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MATHEMATICAL FOUNDATIONS OF THERMODYNAMICS OF IDEAL GAS

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Summary. An axiomatic system of the thermodynamics of ideal gas is proposed. After formulating several definitions and proving some fundamental theorems, adiabatic processes are defined and studied. Then the consistency and independence of axioms is proved.

Keywords: thermodynamics (mathematical foundations), adiabatic processes, axiomatic systems in physics.

AMS Subject Classification: 80A05

1. INTRODUCTION

Thermodynamics is usually explained in a manner which is very unsatisfactory for any mathematician. It is usually not clearly indicated which notions are primitive and which are defined. Two different standpoints are mixed: a statistical standpoint in which a system is considered as a set of particles, and a macroscopic standpoint in which a system is given by means of state quantities. This mixture of the two standpoints leads to inconsistent notions, as e.g. "*an infinitely slowly running process*". The proofs of theorems are often inexact and incomplete, sometimes heuristic considerations are presented as proofs (some remarks to these problems see in [7]). There are papers on thermodynamics – e.g. [1], [5], [6] – which are quite exact from the mathematical point of view, but they are so general and abstract that the application of the respective theories to simple particular cases is considerably difficult.

In this paper we try to lay, in a mathematically exact way, the foundations of thermodynamics of the ideal gas. We deduce the formula for the heat quantity consumed by a process, define adiabatic processes and find formulae for them. In current textbooks of thermodynamics these formulae are usually "deduced" from Gay-Lussac's and Boyle-Mariotte's laws, but some further suppositions are implicitly used (e.g. continuity). Moreover, the way of the deduction is not satisfactory from the logical point of view. The papers which are satisfactory from the theoretical point of view (as the above mentioned papers [1], [5], [6]) discuss the thermodynamics of systems more generally and the case of the ideal gas is mentioned only as an example; the formulae which we deduce are in these examples usually given as definitions. The deduction of these formulae is therefore kept on the level of heuristic considerations of the textbooks of classical thermodynamics. The result then is that an ordinary mathematician accustomed to exact mathematical methods is not able to read these papers.

We try to fill this gap. We introduce primitive notions, give axioms for them and deduce theorems from axioms. At the end, we prove consistency of our system of axioms and independence of the individual axioms on the other ones.

This paper is not intended for physicist (the less so for specialists in thermodynamics); they would hardly find anything new in it. It is intended for mathematicians who have only little knowledge of physics but who want to study some papers about thermodynamics and are not able to penetrate the logical chaos in the fundamental notions.

2. PRIMITIVE NOTIONS AND AXIOMS

If f is a mapping defined on a domain M and $M_1 \subset M$, we denote by $f \mid M_1$ the partial mapping defined on M_1. If we have mappings

$$f: M_1 \to M_2\,, \qquad g: M_2 \to M_3\,,$$

we denote by $g \circ f$ the mapping defined by

$$(g \circ f)(x) = g(f(x)) \quad \text{for every} \quad x \in M_1\,.$$

If J is a compact interval and f real function defined on an interval $K \supset J$, then the total variation of f on J will be denoted by $\mathscr{V}_f^J$.

Our starting point will be the theory of real numbers. We denote by $\mathbf{R}$ the set of all real numbers, by $\mathbf{R}^+$ the set of all real positive numbers, by $\boldsymbol{K}$ the set of all real compact non-degenerate intervals.

Put $\mathbf{S} = \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+$. Elements of the set $\mathbf{S}$ will be called states. We define mappings V, p, T: $\mathbf{S} \to \mathbf{R}^+$ in the following way: given $s = (a_1, a_2, a_3) \in \mathbf{S}$, then $\mathsf{V}(s) = a_1$, $\mathsf{p}(s) = a_2$, $\mathsf{T}(s) = a_3$. The numbers a_1, a_2, a_3 will be called the *volume*, *pressure* and *temperature* of the state s.

If $J \in \boldsymbol{K}$, we define a *process with the base* J as a mapping $\pi: J \to \mathbf{S}$ such that $\mathsf{V} \circ \pi$, $\mathsf{p} \circ \pi$, $\mathsf{T} \circ \pi$ are continuous functions with bounded variation. The set of all processes with the base J will be denote by $\boldsymbol{P}_J$. We put $\boldsymbol{P} = \bigcup_{J \in \boldsymbol{K}} \boldsymbol{P}_J$.

We will discuss thermodynamics as an axiomatic theory described in a way which was outlined by N. Bourbaki (see [2], § 1, n° 4). This method of description of a theory was applied by G. Ludwig to physical theories (see [3], § 7.1).

We will discuss a theory whose structure (see [2], § 1, n° 4 and [3], § 7.1) is given by a principal base formed by a single term $\boldsymbol{X}$, by an auxiliary base $\{\mathbf{R}, \mathbf{R}^+, \boldsymbol{K}, \boldsymbol{P}, \mathbf{S}\}$, by structural terms x_0, μ_0, W, Q which are characterized by the typification

(T1) $$x_0 \in X,$$

(T2) $$\mu_0 \in \mathbf{R}^+,$$

(T3) $$W \subset X \times \mathbf{S},$$

(T4) $$Q \subset X \times P \times \mathbf{R},$$

and by axioms **A1**, ..., **A9**, which we shall formulate below.

We will give an intuitive explication of these notions for readers who are not familiar with Bourbaki's method. We start from the theory of real numbers as a basic theory. The terms of the auxiliary base, i.e. $\mathbf{R}$, $\mathbf{R}^+$, K, P, $\mathbf{S}$, are sets defined in the basic theory, which we use to describe the primitive notions of our new theory. The term X of the principal base is a primitive notion of our theory. Intuitively, X is a set of some objects which we will call *systems*. The relation $x \in X$ will be read "*x is a system*". Further primitive notions are the structural terms x_0, μ_0, W, Q. The typifications (**T1**), ..., (**T4**) gives a characterisation of these primitive notions. So, x_0 is an element of X (or, in other words, x_0 is a system), and μ_0 is a real positive number. The system x_0 will be called the *scale system*, the number μ_0 will be called the *mass quantity of the scale system*. W is a set of pairs (x, s), where x is a system, s is a state. Therefore, W is a binary relation between X and $\mathbf{S}$. If this relation is fulfilled, i.e. if $(x, s) \in W$, we shall say that *s is a possible state of the system x*. Similarly, Q is a ternaty relation between X, P and $\mathbf{R}$. If this relation is fulfilled, i.e. if $(x, \pi, q) \in Q$, we shall say that *the process π of the system x consumes the heat quantity q*.

A process $\pi \in P_J$ is called a *possible process* of a system x, if $(x, \pi(\tau)) \in W$ for every $\tau \in J$.

We are now going to formulate the axioms **A1**, ..., **A9**.

Axiom A1. For any $x \in X$ there exists an $s \in \mathbf{S}$ such that $(x, s) \in W$.

Axiom A2. If $x \in X$, $s_1 \in \mathbf{S}$, $s_2 \in \mathbf{S}$, $(x, s_1) \in W$ and

(2.1) $$\frac{\mathsf{V}(s_1)\,\mathsf{p}(s_1)}{\mathsf{T}(s_1)} = \frac{\mathsf{V}(s_2)\,\mathsf{p}(s_2)}{\mathsf{T}(s_2)},$$

then $(x, s_2) \in W$.

Axiom A3. If π is a possible process of a system x, then there exists a $q \in \mathbf{R}$ such that $(x, \pi, q) \in Q$.

Axiom A4. For any $x \in X$,, $J \in K$, $\pi \in P_J$, where π is a possible process of x, and for any $\varepsilon \in \mathbf{R}^+$ there exists a $\delta \in \mathbf{R}^+$ with the following property:

if $\sigma \in P_J$, σ being a possible process of x, if

(2.2) $$(x, \pi, q) \in Q, \qquad (x, \sigma, r) \in Q,$$

(2.3) $$\mathscr{V}^J_{\mathsf{V}\circ\sigma} \leqq \mathscr{V}^J_{\mathsf{V}\circ\pi}\,, \quad \mathscr{V}^J_{\mathsf{p}\circ\sigma} \leqq \mathscr{V}^J_{\mathsf{p}\circ\pi}\,,$$
and if

(2.4) $$|\mathsf{V}(\pi(\tau)) - \mathsf{V}(\sigma(\tau))| < \delta\,, \quad |\mathsf{p}(\pi(\tau)) - \mathsf{p}(\sigma(\tau))| < \delta$$
for every $\tau \in J$, then

(2.5) $$|q - r| < \varepsilon\,.$$

Remark 2.1. The axiom **A4** says that the heat quantity consumed by a process depends continuously on the process. However, the continuity thus formulated has further consequences (see theorems 3.1 and 3.2).

Axiom A5. If $x \in \boldsymbol{X}$, $a < b < c$ are real numbers, $J = \langle a, b\rangle$, $K = \langle b, c\rangle$, $L = J \cup K$, $\pi_L \in \boldsymbol{P}_L$, $\pi_J = \pi_L \mid J$, $\pi_K = \pi_L \mid K$, $q_J \in \mathbf{R}$, $q_K \in \mathbf{R}$, $(x, \pi_J, q_J) \in Q$, $(x, \pi_K, q_K) \in Q$, then $(x, \pi_L, q_J + q_K) \in Q$.

