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# A note on existence and uniqueness of solutions of neutral functional-differential equations with state-dependent delays

ZDZISLAW JACKIEWICZ

Abstract. Existence and uniqueness theorem for state-dependent delay-differential equations of neutral type is given. This theorem generalizes previous results by Grimm and the author.

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Consider the scalar initial-value problem for state-dependent delay-differential equations of neutral type

(1)

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$$y'(t) = f(t, y(t), y(\alpha(t, y(t))), y'(\beta(t, y(t)))), \quad t \in [a, b],$$
  
$$y(t) = g(t), \quad t \in [\gamma, \alpha],$$

 $\gamma \leq a < b$ , where  $\gamma \leq \alpha(t, y) \leq t$ ,  $\gamma \leq \beta(t, y) \leq t$ , and g is a given initial function. We assume the following:

- (i) g and g' are Lipschitz-continuous with constants  $L_g$  and  $L_{g'}$  respectively;
- (ii)  $f(a, g(a), g(\alpha(a, g(a))), g'(\beta(a, g(a)))) = g'(a)$ , where g'(a) denotes the left hand side derivative.

Moreover, suppose that in their respective domains  $f, \alpha$  and  $\beta$  satisfy the following conditions with nonnegative Lipschitz constants:

(iii)  $|f(t_1, y_1, u_1, z_1) - f(t_2, y_2, u_2, z_2)|$  $\leq L_1(|t_1 - t_2| + |y_1 - y_2| + |u_1 - u_2|) + L_2|z_1 - z_2|, L_2 < 1;$ (iv)  $|\alpha(t_1, y_1) - \alpha(t_2, y_2)| \le A_1 |t_1 - t_2| + A_2 |y_1 - y_2|;$ (v)  $|\beta(t_1, y_1) - \beta(t_2, y_2)| \le B_1 |t_1 - t_2| + B_2 |y_1 - y_2|.$ 

The problem (1) with  $\gamma = a$  was studied by Grimm [1]. He proved an existence result for (1) assuming that f is bounded by some constant  $M, L_2 < 1$ , and  $B_1 + B_2 M \leq 1$ . He also proved a uniqueness result when  $\beta$  is independent of y. In the recent paper [2] the author relaxed this very restrictive assumption at the expense of the additional condition  $L_2(1 + B_1 + B_2 G) < 1$ , where G is some constant depending on f and g. This condition means that the dependence of fon the last argument is not too strong. It is the purpose of this note to improve

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further the results given in [1] and [2]. We prove the existence and uniqueness theorem for (1) (with  $\beta$  depending both on t and y), where the inequality  $L_2(1 + 1)$  $B_1 + B_2 G < 1$  is replaced by the weaker conditions  $L_2 < 1$  and  $B_1 + B_2 G \leq 1$ . For any continuous functions y and z on  $[\gamma, b]$ , put

$$F(t, y, z) := f(t, y(t), y(\alpha(t, y(t))), z(\beta(t, y(t))))$$

and define

$$M := \sup\{|F(t,0,0)| : t \in [a,b]\}; \qquad C_1 := (g'_{[\gamma,a]} + M)/(1 - L_2);$$
  

$$C_2 := 2L_1/(1 - L_2); \qquad Y := (g_{[\gamma,a]} + C_1/C_2)\exp((b - a)C_2);$$
  

$$Z := \max\{C_1 + C_2Y, (M + 2L_1Y)/(1 - L_2)\}; \qquad G := \max\{L_g, Z\};$$
  

$$D := \max\{L_{g'}, L_1(1 + G(1 + A_1 + A_2G))/(1 - L_2(B_1 + B_2G))\}.$$

Here  $x_{[c,d]} := \sup\{|x(t)| : t \in [c,d]\}$  for any function x. We have the following:

**Theorem.** Assume that (i)–(v) hold,  $L_2 < 1$ , and  $B_1 + B_2G \leq 1$ . Then (1) has a solution y whose derivative is Lipschitz-continuous. Moreover, this solution is unique in the space of continuously differentiable functions on  $[\gamma, a]$ .

