# Commentationes Mathematicae Universitatis Carolinae

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# Distinguishing Lindelöfness and inverse Lindelöfness

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*Abstract.* On $\omega_1$ a Hausdorff inverse Lindelöf non Lindelöf topology has been constructed.

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## 1. Introduction

M.V. Matveev [1] calls a topological space $X$ *inversely compact* if for every open cover $\mathcal{U}$ of $X$, one can select a finite cover $\mathcal{V}$ of $X$ which consists of the elements of $\mathcal{U}$ or their complements, but of course $\mathcal{V}$ is prohibited to contain both $U$ and $X \setminus U$ for any $U \in \mathcal{U}$. In the sequel we will use the following

**1.1 Reformulation.** *A space $X$ is inversely compact if for every open cover $\mathcal{U}$ of $X$ there exists a finite function $f : \mathcal{U} \to \{0, 1\}$ such that $\cup\{U^{f(U)} : U \in \mathrm{dom}(f)\} = X$ (by a finite function we mean a function with a finite domain; here and henceforth $U^0 = U$ and $U^1 = X \setminus U$ for any subset $U$ of $X$).*

We obtain *inverse countable compactness* if in this definition we consider only countable covers; likewise, a space $X$ is *inverse Lindelöf* if in this definition we consider countable functions.

In [1] Matveev has obtained the following criterion for inverse compactness:

**1.2 Criterion.** *A space $X$ is inversely compact iff every independent family of closed subsets of $X$ has nonempty intersection.*

Let us remind that a family $\mathcal{F}$ is called *independent* if for every finite function $f : \mathcal{F} \to \{0, 1\}$ the intersection $\cap\{F^{f(F)} : F \in \mathrm{dom}(f)\}$ is nonempty.

Analogous criterion holds for *inverse Lindelöfness*:

**1.3 Criterion.** *A space $X$ is inversely Lindelöf iff every countably independent family of closed subsets of $X$ has nonempty intersection.*

In this connection a family $\mathcal{F}$ is *called countably independent* if for every countable function $f : \mathcal{F} \to \{0, 1\}$ the intersection $\cap\{F^{f(F)} : F \in \mathrm{dom}(f)\}$ is nonempty.

**1.4.** There are *various questions about inverse compactness and inverse Lindelöfness*, for instance:

(Matveev) *Does there exist a $T_2$, inversely compact, noncompact space?*

(Matveev) *Which properties of Hausdorff compact spaces remain valid for inversely compact spaces?* For instance: *Does countable compactness (inverse countable compactness) + inverse Lindelöfness imply (inverse) compactness?*

In this article we answer some such questions.

**1.5 Remark.** The characterization of inverse compactness via independent families of closed sets and also some results of Katětov motivated the author to consider the general idea of introducing properties weaker than compactness: one should demand that only some (not all) centered families of closed sets have nonempty intersection. Inverse compactness can be considered an instance of this general approach.

## 2. The inverse Lindelöfness and CH

**2.1 Theorem.** *Every space of the power smaller than $\mathfrak{C}$ is inversely Lindelöf.*

PROOF: Let us suppose that a space is not inversely Lindelöf, then there exists a countably independent family $\mathcal{F}$ with empty intersection. It is evident that $\mathcal{F}$ is uncountable. Let $\mathcal{E}$ be a some infinite countable subfamily. For every function $e : \mathcal{E} \to \{0,1\}$ we have $K_e = \cap\{E^{e(E)} : E \in \mathcal{E}\} \neq \emptyset$ and if $e_1 \neq e_2$ then $K_{e_1} \cap K_{e_2} = \emptyset$. So $\cup\{K_e : e$ runs the set of all functions from $\mathcal{E}$ into $\{0,1\}\}$ has the power non smaller than $\mathfrak{C}$, so the whole space has the power non smaller than $\mathfrak{C}$ as well.

**2.2 Corollary.** *It is impossible to prove in ZFC the coincidence of inverse Lindelöfness and (usual) Lindelöfness.*

Indeed, under the negation of **CH** a discrete space of the power $\aleph_1$ is inversely Lindelöf, but non Lindelöf.

Now we need the following

**2.3 Lemma.** *On a set of cardinality $m^{\aleph_0}$, $m > \omega$, there exists a countably independent family of cardinality $m$ with empty intersection.*

PROOF: In $2^m$ let us consider the subset $X = \{x \in 2^m : |x^{-1}(1)| \leq \aleph_0\}$, i.e. the $\Sigma$-product of $m$ disconnected two-points with the base point $\overline{0} = (0, \dots)$. For every $\alpha \in m$ let $A_\alpha = \{x \in X : x(\alpha) = 1\}$. It is easy to see the family $\{A_\alpha : \alpha \in m\}$ is as desired.                                                      $\square$

**2.4 Theorem.** *A discrete space is inversely Lindelöf iff its cardinality is smaller than $\mathfrak{C}$.*

PROOF: According to Theorem 2.1 it is sufficient to prove that a discrete space of the power $\mathfrak{C}$ is not inversely Lindelöf (as Matveev [1] proved that inverse compactness and inverse Lindelöfness are closed-hereditary). For this let us take in