Axiom A6. Let $a < b$ be real numbers, $J = \langle a, b\rangle$, and let $\pi \in \boldsymbol{P}_J$, $\sigma \in \boldsymbol{P}_J$ be possible processes of a system x. Let the functions $\mathsf{V} \circ \pi$, $\mathsf{V} \circ \sigma$ be constant on J. Then (2.2) implies

(2.6) $$q(\mathsf{T}(\sigma(b)) - \mathsf{T}(\sigma(a))) = r(\mathsf{T}(\pi(b)) - \mathsf{T}(\pi(a)))\,.$$

Axiom A7. Let $a < b$ be real numbers, $J = \langle a, b\rangle$, and let $\pi \in \boldsymbol{P}_J$, $\sigma \in \boldsymbol{P}_J$ be possible processes of a system x. Let the function $\mathsf{p} \circ \pi$, $\mathsf{p} \circ \sigma$ be constant on J. Then (2.2) implies (2.6).

Axiom A8. Let $a < b$ be real numbers, $J = \langle a, b\rangle$, and let $\pi \in \boldsymbol{P}_J$ be a possible process of a system x. Let the function $\mathsf{V} \circ \pi$ be constant on J and let $\mathsf{T}(\pi(b)) > > \mathsf{T}(\pi(a))$, $(x, \pi, q) \in Q$. Then $q > 0$.

Axiom A9. Let $a < b$ be real numbers, $J = \langle a, b\rangle$, and let $\pi \in \boldsymbol{P}_J$, $\sigma \in \boldsymbol{P}_J$ be possible processes of a system x. Let $\mathsf{V}(\pi(\tau)) = \mathsf{V}(\sigma(\tau))$, $\mathsf{p}(\pi(\tau)) \leqq \mathsf{p}(\sigma(\tau))$ for all $\tau \in J$. Let the function $\mathsf{V} \circ \pi = \mathsf{V} \circ \sigma$ be increasing on J. Let $\mathsf{p}(\pi(a)) = \mathsf{p}(\sigma(a))$ and let $\mathsf{p}(\pi(\tau)) < \mathsf{p}(\sigma(\tau))$ for some $\tau \in J$. Then (2.2) implies $q < r$.

Remark 2.2. The axiom **A9** has the following heuristic justification: the heat consumed by the process π of a system x is proportional to the work produced by the system x during the process π. However, as the volume is an increasing function, this work is positive, and, as the two processes π, σ have the same volume at each moment, the system produces larger work by higher pressure.

Remark 2.3. The structural terms x_0, μ_0 define a system of physical units. It is not necessary to introduce these terms as primitive notions, but otherwise they would appear as parameters in many theorems. It is therefore simpler, from the formal viewpoint, to consider them as primitive notions.

3. SOME SIMPLE CONSEQUENCE OF AXIOMS AND DEFINITIONS

The axioms **A3** and **A4** imply

Theorem 3.1. *If π is a possible process of a system x, then there exists one and only one number $q \in \mathbf{R}$ such that $(x, \pi, q) \in Q$.*

This number q will be denoted by $q_{x,\pi}$.

Proof. I. The existence of the number q follows from the axiom **A3.**

II. Suppose $(x, \pi, q) \in Q$, $(x, \pi, r) \in Q$. Let $\varepsilon \in \mathbf{R}^+$. If we put $\pi = \sigma$ in the axiom **A4**, then (2.2), (2.3) and (2.4) are fulfilled for any $\delta > 0$. Therefore (2.5) also holds. As $\varepsilon > 0$ is arbitrary, we obtain $q = r$.

Theorem 3.2. *Let $x \in X$, $s_1 \in \mathbf{S}$, $s_2 \in \mathbf{S}$, $(x, s_1) \in W$. Then $(x, s_2) \in W$ if and only if (2.1) holds.*

Proof. I. If (2.1) holds, then $(x, s_2) \in W$ by the axiom **A2.**

II. Let $(x, s_1) \in W$, $(x, s_2) \in W$. Put $J = \langle 1, 2 \rangle$ and define processes $\pi \in \boldsymbol{P}_J$, $\sigma \in \boldsymbol{P}_J$ by

$$\pi\colon \tau \mapsto \left(1, \tau, \frac{\tau\, \mathsf{T}(s_1)}{\mathsf{V}(s_1)\, \mathsf{p}(s_1)}\right),$$

$$\sigma\colon \tau \mapsto \left(1, \tau, \frac{\tau\, \mathsf{T}(s_2)}{\mathsf{V}(s_2)\, \mathsf{p}(s_2)}\right).$$

As

$$\frac{\mathsf{V}(\pi(\tau))\, \mathsf{p}(\pi(\tau))}{\mathsf{T}(\pi(\tau))} = \frac{\mathsf{V}(s_1)\, \mathsf{p}(s_1)}{\mathsf{T}(s_1)}, \quad \frac{\mathsf{V}(\sigma(\tau))\, \mathsf{p}(\sigma(\tau))}{\mathsf{T}(\sigma(\tau))} = \frac{\mathsf{V}(s_2)\, \mathsf{p}(s_2)}{\mathsf{T}(s_2)}$$

for all $\tau \in J$, the processes π, σ are, by the axiom **A2**, possible processes of the system x. By Theorem 3.1, there exist uniquely determined numbers q, r such that (2.2) holds. As $\mathsf{V} \circ \pi = \mathsf{V} \circ \sigma$ and $\mathsf{p} \circ \pi = \mathsf{p} \circ \sigma$, (2.3) and (2.4) are also fulfilled (for any $\delta > 0$). Let $\varepsilon \in \mathbf{R}^+$. Then, by the axiom **A4**, (2.5) is fulfilled and therefore $q = r$. By the axiom **A8** we have $q > 0$. But by the axiom **A6** we have

$$q(\mathsf{T}(\sigma(2)) - \mathsf{T}(\sigma(1))) = r(\mathsf{T}(\pi(2)) - \mathsf{T}(\pi(1))),$$

therefore

$$q\left(\frac{2\, \mathsf{T}(s_2)}{\mathsf{V}(s_2)\, \mathsf{p}(s_2)} - \frac{\mathsf{T}(s_2)}{\mathsf{V}(s_2)\, \mathsf{p}(s_2)}\right) = r\left(\frac{2\, \mathsf{T}(s_1)}{\mathsf{V}(s_1)\, \mathsf{p}(s_1)} - \frac{\mathsf{T}(s_1)}{\mathsf{V}(s_1)\, \mathsf{p}(s_1)}\right).$$

As $q = r > 0$, we conclude that (2.1) holds.

Theorem 3.2 implies

Theorem 3.3. *If systems x, y have a possible state in common, then all their possible states are common.*

Theorem 3.4. *For any system x there exists exactly one number $\mu(x)$ satisfying*

$$(3.1) \qquad \mu(x) = \frac{\mathsf{V}(s)\,\mathsf{p}(s)\,\mathsf{T}(s_0)}{(\mathsf{V}s_0)\,\mathsf{p}(s_0)\,\mathsf{T}(s)}\,\mu_0\,,$$

where

$$(3.2) \qquad (x, s) \in W\,, \quad (x_0, s_0) \in W\,.$$

Proof. By (**T1**) and the axiom **A1** there exist states s_0, s satisfying (3.2), therefore for any system x there is a number $\mu(x)$ satisfying (3.1) and (3.2). By Theorem 3.2, this number is independent of the choice of the states s, s_0, as far as they satisfy (3.2).

Definition. The number $\mu(x)$ is called the *mass quantity* of the system x.

Theorem 3.5. *The inequality $\mu(x) > 0$ holds for every system x. Moreover, $\mu(x_0) = \mu_0$.*

Proof. The first assertion follows from (**T2**), (3.1) and from the definition of a state; the second is a consequence of (3.1) and Theorem 3.2.

Theorem 3.2 and the axiom **A1** imply

Theorem 3.6. *There exists a uniquely determined number R such that*

$$(3.3) \qquad R = \frac{\mathsf{p}(s_0)\,\mathsf{V}(s_0)}{\mathsf{T}(s_0)} \cdot \frac{1}{\mu_0}\,,$$

where s_0 is a possible state of the system x_0. Moreover, $R > 0$.

The $R > 0$ follows from (**T2**) and the definition of a state. By substituting (3.3) into (3.1), Theorem 3.4 yields

Theorem 3.7. *If $x \in X$, $s \in \mathbf{S}$, $(x, s) \in W$, then*

$$(3.4) \qquad \mu(x) = \frac{1}{R} \cdot \frac{\mathsf{V}(s)\,\mathsf{p}(s)}{\mathsf{T}(s)}\,.$$

4. SPECIFIC HEAT

For any $J \in \boldsymbol{K}$ denote

$\boldsymbol{P}^{\mathsf{V}}_{J,x} = \{\pi \in \boldsymbol{P}_J\colon \pi \text{ is a possible process of the system } x,\ \mathsf{V} \circ \pi \text{ is constant on } J\}$,

$\boldsymbol{P}^{\mathsf{p}}_{J,x} = \{\pi \in \boldsymbol{P}_J\colon \pi \text{ is a possible process of the system } x,\ \mathsf{p} \circ \pi \text{ is constant on } J\}$.

