Commentationes Mathematicae Universitatis Carolinae

Edgar E. Enochs; Jenda M. G. Overtoun

Copure injective resolutions, flat resolvents and dimensions

Persistent URL: http://dml.cz/dmlcz/118572

Terms of use:

© Charles University in Prague, Faculty of Mathematics and Physics, 1993

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Copure injective resolutions, flat resolvents and dimensions

Edgar E. Enochs, Overtoun M.G. Jenda

Abstract. In this paper, we show the existence of copure injective preenvelopes over noetherian rings and copure flat preenvelopes over commutative artinian rings. We use this to characterize n-Gorenstein rings. As a consequence, if the full subcategory of strongly copure injective (respectively flat) modules over a left and right noetherian ring R has cokernels (respectively kernels), then R is 2-Gorenstein.

Keywords: preenvelopes, copure injective, copure flat, n-Gorenstein, resolutions

*Classification:* 18G05, 18G10, 18G15

1. Introduction.

R will denote an associative ring with a unit element and all modules are unitary. Module and noetherian will mean left R-module and left noetherian respectively.

A submodule A of B is said to be a *copure submodule* if B/A is injective, and a module M is said to be *copure injective* if it is injective with respect to all copure submodules. These modules were introduced and studied in Enochs-Jenda [6]. It is easy to see that copure injective modules are precisely those modules M such that $\operatorname{Ext}^1(E, M) = 0$ for all injective R-modules E. Therefore, over principal ideal domains, copure injective modules are precisely injective modules. We will say that M is *strongly copure injective* if $\operatorname{Ext}^i(E, M) = 0$ for all injective R-modules E for all $i \geq 1$.

A module M is said to be *copure flat* if it is flat with respect to all copure submodules. It is again easy to see that M is copure flat if and only if $\operatorname{Tor}_1(E, M) = 0$ for all injective right R-modules E. We will say that M is *strongly copure flat* if $\operatorname{Tor}_i(E, M) = 0$ for all injective right R-modules E for all $i \geq 1$. A submodule $A \subset B$ is said to be *E-pure* if $0 \to E \otimes A \to E \otimes B$ is exact for all injectives E.

A map $\psi : M \to C$ where C is strongly copure injective (respectively flat) is said to be a *copure injective* (respectively *flat*) *preenvelope* if any diagram
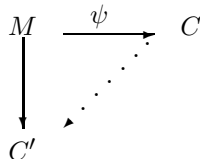
with C' strongly copure injective (respectively flat) can be completed. If further-more, the diagram

$$
\begin{array}{ccc}
M & \xrightarrow{\ \psi\ } & C \\
{\scriptstyle\psi}\big\downarrow & & \\
C & &
\end{array}
$$

can only be completed by automorphisms of C, then the map $\psi : M \to C$ is said to be a *copure injective* (respectively *flat*) *envelope*.

In this paper, we show that if R is noetherian, then every R-module has a copure injective preenvelope (Theorem 2.2). We note that any copure injective preenvelope $M \to C$ is a monomorphism since the injective envelope of M is strongly copure injective.

An exact sequence $0 \to M \to C^\circ \to C^1 \to \cdots$ where $M \to C^\circ$, $Coker(M \to C^\circ) \to C^1$, $Coker(C^{n-1} \to C^n) \to C^{n+1}$ for $n \geq 1$ are copure injective preenvelopes is called a *copure injective resolution* of M. If there is a copure injective resolution $0 \to M \to C^\circ \to C^1 \to \cdots \to C^n \to 0$, we say that M has *copure injective dimension* $(cid) \leq n$. In Theorem 2.5, we show the existence of copure flat preenvelopes over commutative artinian rings. In this case, we define a *copure flat resolvent* (see Enochs [3]) of an R-module M to be a sequence (not necessar-ily exact) $0 \to M \to T^\circ \to T^1 \to \cdots$ where $M \to T^\circ$, $Coker(M \to T^\circ) \to T^1$, $Coker(T^{n-1} \to T^n) \to T^{n+1}$ for $n \geq 1$ are copure flat preenvelopes. The meaning of *copure flat resolvent dimension* of M is now clear.

We define *copure flat dimension* (cfd) of an R-module M for any ring R to be the largest positive integer n such that $\mathrm{Tor}_n(E, M) \neq 0$ for some injective right R-module E. If M is strongly copure flat, then we set $cfdM = 0$.

