarrow V(H)$ such that $(x, y) \in E(G) \Rightarrow (h(x), h(y)) \in E(H)$.

Suppose $G, H \in \mathscr{D}$, h is a homomorphism from G to H; we denote by $h(G)$ the subgraph of H defined by

$$h(G) = (\{h(v);\ v \in V(G)\},\ \{(h(v_1), h(v_2));\ (v_1, v_2) \in E(G)\})\,.$$

Further, we introduce the following classes of graphs:

$A \nrightarrow$ is the class of all digraphs $G \in \mathscr{D}$ such that there is no homomorphism from A to G.

$\rightarrow B$ is the class of all digraphs $G \in \mathscr{D}$ such that there exists a homomorphism from G to B.

An other concept to be defined is the *good characterization* (Edmonds, 1965). In this paper we will understand a good characterization as a relation $A \nrightarrow \; = \; \rightarrow B$ (see [3]).

There are two known classes of good characterizations in $\mathscr{D}$:

$P_n \nrightarrow \; = \; \rightarrow U_n$ (for every $n \geqq 1$) (see [4]) and

$A_{m,n} \nrightarrow \; = \; \rightarrow B_{m,n}$ (for every $m, n, m \geqq 2,\ n \geqq 2$) (see [2]).

2. THE NECESSARY CONDITION

We shall show here one necessary condition for the digraph A when $A \nrightarrow = \rightarrow B$.

Definition 1. Let $C = (V, E)$ be a semicycle (defined as in [1]) on n vertices, let $V = \{v_0, \ldots, v_{n-1}\}$, let $E = E_1 \cup E_2$, where E_1, E_2 are defined by

$E_1 = \{e \in E;\ e = (v_i, v_{i+1});\ i \in \{0, 1, \ldots, n - 1\}\}$,

$E_2 = \{e \in E;\ e = (v_{i+1}, v_i),\ i \in \{0, 1, \ldots, n - 1\}\}$, where $v_n = v_0$.

We call the semicycle C *balanced* iff $|E_1| = |E_2| = k\ (k = n/2)$.

Lemma 1. *The semicycle C is balanced iff there exists a directed tree T and a homomorphism h such that $h(C) = T$.*

Proof. 1) Suppose that $C = (\{v_0, \ldots, v_{2k-1}\}, E)$ is balanced. We shall construct T as a directed path whose vertices will be integers. Define $h(v_0) = 0$. Suppose $h(v_i) = m$. If $(v_i, v_{i+1}) \in E$, define $h(v_{i+1}) = m + 1$. If $(v_{i+1}, v_i) \in E$, define $h(v_{i+1}) = m - 1$. The semicycle C is balanced so that when we go through all the vertices of C (back to v_0), we must be again at the vertex 0. This means that h is a homomorphism from C to the directed path T, where $V(T) = \{r, r + 1, \ldots, -1, 0, 1, \ldots, s\}$ for some integers $r \leqq 0$, $s \geqq 0$.

2) Suppose that $C = (\{v_0, \ldots, v_{n-1}\}, E)$ is a semicycle, T is a directed tree and h is a homomorphism satisfying $h(C) = T$. For every $w \in V(T)$ define the height $H(w)$ by:

a) $H(h(v_0)) = 0$;

b) $H(w) = m + 1$, if there exists a vertex w' such that $(w', w) \in E(T)$, $H(w') = m$;

c) $H(w) = m - 1$, if there exists a vertex w' such that $(w, w') \in E(T)$, $H(w') = m$.

It follows that if $x, y \in V(T)$ satisfy $(x, y) \in E(T)$ then $H(y) = H(x) + 1$.