Theorem 4.1. *Let $x \in X$. Then there exists one and only one number $\tilde{c}_{\mathsf{V}}(x)$ with the following property: if $a < b$, $J = \langle a, b\rangle$, $\pi \in \boldsymbol{P}^{\mathsf{V}}_{J,x}$, then*

$$q_{x,\pi} = \tilde{c}_{\mathsf{V}}(x)\,(\mathsf{T}(\pi(b)) - \mathsf{T}(\pi(a)))\,.$$

Proof. I. Let $x \in X$, $a < b$, $J = \langle a, b \rangle$. Then there exists one and only one number $\tilde{c}^J_{\mathsf{V}}(x)$ with the property: if $\pi \in \boldsymbol{P}^{\mathsf{V}}_{J,x}$, then

$$q_{x,\pi} = \tilde{c}^J_{\mathsf{V}}(x)\left(\mathsf{T}(\pi(b)) - \mathsf{T}(\pi(a))\right). \tag{4.1}$$

Indeed, by the axiom **A1**, there exists a state $s_1 = (V_1, p_1, T_1)$ such that $(x, s_1) \in W$. For $\tau \in J$ put

$$\sigma(\tau) = (V_1, f(\tau), T_1 + \tau - a),$$

where

$$f(\tau) = p_1 \frac{T_1 + \tau - a}{T_1}.$$

By the axiom **A2**, we have $\sigma \in \boldsymbol{P}^{\mathsf{V}}_{J,x}$. Put

$$\tilde{c}^J_{\mathsf{V}}(x) = \frac{q_{x,\sigma}}{b - a}.$$

By Theorem 3.1, $\tilde{c}^J_{\mathsf{V}}(x)$ is uniquely determined and, by the axiom **A6**, (4.1) holds for any $\pi \in \boldsymbol{P}^{\mathsf{V}}_{J,x}$.

II. Let $x \in X$, $J \in \boldsymbol{K}$, $K \in \boldsymbol{K}$. We will prove that

$$\tilde{c}^J_{\mathsf{V}}(x) = \tilde{c}^K_{\mathsf{V}}(x). \tag{4.2}$$

III. Let $J = \langle a, b \rangle$, $K = \langle b, c \rangle$. Denote $L = \langle a, c \rangle$. Let again $s_1 = (V_1, p_1, T_1)$, $(x, s_1) \in W$. Define processes $\pi_L \in \boldsymbol{P}^{\mathsf{V}}_{L,x}$, $\sigma_L \in \boldsymbol{P}^{\mathsf{V}}_{L,x}$ by

$$\pi_L(\tau) = (V_1, f_1(\tau), g_1(\tau)),$$
$$\sigma_L(\tau) = (V_1, f_2(\tau), g_2(\tau)),$$

where

$$g_1(\tau) = T_1 + \frac{\tau - a}{b - a}, \quad g_2(\tau) = T_1 + 2\frac{\tau - a}{b - a} \quad \text{for} \quad \tau \in J,$$

$$g_1(\tau) = T_1 + 1 + 2\frac{\tau - b}{c - b}, \quad g_2(\tau) = T_1 + 2 + \frac{\tau - b}{c - b} \text{ for } \quad \tau \in K,$$

$$f_1(\tau) = \frac{p_1}{T_1} g_1(\tau), \quad f_2(\tau) = \frac{p_1}{T_1} g_2(\tau) \quad \text{for} \quad \tau \in L.$$

By I, we have

$$q_{x,\pi_L} = \tilde{c}^L_{\mathsf{V}}(x)\left(g_1(c) - g_1(a)\right) = 3\,\tilde{c}^L_{\mathsf{V}}(x),$$
$$q_{x,\sigma_L} = \tilde{c}^L_{\mathsf{V}}(x)\left(g_2(c) - g_2(a)\right) = 3\,\tilde{c}^L_{\mathsf{V}}(x)$$

therefore $q_{x,\pi_L} = q_{x,\sigma_L}$. Put

$$\pi_J = \pi_L \mid J, \quad \pi_K = \pi_L \mid K, \quad \sigma_J = \sigma_L \mid J, \quad \sigma_K = \sigma_L \mid K.$$

By the axiom **A5** and by I, we have

$$q_{x,\pi_L} = q_{x,\pi_J} + q_{x,\pi_K} = \tilde{c}^J_{\mathsf{V}}(x)\left(g_1(b) - g_1(a)\right) +$$
$$+ \tilde{c}^K_{\mathsf{V}}(x)\left(g_1(c) - g_1(b)\right) = \tilde{c}^J_{\mathsf{V}}(x) + 2\,\tilde{c}^K_{\mathsf{V}}(x),$$

$$q_{x,\sigma_L} = q_{x,\sigma_J} + q_{x,\sigma_K} = \tilde{c}_V^J(x)\,(g_2(b) - g_2(a)) + \\ + \tilde{c}_V^K(x)\,(g_2(c) - g_2(b)) = 2\,\tilde{c}_V^J(x) + \tilde{c}_V^K(x)\,,$$

therefore

$$\tilde{c}_V^J(x) + 2\,\tilde{c}_V^K(x) = 2\,\tilde{c}_V^J(x) + \tilde{c}_V^K(x)\,,$$

hence (4.2) holds.

IV. Let $J = \langle a, b\rangle$, $K = \langle c, d\rangle$ with $b < c$. Denote $L = \langle b, c\rangle$. By III, we have

$$\tilde{c}_V^J(x) = \tilde{c}_V^L(x)\,, \quad \tilde{c}_V^L(x) = \tilde{c}_V^K(x)$$

and therefore (4.2).

V. Let $J = \langle a, b\rangle$, $K = \langle c, d\rangle$ arbitrary. Choose $A \in \mathbf{R}$, $B \in \mathbf{R}$ such that

$$\max(b, d) < A < B$$

and denote $L = \langle A, B\rangle$. By IV, we have

$$\tilde{c}_V^J(x) = \tilde{c}_V^L(x)\,, \quad \tilde{c}_V^K(x) = \tilde{c}_V^L(x)$$

and therefore (4.2).

Similarly, the axioms **A1**, **A2**, **A5**, **A7** and Theorem 3.1 imply

Theorem 4.2. *Let $x \in \boldsymbol{X}$. Then there exists one and only one number $\tilde{c}_p(x)$ with the following property: if $a < b$, $J = \langle a, b\rangle$, $\pi \in \boldsymbol{P}_{J,x}^p$, then*

$$q_{x,\pi} = \tilde{c}_p(x)\,(\mathsf{T}(\pi(b)) - \mathsf{T}(\pi(a)))\,.$$

Definition. For $x \in \boldsymbol{X}$ put

$$c_V(x) = \tilde{c}_V(x)/\mu(x)\,, \quad c_p(x) = \tilde{c}_p(x)/\mu(x)\,.$$

The number $c_V(x)$ is called the *specific heat of the system x by constant volume,* the number $c_p(x)$ is called the *specific heat of x by constant pressure.*

This definition and Theorem 4.1 imply

Theorem 4.3. *If $x \in \boldsymbol{X}$, $a < b$, $J = \langle a, b\rangle$, $\pi \in \boldsymbol{P}_{J,x}^V$, then*

$$q_{x,\pi} = c_V(x)\,\mu(x)\,(\mathsf{T}(\pi(b)) - \mathsf{T}(\pi(a)))\,.$$

Similarly, Theorem 4.2 implies

Theorem 4.4. *If $x \in \boldsymbol{X}$, $a < b$, $J = \langle a, b\rangle$, $\pi \in \boldsymbol{P}_{J,x}^p$, then*

$$q_{x,\pi} = c_p(x)\,\mu(x)\,(\mathsf{T}(\pi(b)) - \mathsf{T}(\pi(a)))\,.$$

Theorem 4.5. *We have $c_V(x) > 0$ for every $x \in \boldsymbol{X}$.*

Proof. By the axiom **A1**, there exists a possible state $s_1 = (V_1, p_1, T_1)$ of the system x. Put $J = \langle 0, 1\rangle$, choose a number $T_2 > T_1$ and define a process $\pi \in \boldsymbol{P}_J$ by

$$\pi(\tau) = (V_1, f(\tau), T_1 + \tau(T_2 - T_1))\,,$$

where

$$f(\tau) = \frac{p_1}{T_1}(T_1 + \tau(T_2 - T_1)) .$$

π is a possible process of x by the axiom **A2**, therefore $\pi \in \boldsymbol{P}^{\mathsf{V}}_{J,x}$. By Theorem 4.3 we have

$$q_{x,\pi} = c_{\mathsf{V}}(x)\,\mu(x)\,(T_2 - T_1) .$$

But $T_2 - T_1 > 0$, $\mu(x) > 0$ by Theorem 3.5 and $q_{x,\pi} > 0$ by the axiom **A8**, therefore also $c_{\mathsf{V}}(x) > 0$.

5. HEAT CONSUMED BY A PROCESS

Theorem 5.1. *Let $J = \langle a, b\rangle$ and let $\pi \in \boldsymbol{P}_J$ be a possible process of a system x. Let $(x, \pi, q) \in Q$. Then*

$$(5.1)\qquad q = \frac{1}{R}\left\{c_{\mathsf{p}}(x)\int_a^b \mathsf{p}(\pi(t))\,\mathrm{d}\mathsf{V}(\pi(t)) + c_{\mathsf{V}}(x)\int_a^b \mathsf{V}(\pi(t))\,\mathrm{d}\mathsf{p}(\pi(t))\right\},$$

where $\int$ denotes the Riemann-Stieltjes integral.[1])

Proof. I. Let us have a number $\varepsilon > 0$. By the axiom **A4**, there exists a $\delta \in \mathbf{R}^+$ with the following property: if $\sigma \in \boldsymbol{P}_J$ is a possible process of x, if (2.2) and (2.3) hold and if (2.4) is true for all $\tau \in J$, then

$$(5.2)\qquad |q - r| < \tfrac{1}{4}\varepsilon .$$

We choose such a δ, so small that

$$(5.3)\qquad \delta < \frac{R\varepsilon}{1 + 4c_{\mathsf{V}}(x)\,\mathscr{V}^J_{\mathsf{V}\circ\pi}}$$

($c_{\mathsf{V}}(x) > 0$ by Theorem 4.5).