In Section 3, we study copure injective and flat dimensions. The results of this section are then used to characterize n-Gorenstein rings (i.e. R is left and right noetherian, $id_R R \leq n$ and $id R_R \leq n$, (see Iwanaga [8])) in terms of copure injective, copure flat, and copure flat resolvent dimensions.

$\mathrm{Ext}^i(N, M), \mathrm{Tor}_i(N, M)$ will denote $\mathrm{Ext}^i_R(N, M), \mathrm{Tor}^R_i(N, M)$ respectively and the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ will be denoted by M^+.

2. Copure injective and flat preenvelopes.

We start with the following

Lemma 2.1. *Let R be noetherian and let $\aleph_\alpha$ be an infinite cardinal. Then there is an infinite cardinal $\aleph_\beta$ such that if C is a strongly copure injective R-module and $S \subset C$ is a submodule with $CardS \leq \aleph_\alpha$, then there is a strongly copure injective submodule D of C containing S with $CardD \leq \aleph_\beta$.*

PROOF: Since R is noetherian, there is a representative set $\{X_k\}$ of injective R-modules (see Gabriel [7]). Therefore, to show that an R-module M is strongly copure injective, it suffices to show that $\mathrm{Ext}^i(\oplus X_k, M) = 0$ for all $i \geq 1$. So let $\cdots \to P_1 \to P_0 \to \oplus X_k \to 0$ be a projective resolution of $\oplus X_k$. We note that for

each syzygy K_j, $\text{Ext}^i(K_j, C) = 0$ for all $i \geq 1$ since C is strongly copure injective. So for each $j \geq 1$, the diagram

can be completed into a commutative diagram.

We now construct D in steps. First, we set $S = S_0$ and consider the maps $\sigma : K_j \to S_0$. Then σ can be extended to $\sigma' : P_{j-1} \to C$. Let $S_1 = S_0 + \sum \sigma'(P_{j-1})$, sum over $\sigma \in \text{Hom}(K_j, S_0), j \geq 1$. Then we construct S_2 from S_1 in the same way and so on. Now define $S_\alpha = \bigcup_{\beta < \alpha} S_\beta$ if α is a limit ordinal. Let λ be the least limit ordinal which is not cofinal with any ordinal whose cardinality is less or equal to $\sum_j Card K_j$. Hence if $\sigma \in \text{Hom}(K_j, S_\lambda)$, then $\sigma(K_j) \subset S_\beta$ for $\beta < \lambda$ for each $j \geq 1$. Then σ can be extended to $\sigma' : P_{j-1} \to S_{\beta+1}$ with $\beta+1 < \lambda$. So let $D = S_\lambda$. Then we see from the construction that $S \subset D \subset C$, D is strongly copure injective, and $Card D \leq \aleph_\beta$. □

Theorem 2.2. *Let R be noetherian. Then every R-module has a copure injective preenvelope.*

PROOF: The lemma above shows that if M is an R-module with $Card M \leq \aleph_\alpha$, then for any homomorphism $M \to C$ with C strongly copure injective, there is a homomorphism $M \to D, D \subset C$, D strongly copure injective, and $Card D \leq \aleph_\beta$ which agrees with the map $M \to C$. Now we set two such homomorphisms $M \to D, M \to D'$ equivalent if the diagram

can be completed by an isomorphism. If X is the set of representatives of such homomorphisms $M \to D$, then $M \to \prod_{D \in X} D$ is a strongly copure injective preenvelope.

□

Remark. It follows from the proof of Proposition 6.1 of Enochs [3] that if R is noetherian and every projective limit of strongly copure injective R-modules is strongly copure injective, then every R-module has a copure injective envelope.

Lemma 2.3. *Let R be right noetherian, $\{X_k\}$ be a representative set of injective right R-modules on $N = \oplus X_k$. If $\aleph_\alpha$ is an infinite cardinal, then there is an infinite cardinal $\aleph_\beta$ such that if M is a strongly copure flat R-module and $S \subset M$*

is a submodule with Card $S \leq \aleph_\alpha$, then there is a T with $S \subset T \subset M$ and Card $T \leq \aleph_\beta$ such that $S \subset T$ induces the zero map

$$\mathrm{Tor}_i(N, S) \to \mathrm{Tor}_i(N, T) \text{ for all } i \geq 1.$$

PROOF: We first note that M is strongly copure flat if and only if $\mathrm{Tor}_i(N, M) = 0$ for all $i \geq 1$.