Let $e = (v_i, v_{i+1}) \in E$ (so that $e \in E_1$); then $(h(v_i), h(v_{i+1})) \in E(T)$ and $H(h(v_{i+1})) = H(h(v_i)) + 1$. On the other hand, let $e' = (v_{j+1}, v_j) \in E$ (so that $e' \in E_2$); then $(h(v_{j+1}), h(v_j)) \in E(T)$ and $H(h(v_{j+1})) = H(h(v_j)) - 1$. Starting at v_0 and going through all the vertices of C back to v_0, we must, in T, get to $h(v_0)$, with height $H(h(v_0)) = 0$. It follows that $|E_1| = |E_2|$. ∎

Theorem 1. *Let A, B be two digraphs satisfying $A \nrightarrow = \rightarrow B$. Then all semicycles contained in A are balanced.*

Proof. Suppose that there exist digraphs A, B such that $A \nrightarrow = \rightarrow B$ and A contains a non-balanced semicycle C. Put $k = |E(C)|$. Lemma 1 implies that for every homomorphism h the digraph $h(C)$, and then also $h(A)$, contains a semicycle C_1, where $|E(C_1)| \leqq k$. This means that every digraph which is not from the class $A \nrightarrow$ must contain a semicycle C_1, $|E(C_1)| \leqq k$. We shall use here the following theorem: For any natural numbers n, c, there exists a digraph G whose chromatic number $\chi(G) > c$ and such that G contains no semicycle C_1 with $|E(C_1)| \leqq n$.

Putting $c = \chi(B)$ and $n = k$, we see that there exists a digraph G with $G \in A \nrightarrow$ but $G \notin \rightarrow B$. This contradicts $A \nrightarrow = \rightarrow B$. ∎

3. GOOD CHARACTERIZATIONS

Now we shall define an infinite family of digraphs $A_{m,n,p,q}$ – examples are in Fig. 1 – and show that for every $A_{m,n,p,q}$ there exists a digraph $B_{m,n,p,q}$ such that $A_{m,n,p,q} \nrightarrow = \rightarrow B_{m,n,p,q}$.

Definition 2. Let m, n, p, q be integers, $0 \leqq p \leqq n - 2$, $0 \leqq q \leqq m - 2$. By $A_{m,n,p,q}$ we denote the digraph (V, E), where

$V = \{0, \ldots, m, m + 1, \ldots, m + p, -q', \ldots, -1', 0', 1', \ldots, n'\}$,

$E = \{(i, i + 1);\ i = 0, \ldots, m + p - 1\} \cup \{(i', i + 1');\ i = -q, \ldots, n - 1\} \cup$
$\cup \{(0', m)\}$ (see Fig.1).

Let us remark that the classes $A_{m,n,p,q} \nrightarrow$ are mutually different.

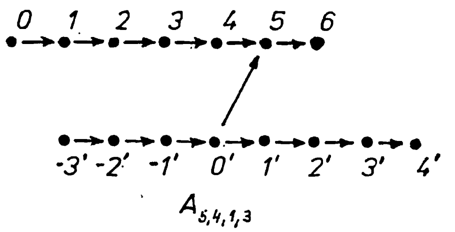


Fig. 1

Definition 3. Let G be a digraph without directed cycles. Define the *valuation* of the vertices of G by $\mu(x) = (d_1, d_2)$, where $d_1(d_2)$ is the length of the longest directed path ending (styrting, respectively) at the vertex x.