II. The functions $\mathsf{V} \circ \pi$, $\mathsf{p} \circ \pi$ are continuous on the compact interval J, therefore there exists a number $\eta_1 > 0$ such that

$$(5.4)\qquad |\mathsf{V}(\pi(\tau_1)) - \mathsf{V}(\pi(\tau_2))| < \delta , \quad |\mathsf{p}(\pi(\tau_1)) - \mathsf{p}(\pi(\tau_2))| < \delta$$

if

$$\tau_1 \in J , \quad \tau_2 \in J , \quad |\tau_1 - \tau_2| < \eta_1 .$$

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[1]) We use the definition of the Riemann-Stieltjes integral given in [4]:
$\int_a^b f\,\mathrm{d}g = A$, if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|A - \sum_{i=1}^{n} f(\xi_i)\,(g(x_i) - g(x_{i-1}))| < \varepsilon$$

for all partitions

$$\mathscr{D}\colon a = x_0 \leqq x_1 \leqq \ldots \leqq x_n = b$$

such that $\max_{i=1,\ldots,n}(x_i - x_{i-1}) < \delta$, and for every choice $\xi_i \in \langle x_{i-1}, x_i\rangle$.

Further, there exists a number $\eta_2 > 0$ with the following property: if we have a partition

$$\mathscr{D}: a = a_0 < a_1 < \ldots < a_n = b \tag{5.5}$$

of the interval J such that

$$\max_{1 \leqq i \leqq n} (a_i - a_{i-1}) < \eta_2$$

and if we have numbers $\xi_1, \ldots, \xi_n$ such that $a_{i-1} \leqq \xi_i \leqq a_i$ for $i = 1, 2, \ldots, n$, then

$$\left| \int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) - \sum_{i=1}^n \mathsf{p}(\pi(\xi_i)) \left(\mathsf{V}(\pi(a_i)) - \mathsf{V}(\pi(a_{i-1}))\right) \right| < \frac{R\varepsilon}{1 + 4|c_{\mathsf{p}}(x)|}, \tag{5.6}$$

$$\left| \int_a^b \mathsf{V}(\pi(t))\, \mathrm{d}\mathsf{p}(\pi(t)) - \sum_{i=1}^n \mathsf{V}(\pi(\xi_i)) \left(\mathsf{p}(\pi(a_i)) - \mathsf{p}(\pi(a_{i-1}))\right) \right| < \frac{R\varepsilon}{1 + 4|c_{\mathsf{V}}(x)|} \tag{5.7}$$

(see [4], Chap. X, § 7, Definition 21).

III. Choose a natural number n such that

$$\Delta = (b - a)/n < \min(\eta_1, \eta_2)\,. \tag{5.8}$$

In the partition (5.5) put

$$a_i = a + i\Delta\,, \quad i = 0, 1, \ldots, n\,, \tag{5.9}$$

and denote

$$J_i = \langle a_{i-1}, a_i \rangle\,. \tag{5.10}$$

If we now have $\tau_1 \in J_i$, $\tau_2 \in J_i$ for some i, $1 \leqq i \leqq n$, then (5.4) holds. If the partition $\mathscr{D}$ is defined by (5.5), (5.8) and (5.9) and $\xi_i \in J_i$ for $i = 1, 2, \ldots, n$, then (5.6) and (5.7) hold.

IV. There exist numbers $c_i \in J_i$, $i = 1, 2, \ldots, n$, such that

$$\mathsf{p}(\pi(c_i)) = \max_{\tau \in J} \mathsf{p}(\pi(\tau))\,. \tag{5.11}$$

For $i = 1, 2, \ldots, n$ denote

$$K_i = \langle a_{i-1}, a_{i-1} + \tfrac{1}{3}\Delta \rangle\,, \quad N_i = \langle a_{i-1} + \tfrac{1}{3}\Delta, a_i - \tfrac{1}{3}\Delta \rangle\,,$$
$$L_i = \langle a_i - \tfrac{1}{3}\Delta, a_i \rangle\,.$$

We define the process $\sigma \in \boldsymbol{P}_J$ in the following way:

$$\mathsf{V}(\sigma(\tau)) = \mathsf{V}(\pi(a_{i-1})) \quad \text{for} \quad \tau \in K_i\,, \tag{5.12a}$$

$$\mathsf{V}(\sigma(\tau)) = \mathsf{V}(\pi(a_{i-1} + 3(\tau - a_{i-1} - \tfrac{1}{3}\Delta))) \quad \text{for} \quad \tau \in N_i\,, \tag{5.12b}$$

$$\mathsf{V}(\sigma(\tau)) = \mathsf{V}(\pi(a_i)) \quad \text{for} \quad \tau \in L_i\,, \tag{5.12c}$$

$$\mathsf{p}(\sigma(\tau)) = \mathsf{p}(\pi(a_{i-1})) + \frac{3(\tau - a_{i-1})}{\Delta} \left(\mathsf{p}(\pi(c_i)) - \mathsf{p}(\pi(a_{i-1}))\right) \quad \text{for} \quad \tau \in K_i\,, \tag{5.13a}$$

$$\mathsf{p}(\sigma(\tau)) = \mathsf{p}(\pi(c_i)) \quad \text{for} \quad \tau \in N_i\,, \tag{5.13b}$$

$$(5.13c)\quad \mathsf{p}(\sigma(\tau)) = \mathsf{p}(\pi(c_i)) + \frac{3(\tau - a_{i-1} - \frac{2}{3}\Delta)}{\Delta}(\mathsf{p}(\pi(a_i)) - \mathsf{p}(\pi(c_i))) \quad \text{for} \quad \tau \in L_i\,,$$

$$(5.14)\qquad \mathsf{T}(\sigma(\tau)) = \frac{\mathsf{V}(\sigma(\tau))\,\mathsf{p}(\sigma(\tau))}{\mathsf{V}(\pi(a))\,\mathsf{p}(\pi(a))}\,\mathsf{T}(\pi(a)) \quad \text{for all} \quad \tau \in J\,.$$

By (5.14) and by the axiom **A2**, σ is a possible process of the system x.

V. We have (2.4) for every $\tau \in J$.

Indeed, if $\tau \in J_i$, then by (5.12) there exists *a* $\tilde{\tau} \in J_i$ such that $\mathsf{V}(\sigma(\tau)) = \mathsf{V}(\pi(\tilde{\tau}))$, and (5.4) implies (with respect to (5.8)) that $|\mathsf{V}(\pi(\tau)) - \mathsf{V}(\pi(\tilde{\tau}))| < \delta$, therefore $|\mathsf{V}(\pi(\tau)) - \mathsf{V}(\sigma(\tau))| < \delta$. Moreover, (5.11) and (5.13) imply that for $\tau \in J_i$ we have

$$\min\left(\mathsf{p}(\pi(a_{i-1})), \mathsf{p}(\pi(a_i))\right) \leqq \mathsf{p}(\sigma(\tau)) \leqq \mathsf{p}(\pi(c_i))\,;$$

by (5.4) we have

$$|\mathsf{p}(\pi(c_i)) - \mathsf{p}(\pi(a_i))| < \delta\,, \quad |\mathsf{p}(\pi(c_i)) - \mathsf{p}(\pi(a_{i-1}))| < \delta\,,$$
$$|\mathsf{p}(\pi(c_i)) - \mathsf{p}(\pi(\tau))| < \delta$$

and therefore $|\mathsf{p}(\pi(\tau)) - \mathsf{p}(\sigma(\tau))| < \delta$.

VI. The inequalities (2.3) are satisfied.

Indeed,

$$\mathscr{V}^{J}_{\mathsf{V}\circ\sigma} = \sum_{i=1}^{n} \mathscr{V}^{J_i}_{\mathsf{V}\circ\sigma}\,, \quad \mathscr{V}^{J}_{\mathsf{V}\circ\pi} = \sum_{i=1}^{n} \mathscr{V}^{J_i}_{\mathsf{V}\circ\pi}\,,$$
$$\mathscr{V}^{J_i}_{\mathsf{V}\circ\sigma} = \mathscr{V}^{K_i}_{\mathsf{V}\circ\sigma} + \mathscr{V}^{N_i}_{\mathsf{V}\circ\sigma} + \mathscr{V}^{L_i}_{\mathsf{V}\circ\sigma}\,,$$

but (5.12) implies

$$\mathscr{V}^{K_i}_{\mathsf{V}\circ\sigma} = \mathscr{V}^{L_i}_{\mathsf{V}\circ\sigma} = 0\,, \quad \mathscr{V}^{N_i}_{\mathsf{V}\circ\sigma} = \mathscr{V}^{J_i}_{\mathsf{V}\circ\pi}\,,$$

therefore

$$\mathscr{V}^{J_i}_{\mathsf{V}\circ\sigma} = \mathscr{V}^{J_i}_{\mathsf{V}\circ\pi}\,, \qquad \mathscr{V}^{J}_{\mathsf{V}\circ\sigma} = \mathscr{V}^{J}_{\mathsf{V}\circ\pi}\,.$$