Let $\cdots \to P_1 \to P_0 \to N \to 0$ be a projective resolution of N. Then Card $\mathrm{Tor}_i(N, S) \leq$ Card $(P_i \otimes N)$. Hence given $\aleph_\alpha$, there exists an infinite cardinal $\aleph_\delta \geq \aleph_\alpha$ such that if Card $S \leq \aleph_\alpha$ then Card $\mathrm{Tor}_i(N, S) \leq \aleph_\delta$ for all $i \geq 1$.

Now note that $0 = \mathrm{Tor}_i(N, M) \cong \lim_\to \mathrm{Tor}_i(N, S')$ $(i \geq 1)$ where the limit is over all S' with $S \subset S' \subset M$ and with S'/S finitely generated. Hence for $z \in \mathrm{Tor}_i(N, S)$, there is an S' such that $\mathrm{Tor}_i(N, S) \to \mathrm{Tor}_i(N, S')$ maps z to 0. Choosing one such S' for each z and letting T be the sum of the S''s chosen, we see that

$$\mathrm{Tor}_i(N, S) \to \mathrm{Tor}_i(N, T)$$

is the zero map for all $i \geq 1$.

Now it is easy to see that we can choose $\aleph_\beta \geq \aleph_\alpha$ so that Card $T \leq \aleph_\beta$ whenever Card $S \leq \aleph_\alpha$ (no matter what choice of S''s we make). $\qquad\square$

Lemma 2.4. *Let R be right noetherian and let $\aleph_\alpha$ be an infinite cardinal. Then there is an infinite cardinal $\aleph_\beta$ such that if M is a strongly copure flat R-module and $S \subset M$ is a submodule with Card $S \leq \aleph_\alpha$, then there is a strongly copure flat submodule T of M containing S with Card $T \leq \aleph_\beta$.*

PROOF: We use the notation in Lemma 2.3 above. Let $\alpha_0 = \alpha$ and $\alpha_1 = \beta$. Let α_1 play the role of α and α_2 be the new β guaranteed by the lemma. Repeating the procedure, we get $\aleph_{\alpha_0}, \aleph_{\alpha_1} \ldots$ with obvious properties. Then given $S \subset M$ with Card $S \leq \aleph_\alpha = \aleph_{\alpha_0}$, let $S_0 = S$ and find $S_1, S_2, \ldots$ with $S_1 \subset S_2 \subset \cdots \subset M$ such that Card $S_j \leq \aleph_{\alpha_j}$ and such that $S_j \subset S_{j+1}$ induces the zero map

$$\mathrm{Tor}_i(N, S_j) \to \mathrm{Tor}_i(N, S_{j+1}) \text{ for all } i \geq 1.$$

So let $T = \bigcup_{j=1}^\infty S_j$. Then

$$\mathrm{Tor}_i(N, T) = \lim_{\to_j} \mathrm{Tor}_i(N, S_j) = 0 \text{ for all } i \geq 1.$$

Thus T is strongly copure flat. Furthermore, if we set $\aleph_\beta = \sup_j \aleph_{\alpha_j}$, then Card $T \leq \aleph_\beta$. $\qquad\square$

Theorem 2.5. *If R is a commutative artinian ring, then every R-module has a copure flat preenvelope.*

PROOF: $E(R/m)$ is finitely generated for each $m \in mSpecR$ since R is commutative artinian. So the product of strongly copure flat R-modules is strongly copure flat by induction on Theorem 2.2 of Lenzing [11]. Then the theorem follows from Lemma 2.4 as in the proof of Theorem 2.2. $\qquad\square$

3. Dimensions.

We start with the following

Lemma 3.1. *Let R be noetherian. Then the following are equivalent for an R-module M.*

   1) *$cidM \leq n$;*
   2) *$\mathrm{Ext}^i(E, M) = 0$ for all injective R-modules E for all $i \geq n + 1$;*
   3) *Every n^{th} cosyzygy of M is strongly copure injective.*