Definition 4. Let $r \geqq 1$, define the *digraph* $F_r = (V_r, E_r)$ by

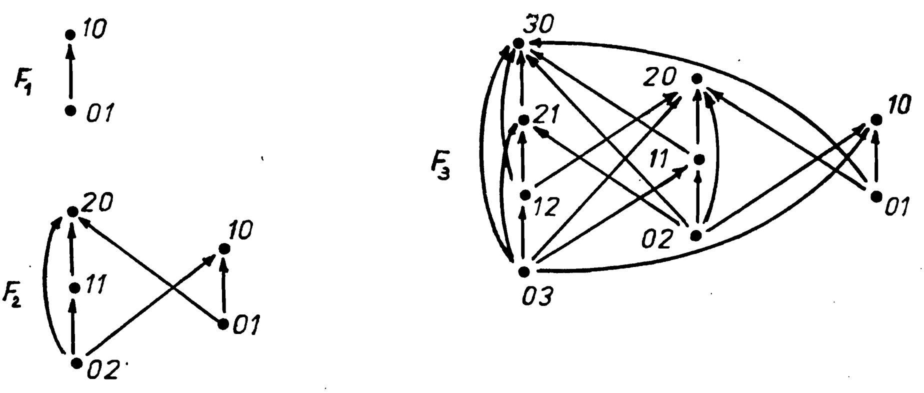


Fig. 2

$V_r = \{ij;\, i \geqq 0,\, j \geqq 0,\, 1 \leqq i + j \leqq r\}$,
$E_r = \{(ij, pq);\, ij, pq \in V_r,\, i < p,\, j > q\}$ (see Fig. 2).

Definition 5. Let m, n, p, q be integers, $0 \leqq p \leqq n - 2$, $0 \leqq q \leqq m - 2$. Define two subsets $V_{L(q)}$ and $V_{U(p)}$ of V_{m+n-2} by

$$V_{L(q)} = \{ij \in V_{m+n-2};\, j \geqq n,\, i \geqq q\}\,,$$
$$V_{U(p)} = \{ij \in V_{m+n-2};\, i \geqq m,\, j \geqq p\}$$

(in Fig. 3, the vertex sets V_{m+n-2}, $V_{L(q)}$ and $V_{U(p)}$ are shown for $m = 7$, $n = 5$, $p = 2$,

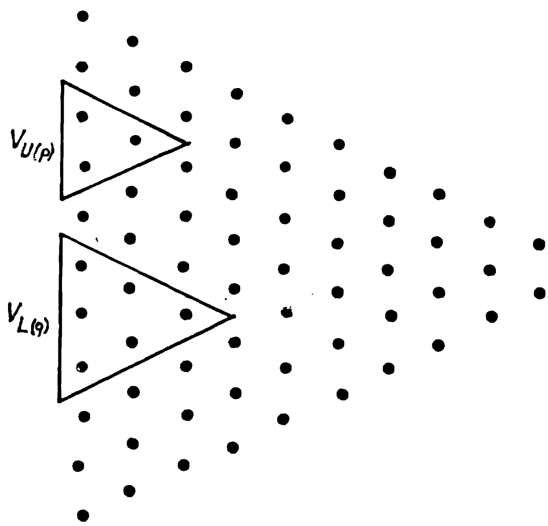


Fig. 3

$q = 3$), define the set $\bar{E}$ by

$$\bar{E} = \{(v_1, v_2) \in E_{m+n-2};\, v_1 \in V_{L(q)},\, v_2 \in V_{U(p)}\}\,.$$

Define now the *digraph* $B_{m,n,p,q} = (V', E')$ by $V' = V_{m+n-2}$, $E' = E_{m+n-2} \setminus \bar{E}$. In Fig. 4 we see $B_{2,4,1,0}$.

The following lemma appears already in [2]:

Lemma 2. *Let G be a digraph without directed cycles. Suppose for every $x \in V(G)$, $d_1 + d_2 \leqq r$ holds, where $(d_1, d_2) = \mu(x)$. In other words, its longest directed path contains at most r edges. There exists a homomorphism $h: G \to F_r$ satisfying the following conditions:*

Let $x \in V(G)$, $\mu(x) = (d_1, d_2)$.

a) *If $d_1 = d_2 = 0$, then $h(x) = 01$.*

b) *Otherwise $(1 \leqq d_1 + d_2 \leqq r)$, $h(x) = d_1 d_2$.*

Proof. For every $x \in V(G)$ the vertex $h(x)$ exists (see Definition 4). Let $v_1, v_2 \in V(G)$, $\mu(v_1) = (d_1, d_2)$, $\mu(v_2) = (d'_1, d'_2)$. Let $(v_1, v_2) \in E(G)$. Then $d_1 < d'_1$ and