Moreover,

$$\mathscr{V}^{J}_{\mathsf{p}\circ\sigma} = \sum_{i=1}^{n} \mathscr{V}^{J_i}_{\mathsf{p}\circ\sigma}\,, \qquad \mathscr{V}^{J}_{\mathsf{p}\circ\pi} = \sum_{i=1}^{n} \mathscr{V}^{J_i}_{\mathsf{p}\circ\pi}\,,$$
$$\mathscr{V}^{J_i}_{\mathsf{p}\circ\sigma} = \mathscr{V}^{K_i}_{\mathsf{p}\circ\sigma} + \mathscr{V}^{N_i}_{\mathsf{p}\circ\sigma} + \mathscr{V}^{L_i}_{\mathsf{p}\circ\sigma}\,,$$

but (5.13) implies

$$\mathscr{V}^{K_i}_{\mathsf{p}\circ\sigma} = |\mathsf{p}(\pi(c_i)) - \mathsf{p}(\pi(a_{i-1}))|\,, \quad \mathscr{V}^{N_i}_{\mathsf{p}\circ\sigma} = 0\,,$$
$$\mathscr{V}^{L_i}_{\mathsf{p}\circ\sigma} = |\mathsf{p}(\pi(a_i)) - \mathsf{p}(\pi(c_i))|\,,$$

therefore

$$\mathscr{V}^{J_i}_{\mathsf{p}\circ\sigma} = |\mathsf{p}(\pi(a_i)) - \mathsf{p}(\pi(c_i))| + |\mathsf{p}(\pi(c_i)) - \mathsf{p}(\pi(a_{i-1}))| \leqq \mathscr{V}^{J_i}_{\mathsf{p}\circ\pi}\,,$$

therefore

$$\mathscr{V}^{J}_{\mathsf{p}\circ\sigma} \leqq \mathscr{V}^{J}_{\mathsf{p}\circ\pi}\,.$$

VII. From I, IV, V and VI we obtain

$$(5.15)\qquad |q_{x,\pi} - q_{x,\sigma}| < \tfrac{1}{4}\varepsilon\,.$$

VIII. We will calculate $q_{x,\sigma}$. By the axiom **A5**, we have

$$q_{x,\sigma} = \sum_{i=1}^{n} q_{x,\sigma|J_i} = \sum_{i=1}^{n} (q_{x,\sigma|K_i} + q_{x,\sigma|N_i} + q_{x,\sigma|L_i}) \,. \tag{5.16}$$

Theorem 4.3 with respect to (5.12a), (5.13a), (5.14) and to Theorem 3.7 implies

$$\begin{aligned} q_{x,\sigma,|K_i} &= c_{\mathsf{V}}(x)\, \mu(x) \left(\mathsf{T}(\sigma(a_{i-1} + \tfrac{1}{3}\Delta)) - \mathsf{T}(\sigma(a_{i-1}))\right) = \\ &= c_{\mathsf{V}}(x)\, \mu(x) \{\mathsf{V}(\sigma(a_{i-1} + \tfrac{1}{3}\Delta))\, \mathsf{p}(\sigma(a_{i-1} + \tfrac{1}{3}\Delta)) - \\ &- \mathsf{V}(\sigma(a_{i-1}))\, \mathsf{p}(\sigma(a_{i-1}))\} \frac{\mathsf{T}(\pi(a))}{\mathsf{V}(\pi(a))\, \mathsf{p}(\pi(a))} = \\ &= c_{\mathsf{V}}(x)\, \mu(x)\, \mathsf{V}(\pi(a_{i-1})) \left(\mathsf{p}(\pi(c_i)) - \mathsf{p}(\pi(a_{i-1}))\right) \frac{1}{R} \frac{1}{\mu(x)} = \\ &= \frac{1}{R} c_{\mathsf{V}}(x)\, \mathsf{V}(\pi(a_{i-1})) \left(\mathsf{p}(\pi(c_i)) - \mathsf{p}(\pi(a_{i-1}))\right), \end{aligned}$$

and similarly (with respect to (5.12c) and (5.13c)),

$$\begin{aligned} q_{x,\sigma|L_i} &= c_{\mathsf{V}}(x)\, \mu(x) \left(\mathsf{T}(\sigma(a_i)) - \mathsf{T}(\sigma(a_i - \tfrac{1}{3}\Delta))\right) = \\ &= \frac{1}{R} c_{\mathsf{V}}(x)\, \mathsf{V}(\pi(a_i)) \left(\mathsf{p}(\pi(a_i)) - \mathsf{p}(\pi(c_i))\right); \end{aligned}$$

Theorem 4.4 with respect to (5.12b), (5.13b), (5.14) and to Theorem 3.7 implies

$$\begin{aligned} q_{x,\sigma|N_i} &= c_{\mathsf{p}}(x)\, \mu(x) \left(\mathsf{T}(\sigma(a_i - \tfrac{1}{3}\Delta)) - \mathsf{T}(\sigma(a_{i-1} + \tfrac{1}{3}\Delta))\right) = \\ &= c_{\mathsf{p}}(x)\, \mu(x) \{\mathsf{V}(\sigma(a_i - \tfrac{1}{3}\Delta))\, \mathsf{p}(\sigma(a_i - \tfrac{1}{3}\Delta)) - \\ &- \mathsf{V}(\sigma(a_{i-1} + \tfrac{1}{3}\Delta))\, \mathsf{p}(\sigma(a_{i-1} + \tfrac{1}{3}\Delta))\} \frac{\mathsf{T}(\pi(a))}{\mathsf{V}(\pi(a))\, \mathsf{p}(\pi(a))} = \\ &= c_{\mathsf{p}}(x)\, \mu(x)\, \mathsf{p}(\pi(c_i)) \left(\mathsf{V}(\pi(a_i)) - \mathsf{V}(\pi(a_{i-1}))\right) \frac{1}{R} \frac{1}{\mu(x)} = \\ &= \frac{1}{R} c_{\mathsf{p}}(x)\, \mathsf{p}(\pi(c_i)) \left(\mathsf{V}(\pi(a_i)) - \mathsf{V}(\pi(a_{i-1}))\right). \end{aligned}$$

Thus we obtain

$$\begin{aligned} q_{x,\sigma|J_i} &= q_{x,\sigma|K_i} + q_{x,\sigma|N_i} + q_{x,\sigma|L_i} = \\ &= \frac{1}{R} c_{\mathsf{V}}(x) \{\mathsf{V}(\pi(a_{i-1})) \left(\mathsf{p}(\pi(c_i)) - \mathsf{p}(\pi(a_{i-1}))\right) + \\ &+ \mathsf{V}(\pi(a_i)) \left(\mathsf{p}(\pi(a_i)) - \mathsf{p}(\pi(c_i))\right)\} + \\ &+ \frac{1}{R} c_{\mathsf{p}}(x)\, \mathsf{p}(\pi(c_i)) \left(\mathsf{V}(\pi(a_i)) - \mathsf{V}(\pi(a_{i-1}))\right) = \end{aligned}$$

$$= \frac{1}{R} c_V(x) V(\pi(a_{i-1})) (p(\pi(a_i)) - p(\pi(a_{i-1}))) +$$

$$+ \frac{1}{R} c_p(x) p(\pi(c_i)) (V(\pi(a_i)) - V(\pi(a_{i-1}))) +$$

$$+ \frac{1}{R} c_V(x) (V(\pi(a_i)) - V(\pi(a_{i-1}))) (p(\pi(a_i)) - p(\pi(c_i))) .$$

If we substitute this result into (5.16), we obtain

$$q_{x,\sigma} = \frac{1}{R} c_V(x) \sum_{i=1}^{n} V(\pi(a_{i-1})) (p(\pi(a_i)) - p(\pi(a_{i-1}))) + \tag{5.17}$$

$$+ \frac{1}{R} c_p(x) \sum_{i=1}^{n} p(\pi(c_i)) (V(\pi(a_i)) - V(\pi(a_{i-1}))) +$$

$$+ \frac{1}{R} c_V(x) \sum_{i=1}^{n} (V(\pi(a_i)) - V(\pi(a_{i-1}))) (p(\pi(a_i)) - p(\pi(c_i))) .$$

IX. By III, we have $|p(\pi(a_i)) - p(\pi(c_i))| < \delta$ for $i = 1, 2, \ldots, n$. Therefore, from (5.3) we obtain

$$\left|\frac{1}{R} c_V(x) \sum_{i=1}^{n} (V(\pi(a_i)) - V(\pi(a_{i-1}))) (p(\pi(a_i)) - p(\pi(c_i)))\right| \leqq$$

$$\leqq \frac{1}{R} c_V(x) \sum_{i=1}^{n} |V(\pi(a_i)) - V(\pi(a_{i-1}))| \cdot |p(\pi(a_i)) - p(\pi(c_i))| <$$

$$< \frac{1}{R} c_V(x) \frac{R\varepsilon}{1 + 4c_V(x) \mathscr{V}^J_{V\circ\pi}} \sum_{i=1}^{n} |V(\pi(a_i)) - V(\pi(a_{i-1}))| \leqq$$

$$\leqq c_V(x) \frac{\varepsilon}{1 + 4c_V(x) \mathscr{V}^J_{V\circ\pi}} \mathscr{V}^J_{V\circ\pi} < \tfrac{1}{4}\varepsilon .$$

X. Using (5.17), we obtain

$$\left|q_{x,\pi} - \frac{1}{R}\left\{c_p(x) \int_a^b p(\pi(t))\, dV(\pi(t)) + c_V(x) \int_a^b V(\pi(t))\, dp(\pi(t))\right\}\right| \leqq$$

$$\leqq |q_{x,\pi} - q_{x,\sigma}| + \left|q_{x,\sigma} - \frac{1}{R}\left\{c_p(x) \int_a^b p(\pi(t))\, dV(\pi(t)) + c_V(x) \int_a^b V(\pi(t))\, dp(\pi(t))\right\}\right| =$$

$$= |q_{x,\pi} - q_{x,\sigma}| + \left|\frac{1}{R} c_V(x) \sum_{i=1}^{n} V(\pi(a_{i-1})) (p(\pi(a_i)) - p(\pi(a_{i-1}))) +\right.$$

$$+ \frac{1}{R} c_p(x) \sum_{i=1}^{n} p(\pi(c_i)) (V(\pi(a_i)) - V(\pi(a_{i-1}))) + \frac{1}{R} c_V(x) \sum_{i=1}^{n} (V(\pi(a_i)) - V(\pi(a_{i-1}))) \cdot$$

$$\left. \cdot (p(\pi(a_i)) - p(\pi(c_i))) - \frac{1}{R}\left\{c_p(x) \int_a^b p(\pi(t))\, dV(\pi(t)) + c_V(x) \int_a^b V(\pi(t))\, dp(\pi(t))\right\}\right| \leqq$$

$$\leqq |q_{x,\pi} - q_{x,\sigma}| + \left|\frac{1}{R} c_V(x) \left\{\sum_{i=1}^{n} V(\pi(a_{i-1}))\,(p(\pi(a_i)) - p(\pi(a_{i-1}))) - \right.\right.$$

$$\left.\left. - \int_a^b V(\pi(t))\,dp(\pi(t))\right\}\right| + \left|\frac{1}{R} c_p(x) \left\{\sum_{i=1}^{n} p(\pi(c_i))\,(V(\pi(a_i)) - V(\pi(a_{i-1}))) - \right.\right.$$

$$\left.\left. - \int_a^b p(\pi(t))\,dV(\pi(t))\right\}\right| + \left|\frac{1}{R} c_V(x) \sum_{i=1}^{n} (V(\pi(a_i)) - V(\pi(a_{i-1})))\,(p(\pi(a_i)) - p(\pi(c_i)))\right| .$$