PROOF: $1 \Leftrightarrow 2$. Let $0 \to M \to C^\circ \to C^1 \to \cdots \to C^{n-1} \to D \to 0$ be an exact sequence with $C^\circ, C^1, \cdots, C^{n-1}$ strongly copure injective. Then $\mathrm{Ext}^{j+n}(E, M) \cong \mathrm{Ext}^j(E, D)$. Thus the result follows. The same proof gives $2 \Leftrightarrow 3$. $\square$

Remark. We note that the copure injective dimension of an R-module M can be considered as the largest positive integer n such that $\mathrm{Ext}^n(E, M) \neq 0$ for some injective module E. Taking this as a definition of copure injective dimension, one may drop the noetherian condition in Lemma 3.1 above.

Corollary 3.2. *If $idM < \infty$, then $cidM = idM$.*

PROOF: $cidM \leq idM$ follows from the lemma above since injectives are copure injectives. Now suppose $cidM = n$. Then $\mathrm{Ext}^i(E, M) = 0$ for all injectives E and for all $i \geq n + 1$. But $idM < \infty$. So $idM \leq n$ by Lemma 2.2 of Jenda [9]. $\square$

We also have the following

Lemma 3.3. *The following are equivalent for a module M over any ring R.*

   1) *$cfdM \leq n$;*
   2) *$\mathrm{Tor}_i(E, M) = 0$ for all injective right R-modules E for all $i \geq n + 1$;*
   3) *Every n^{th} syzygy of M is strongly copure flat.*

PROOF: $1 \Leftrightarrow 2$ by definition.

$2 \Leftrightarrow 3$. We simply note that if K_n is an nth syzygy of M, then $\mathrm{Tor}_{j+n}(E, M) \cong \mathrm{Tor}_j(E, K_n)$. So the result follows. $\square$

Remark. If R is noetherian and M is a finitely generated R-module, then

$$\mathrm{Tor}_i(E, M) \cong \mathrm{Tor}_i(\mathrm{Hom}(R, E), M) \cong \mathrm{Hom}(\mathrm{Ext}^i(M, R), E)$$

for any injective right R-module E. So in this case, copure flat dimension coincides with Gorenstein dimension $(G - dim)$ if the latter is finite (see Auslander-Bridger [1, p. 95]).

Lemma 3.4. *Let M be an R-module. Then*

$$cfdM = cidM^+.$$

PROOF: This follows from the standard isomorphism $\mathrm{Tor}_i(E, M)^+ \cong \mathrm{Ext}^i(E, M^+)$. $\square$

Proposition 3.5. *Let* $0 \to M \to C^\circ \to C^1 \to \cdots$ *be a copure injective resolution of an R-module M. Then the sequence* $0 \to T \otimes M \to T \otimes C^\circ \to T \otimes C^1 \to \cdots$ *is exact for all right strongly copure flat R-modules T.*

PROOF: Let $S^i(M) = Ker(C^i \to C^{i+1}), i \geq 1$ and $S^\circ(M) = M$. We consider the short exact sequence $0 \to S^i(M) \to C^i \to S^{i+1}(M) \to 0$. Then $0 \to T \otimes S^i(M) \to T \otimes C^i \to T \otimes S^{i+1}(M) \to 0$ is exact if and only if $0 \to (T \otimes S^{i+1}(M))^+ \to (T \otimes C^i)^+ \to (T \otimes S^i(M))^+ \to 0$ is exact. But the latter is equivalent to $0 \to \mathrm{Hom}(S^{i+1}(M), T^+) \to \mathrm{Hom}(C^i, T^+) \to \mathrm{Hom}(S^i(M), T^+) \to 0$ being exact. So the result follows since T^+ is strongly copure injective by Lemma 3.4.   □

Lemma 3.6. *Let R be a commutative artinian ring. Then*

$$cidM = cfdM^+$$

PROOF: We note that if E is injective, then

$$\mathrm{Tor}_i(E, M^+) \cong \mathrm{Tor}_i(E(R/m), M^+), \text{ over } m \in mSpecR$$
$$\cong \mathrm{Ext}^i(E(R/m), M)^+ \text{ since } E(R/m)$$

is finitely generated.