$d_2 > d'_2$ (see Definition 3). The definition of E_r implies that $(d_1 d_2, d'_1 d'_2) \in E_r$. This means that $(v_1, v_2) \in E(G) \Rightarrow (h(v_1), h(v_2)) \in E_r$ holds. ■

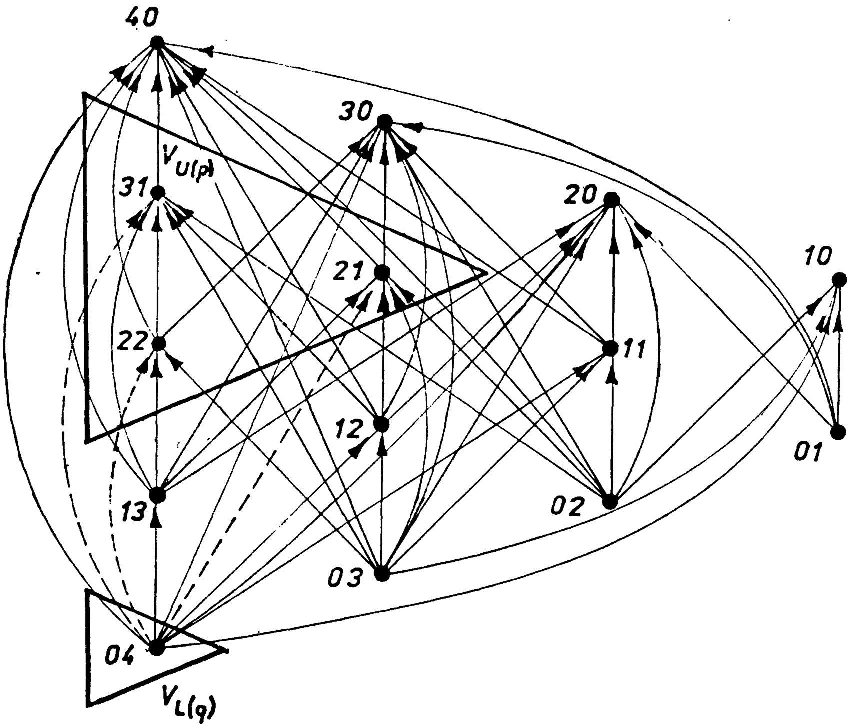


Fig. 4

Theorem 2. *Let m, n, p, q be integers, $0 \leqq p \leqq n - 2$, $0 \leqq q \leqq m - 2$. Then (in the class $\mathscr{D}$)*

$$A_{m,n,p,q} \nrightarrow = \rightarrow B_{m,n,p,q}\,.$$

Proof. a) First we shall prove that $A_{m,n,p,q} \nrightarrow \subseteq \rightarrow B_{m,n,p,q}$. Let $G \in A_{m,n,p,q} \nrightarrow$. This means that there is no homomorphism from $A_{m,n,p,q}$ to G. So, G contains no directed path P_k with $k \geqq m + n - 1$ and no directed cycle. Hence, by Lemma 2,

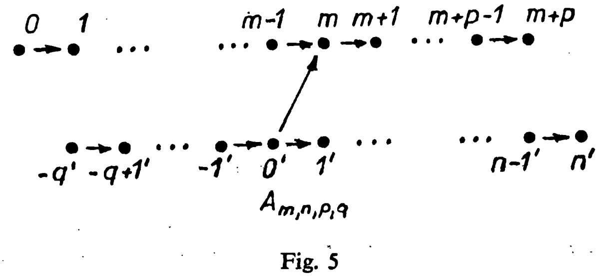
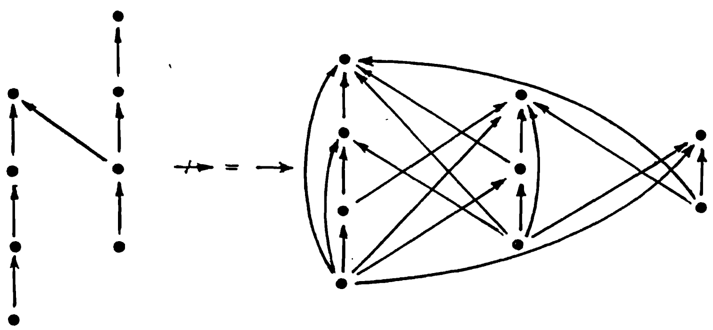