By (5.15), (5.7), (5.6) and by IX, we conclude that

$$\left|q_{x,\pi} - \frac{1}{R}\left\{c_p(x) \int_a^b p(\pi(t))\,dV(\pi(t)) + c_V(x) \int_a^b V(\pi(t))\,dp(\pi(t))\right\}\right| <$$

$$< \tfrac{1}{4}\varepsilon + \frac{1}{R}\,|c_V(x)|\,\frac{R\varepsilon}{1 + 4|c_V(x)|} + \frac{1}{R}\,|c_p(x)|\,\frac{R\varepsilon}{1 + 4|c_p(x)|} + \tfrac{1}{4}\varepsilon < \varepsilon .$$

As ε was an arbitrary positive number, we obtain (5.1) from the last inequality.

Integrating by parts (see [4], Chap. X, § 7, Theorem 147), we can give two other forms of (5.1):

Theorem 5.2. *Let $J = \langle a, b\rangle$ and let $\pi \in \boldsymbol{P}_J$ be a possible process of a system x. Let $(x, \pi, q) \in Q$. Then*

$$q = \frac{1}{R}\left\{(c_p(x) - c_V(x)) \int_a^b p(\pi(t))\,dV(\pi(t)) + \right. \tag{5.18}$$

$$\left. + c_V(x)\,(V(\pi(b))\,p(\pi(b)) - V(\pi(a))\,p(\pi(a)))\right\},$$

$$q = \frac{1}{R}\left\{(c_V(x) - c_p(x)) \int_a^b V(\pi(t))\,dp(\pi(t)) + \right. \tag{5.19}$$

$$\left. + c_p(x)\,(V(\pi(b))\,p(\pi(b)) - V(\pi(a))\,p(\pi(a)))\right\}.$$

Theorem 5.3. *We have $c_p(x) > c_V(x) > 0$ for every $x \in \boldsymbol{X}$.*

Proof. By the axiom **A1**, there exists a possible state $s_1 = (V_1, p_1, T_1)$ of the system x. Put $J = \langle 0, 2\rangle$ and define processes $\pi \in \boldsymbol{P}_J$, $\sigma \in \boldsymbol{P}_J$ in the following way:

$$\pi(t) = (V_1 + t,\, p_1,\, (V_1 + t)\,T_1/V_1) \qquad \text{for} \quad t \in \langle 0, 2\rangle ,$$

$$\sigma(t) = (V_1 + t,\, p_1 + t,\, (p_1 + t)\,(V_1 + t)\,T_1/p_1V_1) \qquad \text{for} \quad t \in \langle 0, 1\rangle ,$$

$$\sigma(t) = (V_1 + t,\, p_1 + 2 - t,\, (p_1 + 2 - t)\,(V_1 + t)\,T_1/p_1V_1) \quad \text{for} \quad t \in \langle 1, 2\rangle .$$

By the axiom **A2**, π, σ are possible processes of x. By (5.18), we have

$$q_{x,\pi} = \frac{1}{R}\left\{(c_p(x) - c_V(x)) \int_0^2 p_1\,d(V_1 + t) + c_V(x)\,((V_1 + 2)\,p_1 - V_1p_1)\right\},$$

$$q_{x,\sigma} = \frac{1}{R}\left\{(c_p(x) - c_V(x))\left(\int_0^1 p_1 + t)\, d(V_1 + t) + \int_1^2 (p_1 + 2 - t)\, d(V_1 + t)\right) + \right.$$
$$\left. + c_V(x)\left((V_1 + 2)\, p_1 - V_1 p_1\right)\right\}.$$

As $q_{x,\pi} < q_{x,\sigma}$ by the axiom **A9** and $R > 0$ by Theorem 3.6, this yields

$$(c_p(x) - c_V(x)) \int_0^2 p_1\, d(V_1 + t) <$$
$$< (c_p(x) - c_V(x))\left(\int_0^1 (p_1 + t)\, d(V_1 + t) + \int_1^2 (p_1 + 2 - t)\, d(V_1 + t)\right).$$

But

$$\int_0^2 p_1\, d(V_1 + t) = \int_0^2 p_1\, dt = 2p_1,$$
$$\int_0^1 (p_1 + t)\, d(V_1 + t) + \int_1^2 (p_1 + 2 - t)\, d(V_1 + t) =$$
$$= \int_0^1 (p_1 + t)\, dt + \int_1^2 (p_1 + 2 - t)\, dt = 2p_1 + 1,$$

therefore

$$(c_p(x) - c_V(x)) \,.\, 2p_1 < (c_p(x) - c_V(x))\,(2p_1 + 1)$$

hence $c_p(x) > c_V(x)$. Finally, $c_V(x) > 0$ by Theorem 4.5.

6. AUXILIARY THEOREMS

In the Chapter 7 we will study special processes called *adiabatic*. For this purpose we need some theorems about the Riemann-Stieltjes integral that we are now going to express. The proofs of these theorems we leave to the reader.

Theorem 6.1. *Let f, g, h be continuous real functions on an interval $\langle a, b\rangle$ and let g, h have bounded variation on $\langle a, b\rangle$. Then*

$$\int_a^b f\, d(gh) = \int_a^b fg\, dh + \int_a^b fh\, dg\,.$$

Theorem 6.2. *Let f, g be continuous real functions on $\langle a, b\rangle$, let g have bounded variation on $\langle a, b\rangle$ and let $g(t) > 0$ for all $t \in \langle a, b\rangle$. Let γ be a real number. Then*

$$\int_a^b f\, d(g^\gamma) = \gamma \int_a^b f g^{\gamma - 1}\, dg\,.$$

Theorem 6.3. *Let f, g be real functions defined on $\langle a, b\rangle$. Let f be continuous on $\langle a, b\rangle$ and let $f(t) > 0$ for all $t \in \langle a, b\rangle$. Let g have bounded variation on $\langle a, b\rangle$. Then*

$$\int_\alpha^\beta f \,\mathrm{d}g = 0$$

for all α, β such that $a \leqq \alpha < \beta \leqq b$ if and only if g is constant on $\langle a, b\rangle$.

Now, we are going to express several theorems that will be useful in Chapters 8 and 9.

Theorem 6.4. *Let f, g be real functions defined on $\langle a, b\rangle$. Let f be continuous on $\langle a, b\rangle$ and g increasing on $\langle a, b\rangle$. Let $f(t) \geqq 0$ for all $t \in \langle a, b\rangle$ and let $f(c) > 0$ for some $c \in \langle a, b\rangle$.*

Then

$$\int_a^b f \,\mathrm{d}g > 0\,.$$

Theorem 6.5. *Let f, h, g be real functions defined on $\langle a, b\rangle$. Let f, h be continuous on $\langle a, b\rangle$ and g increasing on $\langle a, b\rangle$. Let $f(t) \geqq h(t)$ for all $t \in \langle a, b\rangle$ and let $f(c) > h(c)$ for some $c \in \langle a, b\rangle$.*

Then

$$\int_a^b f \,\mathrm{d}g > \int_a^b h \,\mathrm{d}g\,.$$

Theorem 6.6. *Let f_0, g_0 be real continuous functions on $\langle a, b\rangle$ with bounded variation. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ with the following property: if f, g are real continuous functions on $\langle a, b\rangle$ such that*

$$|f_0(t) - f(t)| < \delta\,, \quad |g_0(t) - g(t)| < \delta \quad \textit{for all} \quad t \in \langle a, b\rangle$$

and

$$\mathscr{V}_g^{\langle a,b\rangle} \leqq \mathscr{V}_{g_0}^{\langle a,b\rangle}\,,$$

then

$$\left|\int_a^b f_0 \,\mathrm{d}g_0 - \int_a^b f \,\mathrm{d}g\right| < \varepsilon\,.$$

7. ADIABATIC PROCESSES

Definition. Let $K \in \boldsymbol{K}$. A process $\pi \in \boldsymbol{P}_K$ is called an *adiabatic process of a system x* if it is a possible process of x and $(x, \pi \mid J, 0) \in Q$ for every compact interval J such that $J \subset K$.