On the other hand, $\mathrm{Ext}^i(E, M) \cong \prod \mathrm{Ext}^i(E(R/m), M)$. So $\mathrm{Tor}_i(E, M^+) = 0$ if and only if $\mathrm{Ext}^i(E, M) = 0$.   □

Proposition 3.7. *Let R be a commutative artinian ring and* $0 \to M \to T^\circ \to T^1 \to \cdots$ *be a copure flat resolvent of M. Then* $0 \to M \otimes C \to T^\circ \otimes C \to T^1 \otimes C \to \cdots$ *is exact for all strongly copure injective R-modules C.*

PROOF: The result follows as in Proposition 3.5 above since C^+ is strongly copure flat by Lemma 3.6.   □

4. Gorenstein rings.

We are now in a position to prove the following

Theorem 4.1. *The following are equivalent for a left and right noetherian ring R.*

1) *R is n-Gorenstein.*
2) $cidM \leq n$ *for all R-modules (left and right) M.*
3) *Every* n^{th} *cosyzygy of an R-module (left and right) is strongly copure injective.*
4) $cfdM \leq n$ *for all R-modules (left and right) M.*
5) $cfdM \leq n$ *for all finitely generated R-modules (left and right) M.*
6) *Every* n^{th} *syzygy of an R-module (left and right) is strongly copure flat.*
7) *Projective resolutions of the* n^{th} *syzygies of finitely generated R-modules (left and right) are projective resolvents.*

Furthermore, if R is commutative artinian, then the above statements are equivalent to

8) *Copure flat resolvent dimension of each R-module is at most n.*

PROOF: $1 \Leftrightarrow 2 \Leftrightarrow 3$. R n-Gorenstein means $pdE \leq n$ for all injective R-modules (left and right) by Jensen [10, Theorem 5.9]. So the equivalences follow from Lemma 3.1.

$1 \Leftrightarrow 4 \Leftrightarrow 6$. This easily follows from Lemma 3.3 as above and Enochs-Jenda [4, Theorem 4.4].

$4 \Leftrightarrow 5$ is trivial since Tor commutes with direct limits.

$5 \Leftrightarrow 7$. If M is finitely generated, then $\mathrm{Ext}^i(M, -)$ commutes with direct sums (see Lenzing [11, Theorem 2]). So $\mathrm{Ext}^i(M, R) = 0$ if and only if $\mathrm{Ext}^i(M, P) = 0$ for all projectives P. Therefore by the remark after Lemma 3.3, $\mathrm{Tor}_i(E, M) = 0$ for all injectives E if and only if $\mathrm{Ext}^i(M, P) = 0$ for all projectives P, So the result follows as in Lemma 2.1 of Enochs-Jenda [5].

$2 \Leftrightarrow 8$. Propositions 3.5 and 3.7 show that if R is a commutative artinian ring and M, N are R-modules, then $M \otimes N$ is right balanced by *copure inj $\times$ copure flat* in the language of Enochs-Jenda [4] and so right derived functors $\mathrm{Tor}^i(M, N)$ can be computed using either copure injective resolutions of M or copure flat resolvents of N. So $cidM \leq n$ if and only if copure flat resolvent dimension of N is at most n.
$\square$

Corollary 4.2. *Let R be left and right noetheiran. Then the following are equivalent.*

1) *R is 1-Gorenstein.*
2) *Every copure injective R-module (left and right) is strongly copure injective.*
3) *Every copure flat R-module (left and right) is strongly copure flat.*
4) *Every homomorphic image of a copure injective R-module (left and right) is copure injective.*
5) *Every submodule of a copure flat R-module (left and right) is copure flat.*

In this case, the full subcategories of copure injective modules (left and right) and copure flat modules (left and right) have cokernels and kernels respectively.

PROOF: $1 \Rightarrow 2, 3$ follow from the theorem above.

$2 \Rightarrow 1$. Let M be an R-module. Consider the short injective resolution $0 \to M \to E^\circ \to D \to 0$. Let $E_\circ \to D$ be an injective precover and $K = Ker(E_\circ \to D)$. Then $E_\circ \to D$ is surjective and so $\mathrm{Ext}^{i+1}(E, K) \cong \mathrm{Ext}^i(E, D)$ for $i \geq 1$. But K is copure injective (see Enochs-Jenda [6, Lemma 2.1]). So D is strongly copure injective. Thus every first cosyzygy is strongly copure injective. So the result follows from the theorem.