**PROOF:** For  $h \in J := \{h \mid h = (b-a)/n, n \ge n_0\}$ , where  $n_0$  is a positive integer, put  $t_i = a + ih$ , i = 0, 1, ..., n, and as in [2] define the modified Euler sequences  $\{y_h\}_{h\in J}$  and  $\{z_h\}_{h\in J}$  by

(2)  
$$y_h(t_i + rh) = y_h(t_i) + rhz_h(t_i),$$
$$z_h(t_i + rh) = (1 - r)z_h(t_i) + rz_h(t_{i+1}),$$
$$z_h(t_{i+1}) = F(t_{i+1}, y_h, z_h),$$

 $i = 0, 1, ..., n - 1, r \in (0, 1]$ , where  $y_h(t) = g(t)$  and  $z_h(t) = g'(t)$  for  $t \in (0, 1]$  $[\gamma, a]$ . Note that (2) is, in general, implicit in  $z_h$ , but in view of  $L_2 < 1$  it has a unique solution  $(y_h, z_h)$  for any  $h \in J$ . We will first show that  $\{y_h\}_{h \in J}$  and  $\{z_h\}_{h\in J}$  are relatively compact in the space  $C[\gamma, b]$  of continuous functions on  $[\gamma, b]$ . Proceeding as in [2] it follows that  $\{y_h\}_{h\in J}$  and  $\{z_h\}_{h\in J}$  are uniformly bounded by Y and Z, respectively, and that  $\{y_h\}_{h\in J}$  are uniformly Lipschitzcontinuous with the constant G. The proof that  $\{z_h\}_{h\in J}$  are also uniformly Lipschitz-continuous is more delicate than in [2]. The proof is by induction. Assume that

(3) 
$$|z_h(t_1) - z_h(t_2)| \le D|t_1 - t_2|, \quad t_1, t_2 \in [\gamma, t_i],$$

and we will show that this inequality is also true for  $t_1, t_2 \in [\gamma, t_{i+1}]$  (obviously (3) holds for  $t_1, t_2 \in [\gamma, t_0]$ ). Define on  $[\gamma, t_{i+1}]$  the iterations  $z_h^{[\nu]}(t) = z_h(t)$  for  $t \in [\gamma, t_i], \nu = 0, 1, ..., \text{ and }$ 

$$\begin{split} z_h^{[0]}(t_i + rh) &= z_h(t_i), \\ z_h^{[\nu+1]}(t_{i+1}) &= F(t_{i+1}, y_h, z_h^{[\nu]}), \\ z_h^{[\nu+1]}(t_i + rh) &= (1 - r)z_h(t_i) + rz_h^{[\nu+1]}(t_{i+1}), \end{split}$$

 $r \in (0,1], \nu = 0, 1, \ldots$  It follows by the induction with respect to  $\nu$  that  $\{z_h^{[\nu]}\}_{\nu=0}^{\infty}$  are uniformly bounded by Z and uniformly Lipschitz-continuous on  $[\gamma, t_{i+1}]$  with the same constant D. Indeed, this is true for  $\nu = 0$  and, assuming that it is true for  $\nu$ , routine manipulations yield

$$|z_h^{[\nu+1]}(t_{i+1})| \le M + 2L_1Y + L_2Z \le Z,$$

and

$$|z_h^{[\nu+1]}(t_{i+1}) - z_h^{[\nu+1]}(t_i)| \le L_1(1 + G(1 + A_1 + A_2G))h + L_2D(B_1 + B_2G)h \le Dh.$$

The last inequality follows from the definition of D. In view of the Ascoli-Arzela theorem the sequence  $\{z_h^{[\nu]}\}_{\nu=0}^{\infty}$  is relatively compact in  $C[\gamma, t_{i+1}]$  and since the solution  $(y_h, z_h)$  of (2) is unique, we have  $z_h^{[\nu]} - z_{h[\gamma, t_{i+1}]} \to 0$  as  $\nu \to \infty$ . Therefore,  $\{z_h\}_{h\in J}$  are uniformly Lipschitz-continuous on  $[\gamma, t_{i+1}]$  with the same constant D. By induction with respect to i, this is also true on  $[\gamma, b]$ . Consequently,  $\{y_h\}_{h\in J}$  and  $\{z_h\}_{h\in J}$  are relatively compact in  $C[\gamma, b]$  and from this point the proof is exactly the same as the proof of Theorem 2 in [2]. We prove the existence by showing that there is a subsequence of  $\{y_h\}_{h\in J}$  convergent to the solution y of (1) and we prove the uniqueness by contradiction.

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Department of Mathematics, Arizona State University, Tempe, Arizona 85287-1804, USA

E-mail: jackiewi@math.la.asu.edu

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