Theorems 5.1 and 5.2 immediately yield

Theorem 7.1. *Let $K = \langle a, b\rangle$ and let $\pi \in \boldsymbol{P}_K$ be a possible process of a system x. Then π is an adiabatic process of x if and only if for any pair of real numbers (α, β) such that $a \leqq \alpha < \beta \leqq b$ one of the following relations holds:*

$$c_{\mathrm{p}}(x) \int_\alpha^\beta \mathrm{p}(\pi(t))\, \mathrm{d}\mathrm{V}(\pi(t)) + c_{\mathrm{V}}(x) \int_\alpha^\beta \mathrm{V}(\pi(t))\, \mathrm{d}\mathrm{p}(\pi(t)) = 0\,; \tag{7.1}$$

$$(c_{\mathrm{p}}(x) - c_{\mathrm{V}}(x)) \int_\alpha^\beta \mathrm{p}(\pi(t))\, \mathrm{d}\mathrm{V}(\pi(t)) + \\ + c_{\mathrm{V}}(x)\, (\mathrm{V}(\pi(\beta))\, \mathrm{p}(\pi(\beta)) - \mathrm{V}(\pi(\alpha))\, \mathrm{p}(\pi(\alpha))) = 0\,; \tag{7.2}$$

$$(c_{\mathrm{V}}(x) - c_{\mathrm{p}}(x)) \int_\alpha^\beta \mathrm{V}(\pi(t))\, \mathrm{d}\mathrm{p}(\pi(t)) + \\ + c_{\mathrm{p}}(x)\, (\mathrm{V}(\pi(\beta))\, \mathrm{p}(\pi(\beta)) - \mathrm{V}(\pi(\alpha))\, \mathrm{p}(\pi(\alpha))) = 0\,. \tag{7.3}$$

This theorem together with the theorems from Chapter 6 yields

Theorem 7.2. *Let $K = \langle a, b\rangle$ and let $\pi \in \boldsymbol{P}_K$ be a possible process of the system x. Then π is an adiabatic process of x if and only if the function*

$$(\mathrm{p} \circ \pi)\, (\mathrm{V} \circ \pi)^{c_{\mathrm{p}}(x)/c_{\mathrm{V}}(x)} \tag{7.4}$$

is constant on a, b.

Proof. For $t \in \langle a, b\rangle$ denote

$$L(t) = \mathrm{p}(\pi(t))\, \mathrm{V}(\pi(t))^{c_{\mathrm{p}}(x)/c_{\mathrm{V}}(x)}\,. \tag{7.5}$$

The function L is continuous with bounded variation on $\langle a, b\rangle$. By Theorem 7.1, π is an adiabatic process of x if and only if (7.1) holds for all α, β such that $a \leqq \alpha < \beta \leqq b$. However, (7.5) implies

$$\mathscr{I}(\alpha, \beta) = c_{\mathrm{p}}(x) \int_\alpha^\beta \mathrm{p}(\pi(t))\, \mathrm{d}\mathrm{V}(\pi(t)) + c_{\mathrm{V}}(x) \int_\alpha^\beta \mathrm{V}(\pi(t))\, \mathrm{d}\mathrm{p}(\pi(t)) =$$

$$= c_{\mathrm{p}}(x) \int_\alpha^\beta L(t)\, \mathrm{V}(\pi(t))^{-c_{\mathrm{p}}(x)/c_{\mathrm{V}}(x)}\, \mathrm{d}\mathrm{V}(\pi(t)) + c_{\mathrm{V}}(x) \int_\alpha^\beta \mathrm{V}(\pi(t))\, \mathrm{d}(L(t)\, \mathrm{V}(\pi(t))^{-c_{\mathrm{p}}(x)/c_{\mathrm{V}}(x)})\,.$$

The functions L, $\mathrm{V} \circ \pi$ and $(\mathrm{V} \circ \pi)^{-c_{\mathrm{p}}(x)/c_{\mathrm{V}}(x)}$ are continuous with bounded variation on $\langle a, b\rangle$ and $\mathrm{V}(\pi(t)) > 0$ for all $t \in \langle a, b\rangle$. Theorems 6.1 and 6.2 imply

$$\mathscr{I}(\alpha, \beta) = c_{\mathrm{p}}(x) \int_\alpha^\beta L(t)\, \mathrm{V}(\pi(t))^{-c_{\mathrm{p}}(x)/c_{\mathrm{V}}(x)}\, \mathrm{d}\mathrm{V}(\pi(t)) + \\ + c_{\mathrm{V}}(x) \int_\alpha^\beta \mathrm{V}(\pi(t))\, L(t)\, \mathrm{d}(\mathrm{V}(\pi(t))^{-c_{\mathrm{p}}(x)/c_{\mathrm{V}}(x)}) +$$

$$+ c_{\mathrm{V}}(x) \int_\alpha^\beta \mathrm{V}(\pi(t))^{1-c_{\mathrm{p}}(x)/c_{\mathrm{V}}(x)}\, \mathrm{d}L(t) = c_{\mathrm{p}}(x) \int_\alpha^\beta L(t)\, \mathrm{V}(\pi(t))^{-c_{\mathrm{p}}(x)/c_{\mathrm{V}}(x)}\, \mathrm{d}(\mathrm{V}(\pi(t)) +$$

$$+ c_V(x)\left(-\frac{c_p(x)}{c_V(x)}\right)\int_\alpha^\beta V(\pi(t))\, L(t)\, V(\pi(t))^{-[c_p(x)/c_V(x)]-1}\, dV(\pi(t)) +$$

$$+ c_V(x)\int_\alpha^\beta V(\pi(t))^{1-[c_p(x)/c_V(x)]}\, dL(t) = c_V(x)\int_\alpha^\beta V(\pi(t))^{1-[c_p(x)/c_V(x)]}\, dL(t)\,.$$

But, by Theorem 6.3, $\mathscr{J}(\alpha, \beta) = 0$ for all α, β such that $a \leqq \alpha < \beta \leqq b$ if and only if L is constant on $\langle a, b\rangle$.

For a system x, denote

$$\gamma(x) = c_p(x)/c_V(x)\,.$$

Theorem 5.3 implies $\gamma(x) > 1$ for any system x. Theorem 7.2 together with Theorem 3.7 yields

Theorem 7.3. *If π is an adiabatic process of a system x, then the following three functions are constant:*

(7.6) $$(p \circ \pi)\,(V \circ \pi)^{\gamma(x)}\,,$$

(7.7) $$(T \circ \pi)\,(V \circ \pi)^{\gamma(x)-1}\,,$$

(7.8) $$(T \circ \pi)^{\gamma(x)}\,(p \circ \pi)^{1-\gamma(x)}\,.$$

If one of these functions is constant, then π is adiabatic.

Proof. Put $p(\pi(t))\, V(\pi(t))^{\gamma(x)} = L(t)$. As by (3.4) we have

$$V(\pi(t)) = R\,\mu(x)\, T(\pi(t))/p(\pi(t))\,, \quad p(\pi(t)) = R\,\mu(x)\, T(\pi(t))/V(\pi(t))\,,$$

we obtain

$$T(\pi(t))\, V(\pi(t))^{\gamma(x)-1} = L(t)/(R\,\mu(x))\,,$$

$$T(\pi(t))^{\gamma(x)}\, p(\pi(t))^{1-\gamma(x)} = L(t)/(R^{\gamma(x)}\,\mu(x)^{\gamma(x)})\,,$$

therefore either all three functions are constant or all three functions are non-constant. However, (7.6) is identical with (7.4).

8. CONSISTENCY OF THE SYSTEM OF AXIOMS

We will prove that the theory formed by typifications **(T1)** ..., **(T4)** and by axioms **A1**, ..., **A9** is consistent, provided the theory of real numbers is consistent. The proof will be carried out by constructing, in terms of the theory of real numbers, a model fulfilling all the above axioms.

In our model, the term X will be the set $\{1\}$, containing the single number 1. The structural terms will be defined as follows:

(*T1) $x_0 = 1$;

(*T2) $\mu_0 = 1$;

(*T3) $(x, (a_1, a_2, a_3)) \in W$ if and only if $x = 1$ and $a_1 a_2 / a_3 = 1$;

(*T4) $(x, \pi, q) \in Q$ if and only if $x = 1$, $\pi \in \boldsymbol{P}_{\langle a,b \rangle}$, $(x, \pi) \in W$ and

$$q = 2 \int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) + \int_a^b \mathsf{V}(\pi(t))\, \mathrm{d}\mathsf{p}(\pi(t)) . \tag{8.1}$$

Theorem 8.1. *The model described above fulfils all axioms* **A1**, ..., **A9**.

Proof. The validity of axioms **A1**, **A2**, **A3** and **A5** is obvious.