$3 \Rightarrow 1$. Now we consider the short projective resolution $0 \to K \to P_\circ \to M \to 0$. Let $K \to F^\circ$ be a flat preenvelope. Then $K \to F^\circ$ is $1-1$ and so $\mathrm{Tor}_{i+1}(E, F^\circ/K) \cong \mathrm{Tor}_i(E, K)$ for $i \geq 1$. But F°/K is copure flat by Enochs-Jenda [4, Proposition 3.3]. So K is strongly copure flat. Thus the result follows from the theorem.

$1 \Leftrightarrow 4$. The same proof as the commutative version in Enochs-Jenda [6, Corollary 3.4] noting that R is 1-Gorenstein if and only if $pdE \leq 1$ for all injective modules (left and right) E.

$1 \Rightarrow 5$. Let S be a submodule of a copure flat R-module M and E be an injective right R-mdoule. We have an exact sequence

$$\cdots \to \mathrm{Tor}_2(E, M) \to \mathrm{Tor}_2(E, M/S) \to \mathrm{Tor}_1(E, S) \to \mathrm{Tor}_1(E, M) = 0.$$

But $\mathrm{Tor}_2(E, M/S) = 0$ since $pdE_R \leq 1$. So S is copure flat. Similarly for right R-modules.

$5 \Rightarrow 1$. Consider a short projective resolution $0 \to K \to P_\circ \to M \to 0$ of M. Then $\mathrm{Tor}_2(E, M) \cong \mathrm{Tor}_1(E, K)$. So if every submodule of copure flat is copure flat, then $\mathrm{Tor}_2(E, M) = 0$ for all R-modules M and for all injective right R-modules E. Therefore $pdE_R \leq 1$ for all such E and so $id_R R \leq 1$. Similarly $id R_R \leq 1$.

The remaining statements of the corollary easily follow from the parts (4) and (5).

$\square$

The following is analogous to a result of Bernecker [2].

Corollary 4.3. *Let R be left and right noetherian. If the full subcategory of strongly copure injective (respectively flat) R-modules has cokernels (respectively kernels), then R is 2-Gorenstein.*

PROOF: Let $C \to D$ be a cokernel of $C' \to C$ with C', C strongly copure injective and E be an injective generator. Then $0 \to \mathrm{Hom}(D, E) \to \mathrm{Hom}(C, E) \to \mathrm{Hom}(C', E)$ is exact implies that $C \to D$ is onto. So cokernels are surjective. Therefore, every 2nd cosyzygy is strongly copure injective by assumption. Thus R is 2-Gorenstein by the theorem above. Similarly for strongly copure flat modules.

$\square$

Remark. It is easy to check that if the full subcategory of strongly copure injective (respectively flat) R-modules (left and right) has kernels (respectively cokernels), then it has cokernels (respectively kernels) and thus R is 2-Gorenstein.

References

[1] Auslander M., Bridger M., *Stable module theory*, Memories of the Amer. Math. Soc. **94** (1969), 1–146.

[2] Bernecker H., *Flatness and absolute purity applying functor categories to ring theory*, J. Algebra **44** (1977), 411–419.

[3] Enochs E., *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), 189–209.

[4] Enochs E. , Jenda O., *Balanced functors applied to modules*, J. Algebra **92** (1985), 303–310.

[5] ———, *Resolvents and dimensions of modules and rings*, Arch. Math. **56** (1991), 528–532.

[6] ———, *Copure injective modules*, Quaest. Mathematicae **14** (1991), 401–409.

[7] Gabriel P., *Objects injectifs dans les catégories abéliennes*, Sém. Dubreil, 1958/59, 17/01–17/32, Paris, 1960.

[8] Iwanaga Y., *On rings with finite self-injective dimension II*, Tsukuba J. Math. **4** (1980), 107–113.

[9] Jenda O., *The dual of the grade of a module*, Arch. Math. **51** (1988), 297–302.

[10] Jensen C., *Les foncteurs derives de $\varprojlim$ et leurs applications en theorie des modules*, Lecture Notes in Mathematics **254**, Springer, 1972.

[11] Lenzing H., *Endlich präsentierbare modulen*, Arch. Math. **20** (1969), 262–266.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506-0027, USA

DEPARTMENT OF ALGEBRA, COMBINATORICS, AND ANALYSIS, AUBURN UNIVERSITY, AL 36849-5307, USA