Fig. 5

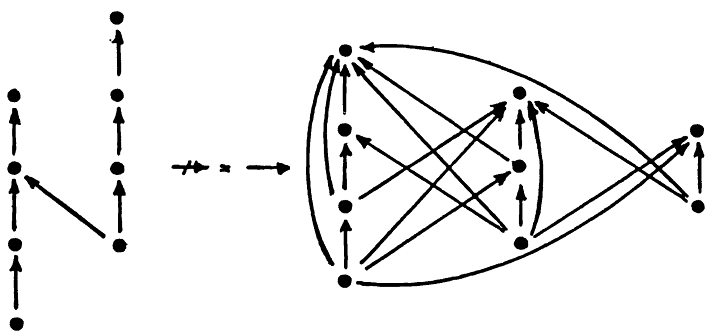
there exists a homomorphism h from G to F_{m+n-2}. Suppose now that there exists no homomorphism from G to $B_{m,n,p,q}$. That means that there exists an edge $(v_1, v_2) \in$ $\in E(G)$ such that $h(v_1) \in V_{L(q)}$, $h(v_2) \in V_{U(p)}$. There exists a directed path P_s $(s \geqq m)$ ending v_2 and another directed path $P_{s'}$ $(s' \geqq p)$ starting v_2. There exists also a directed path P_t $(t \geqq q)$ ending v_1 and another directed path $P_{t'}$ $(t' \geqq n)$ starting v_1. This is a contradiction with the assumption $G \in A_{m,n,p,q} \nrightarrow$. It means that $A_{m,n,p,q} \nrightarrow$ $\nrightarrow \subseteq \rightarrow B_{m,n,p,q}$ holds.

b) It remains to prove that $\rightarrow B_{m,n,p,q} \subseteq A_{m,n,p,q} \nrightarrow$. Let $G \in \rightarrow B_{m,n,p,q}$. Suppose that $G \notin A_{m,n,p,q} \nrightarrow$. From these two statements it follows that there exists a homomorphism h from $A_{m,n,p,q}$ to $B_{m,n,p,q}$. There is a directed path P_m ending at the vertex m and also another directed path P_p starting at m (see Fig. 5); this means that $h(m) \in$ $\in V_{U(p)}$. There is a directed path P_q ending at the vertex $0'$ and also another directed path P_n starting at $0'$; this means that $h(0') \in V_{L(q)}$.



$A_{3,2,0,1}$ $B_{3,2,0,1}$

Fig. 6a



$A_{2,3,1,0}$ $B_{2,3,1,0}$

Fig. 6b

We have $(0', m) \in E(A_{m,n,p,q})$, but there are no edges between $V_{L(q)}$ and $V_{U(p)}$ in $B_{m,n,p,q}$. This contradiction implies that if $G \in \rightarrow B_{m,n,p,q}$, then $G \in A_{m,n,p,q} \nrightarrow$. The proof is complete. ■

Some examples of good characterizations are seen in Fig. 6.

Remark. Theorem 2 shows that there exist good characterizations $A \nrightarrow\, = \,\rightarrow B$, where A contains vertices of degree 3. This suggests that the following conjecture is perhaps not too adventurous:

Conjecture. For every orientation of a tree T there exists a digraph B such that $T \nrightarrow\, = \,\rightarrow B$.

References

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