Verification of the axiom **A4**. Let $x \in \boldsymbol{X}$, $J = \langle a, b \rangle$ and let $\pi \in \boldsymbol{P}_J$ be a possible process of x. If $\varepsilon > 0$, then, by Theorem 6.6, there exists a $\delta_1 > 0$ such that if $\sigma \in \boldsymbol{P}_J$ is a possible process of x such that

$$|\mathsf{V}(\pi(t)) - \mathsf{V}(\sigma(t))| < \delta_1 , \quad |\mathsf{p}(\pi(t)) - \mathsf{p}(\sigma(t))| < \delta_1 \quad \text{for all} \quad t \in J$$

and if $\mathscr{V}^J_{\mathsf{V}\circ\sigma} \leqq \mathscr{V}^J_{\mathsf{V}\circ\pi}$ then

$$\left| \int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) - \int_a^b \mathsf{p}(\sigma(t))\, \mathrm{d}\mathsf{V}(\sigma(t)) \right| < \tfrac{1}{3}\varepsilon ,$$

and there exists a $\delta_2 > 0$ such that if $\sigma \in \boldsymbol{P}_J$ is a possible process of x such that

$$|\mathsf{V}(\pi(t)) - \mathsf{V}(\sigma(t))| < \delta_2 , \quad |\mathsf{p}(\pi(t)) - \mathsf{p}(\sigma(t))| < \delta_2 \quad \text{for all} \quad t \in J$$

and if $\mathscr{V}^J_{\mathsf{p}\circ\sigma} \leqq \mathscr{V}^J_{\mathsf{p}\circ\pi}$ then

$$\left| \int_a^b \mathsf{V}(\pi(t))\, \mathrm{d}\mathsf{p}(\pi(t)) - \int_a^b \mathsf{V}(\sigma(t))\, \mathrm{d}\mathsf{p}(\sigma(t)) \right| < \tfrac{1}{3}\varepsilon .$$

If we put $\delta = \min(\delta_1, \delta_2)$ and if $\sigma \in \boldsymbol{P}_J$ is a possible process of x such that (2.2), (2.3) and (2.4) hold, then (*T4) gives

$$|q - r| = \left| 2 \int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) + \int_a^b \mathsf{V}(\pi(t))\, \mathrm{d}\mathsf{p}(\pi(t)) - \right.$$

$$\left. - 2 \int_a^b \mathsf{p}(\sigma(t))\, \mathrm{d}\mathsf{V}(\sigma(t)) - \int_a^b \mathsf{V}(\sigma(t))\, \mathrm{d}\mathsf{p}(\sigma(t)) \right| \leqq$$

$$\leqq 2 \left| \int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) - \int_a^b \mathsf{p}(\sigma(t))\, \mathrm{d}\mathsf{V}(\sigma(t)) \right| +$$

$$+ \left| \int_a^b \mathsf{V}(\pi(t))\, \mathrm{d}\mathsf{p}(\pi(t)) - \int_a^b \mathsf{V}(\sigma(t))\, \mathrm{d}\mathsf{p}(\sigma(t)) \right| < \tfrac{2}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon ,$$

therefore (2.5) holds.

Verification of axioms **A6** and **A8**. Let $J = \langle a, b \rangle$, let $\pi \in \boldsymbol{P}_J$ be a possible process of x and let $\mathsf{V}(\pi(t)) = V_0$ for all $t \in J$. Let $(x, \pi, q) \in Q$. Then, by (8.1), we obtain

$$q = \int_a^b V_0 \, \mathrm{d}\mathsf{p}(\pi(t)) = V_0(\mathsf{p}(\pi(b)) - \mathsf{p}(\pi(a)))\,.$$

But (***T3**) yields $\mathsf{T}(\pi(t)) = \mathsf{V}(\pi(t))\,.\,\mathsf{p}(\pi(t))$ for all $t \in J$ and therefore

(8.2) $$q = \mathsf{T}(\pi(b)) - \mathsf{T}(\pi(a))\,.$$

This implies the axiom **A8**.

In the same way, if $\sigma \in \boldsymbol{P}_J$ is a possible process of x, $\mathsf{V} \circ \sigma$ is constant on J and $(x, \sigma, r) \in Q$, we obtain

(8.3) $$r = \mathsf{T}(\sigma(b)) - \mathsf{T}(\sigma(a))\,.$$

The axiom **A6** follows from (8.2) and (8.3).

Verification of the axiom **A7** is analogous to the verification of the axiom **A6**.

Verification of the axiom **A9**. Let $J = \langle a, b \rangle$, let $\pi \in \boldsymbol{P}_J$, $\sigma \in \boldsymbol{P}_J$ be possible processes of x, let $\mathsf{V}(\pi(t)) = \mathsf{V}(\sigma(t))$, $\mathsf{p}(\pi(t)) \leqq \mathsf{p}(\sigma(t))$ for all $t \in J$, let $\mathsf{V} \circ \pi = \mathsf{V} \circ \sigma$ be increasing on J, $\mathsf{p}(\pi(a)) = \mathsf{p}(\sigma(a))$, let $\mathsf{p}(\pi(\tau)) < \mathsf{p}(\sigma(\tau))$ for some $\tau \in J$, and let (2.2) hold. Then (***T4**) implies (by integrating by parts in the second integral (8.1))

$$q = \int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) + \mathsf{V}(\pi(b))\, \mathsf{p}(\pi(b)) - \mathsf{V}(\pi(a))\, \mathsf{p}(\pi(a))$$

and similarly,

$$r = \int_a^b \mathsf{p}(\sigma(t))\, \mathrm{d}\mathsf{V}(\sigma(t)) + \mathsf{V}(\sigma(b))\, \mathsf{p}(\sigma(b)) - \mathsf{V}(\sigma(a))\, \mathsf{p}(\sigma(a))\,.$$

Further,

$$\mathsf{V}(\pi(a))\, \mathsf{p}(\pi(a)) = \mathsf{V}(\sigma(a))\, \mathsf{p}(\sigma(a))\,, \quad \mathsf{V}(\pi(b))\, \mathsf{p}(\pi(b)) \leqq \mathsf{V}(\sigma(b))\, \mathsf{p}(\sigma(b))$$

and by Theorem 6.5 we have

$$\int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) < \int_a^b \mathsf{p}(\sigma(t))\, \mathrm{d}\mathsf{V}(\sigma(t))\,,$$

therefore $q < r$.

9. INDEPENDENCE OF THE AXIOMS

We will now prove that axioms **A1**, ..., **A9** are independent (if the structural terms are given by typifications ((**T1**), ..., (**T4**)). To this end we shall construct, in terms of the theory of real numbers, models, each of them fulfilling all axioms **A1**, ..., **A9** except one. We leave the verification of the axioms to the reader, noting only that Theorems 6.5 and 6.6 will be useful.

In each of these models, the term X will be the set $\{1\}$, containing the single number 1. The structural terms x_0, μ_0 will be always given by (***T1**) and (***T2**) from Chapter 8. Each model will be defined by giving the structural terms W and Q.

Independence of the axiom **A1**.

$$W = \emptyset\,, \quad Q = \emptyset \quad \text{(empty sets)}\,.$$

Independence of the axiom **A2**.

$$(x_1, (a_1, a_2, a_3)) \in W \Leftrightarrow x = 1\,, \quad a_1 = a_2 = a_3 = 1\,,$$
$$(x, \pi, q) \in Q \Leftrightarrow x = 1\,, \quad \pi \text{ is constant}\,, \quad q = 0\,.$$

In all the other models, the term W will be defined by (***T3**). It remains to define the term Q in each model.

Independence of the axiom **A3**. $Q = \emptyset$ (empty set).

Independence of the axiom **A4**.
$(x, \pi, q) \in Q$ if and only if $x = 1$, $\pi \in \boldsymbol{P}_{\langle a,b \rangle}$ and either (8.1) holds or none of the functions $\mathsf{p} \circ \pi$, $\mathsf{V} \circ \pi$ is monotone on $\langle a, b \rangle$.

Independence of the axiom **A5**.
$(x, \pi, q) \in Q$ if and only if $(x, \pi) \in W$, $\pi \in \boldsymbol{P}_{\langle a,b \rangle}$ and

$$q = (b - a)\left(2 \int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) + \int_a^b \mathsf{V}(\pi(t))\, \mathrm{d}\mathsf{p}(\pi(t))\right).$$

Independence of the axiom **A6**.
$(x, \pi, q) \in Q$ if and only if $(x, \pi) \in W$, $\pi \in \boldsymbol{P}_{\langle a,b \rangle}$ and

$$q = 2 \int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) + \int_a^b (\mathsf{V}(\pi(t)) + \operatorname{arctg} \mathsf{V}(\pi(t)))\, \mathrm{d}p(\pi(t))\,.$$

Remark. 9.1. This model represents the theory in which the specific heat by constant volume depends on the temperature.

Independence of the axiom **A7**.
$(x, \pi, q) \in Q$ if and only if $(x, \pi) \in W$, $\pi \in \boldsymbol{P}_{\langle a,b \rangle}$ and

$$q = 2 \int_a^b (1 + \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t)) + \int_a^b \mathsf{V}(\pi(t))\, \mathrm{d}\mathsf{p}(\pi(t))\,.$$

Independence of the axiom **A8**.
$(x, \pi, q) \in Q$ if and only if $(x, \pi) \in W$, $\pi \in \boldsymbol{P}_{\langle a,b \rangle}$ and

$$q = \int_a^b \mathsf{p}(\pi(t))\, \mathrm{d}\mathsf{V}(\pi(t))\,.$$

Independence of the axiom **A9**.
$(x, \pi, q) \in Q$ if and only if $(x, \pi) \in W$, $\pi \in \boldsymbol{P}_{\langle a,b \rangle}$ and

$$q = \int_a^b \mathsf{p}(\pi(t))\, \mathsf{d}\mathsf{V}(\pi(t)) + 2 \int_a^b \mathsf{V}(\pi(t))\, \mathsf{d}\mathsf{p}(\pi(t))\,.$$

Remark 9.2. In this model, the specific heat by constant volume is greater than the specific heat by constant pressure.

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Souhrn

MATEMATICKÉ ZÁKLADY THERMODYNAMIKY IDEÁLNÍHO PLYNU

MILOSLAV JŮZA

Článek je pokusem o matematicky přesné zavedení základních pojmů thermodynamiky ideálního plynu. Je zaveden systém axiomů a z něho odvozeno několik základních vět. Pak jsou definovány a studovány adiabatické procesy. Nakonec je dokázána bezespornost a nezávislost používaných axiomů.

Резюме

МАТЕМАТИЧЕСКИЕ ОСНОВАНИЯ ТЕРМОДИНАМИКИ ИДЕАЛЬНОГО ГАЗА

MILOSLAV JŮZA

Статья является попыткой найти математически точную формулировку основных понятий термодинамики идеального газа. В основу положена система аксиом и из неё выведено несколько теорем. После этого определены и изучены адиабатические процессы. В заключение доказаны непротиворечивость и независимость аксиом.

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