**Lemma 2.4** $m = \mathfrak{C}$ and consider $X$ with discrete topology, then $\{A_\alpha : \alpha \in m\}$ is the countably independent family of closed subsets with empty intersection. $\square$

**2.5 Corollary.** *Under CH every uncountable discrete space is not inversely Lindelöf.*

**2.6 Corollary.** *CH is equivalent to the non inverse Lindelöfness of a discrete space of cardinality $\aleph_1$.*

**2.7 Corollary.** *For discrete space of cardinality $\aleph_1$ the property to be inversely Lindelöf is undetermined in ZFC.*

**2.8 Lemma.** *Let $m$ be a regular cardinal and $m^{\aleph_0} = m$, then on $m$ there exists a countably independent family $\{A_\alpha : \alpha \in m\}$ such that $\alpha \cap A_\alpha = \emptyset$ for every $\alpha \in m$.*

PROOF: We shall construct such family by the transfinite induction. Let us consider a general step of this induction.

Let us suppose that for some $\alpha \in m$ one has defined yet subsets $A_{\beta\gamma}$ for all $\beta < \alpha$ and $\beta \leq \gamma < \alpha$ and also all $|A_{\beta\gamma}| < m$, some other inductive assumptions yet unspecified are fulfilled as well.

Let $e$ be any countable function from $m$ into $\{0,1\}$ and $\mathrm{dom}(e) \subset \alpha$. Let $\delta = \mathrm{Sup}(\cup\{A_{\beta\gamma} : \beta \leq \gamma < \alpha\})$ and $\theta = \max\{\delta, \alpha\} + 1$. For $\beta < \alpha$ let us set $A'_{\beta\alpha} = \cup\{A_{\beta\gamma} : \beta \leq \gamma < \alpha\}$. Further let us set $A_{\beta\alpha} \cup \{\theta\}$ if $e(\beta) = 0$ and $A_{\beta\alpha} = A'_{\beta\alpha}$ if $e(\beta) = 1$. Let us also set $A_{\alpha\alpha} = \{\theta\}$. The general step of transfinite induction has been completed. Now let $A_\alpha = \cup\{A_{\alpha\gamma} : \alpha \leq \gamma < m\}$ for every $\alpha \in m$. $\square$

**2.9 Theorem** [CH]. *$\omega_1$ with ordered topology is not inversely Lindelöf.*

PROOF: There exists an uncountable discrete subspace $Z = \{z_\beta : \beta < \omega_1\} \subset \omega_1$. Let us take in Lemma 2.8 $m = \omega_1$, then under CH the conditions of this lemma are fulfilled. Let $\{A_\alpha : \alpha \in \omega_1\}$ be the corresponding family of this lemma. Let us set $E_\alpha = \{z_\beta : \beta \in A_\alpha\}$ and $F_\alpha = \overline{E_\alpha}$. It is easy to verify that $\{F_\alpha : \alpha \in \omega_1\}$ is a countably independent family of closed subsets of $\omega_1$ with empty intersection. $\square$

**2.10 Corollary.** *The implication "Countable compactness (inverse countable compactness) + inverse Lindelöfness" $\rightarrow$ "Inverse compactness" is improvable in ZFC.*

As this implication does not take place for $\omega_1$ with ordered topology under the negation of CH (let us note that this space is not inversely compact—see the proof in [2]).

## 3. The inverse Lindelöfness and Ostaszewskii's space

**3.1 Theorem.** *If each closed subset of a space is countable or its complement is countable, then this space is inversely Lindelöf.*

PROOF: Let $\mathcal{F}$ be a countably independent family of closed subsets of the considered space $X$ with empty intersection and $F$ be an arbitrary member of $\mathcal{F}$. As it is easy to see, $F$ cannot be countable, so $X \setminus F$ is countable. Let $\mathcal{A}$ be any countable infinite subfamily of $\mathcal{F} \setminus \{F\}$. For every function $e : \mathcal{A} \to \{0,1\}$ $K_e = (X \setminus F) \cap (\cap\{A^{e(A)} : A \in \mathcal{A}\}) \neq \emptyset$ and all these $K_e$ are disjoint. But this contradicts countability of the set $X \setminus F$.                                         $\square$

**3.2 Corollary.** *An Ostaszewskii's space is inversely Lindelöf.*

Let us note that in [2] it was proved that no Ostaszewskii's space is inversely compact.

## 4. The inverse Lindelöfness of $\omega_1 \setminus \{x\}$ for some $x$

**4.1 Theorem.** *There exists some $x$ such that the space $\omega^* \setminus \{x\}$ is not inversely Lindelöf.*

PROOF: We use Theorem 4.4.4 of [3] stating that every $P$-space of weight non greater that $\mathfrak{C}$ can be embedded in $\omega^*$.

For every $\alpha \in \omega_1$ let $A_\alpha$ be a discrete space of the power $\mathfrak{C}$ and $X = \{*\} \cup (\cup\{A_\alpha : \alpha \in \omega_1\})$, where $*$ is a special point with basic neighbourhoods $B_\beta = \{*\} \cup (\cup\{A_\alpha : \alpha \in \omega_1 \setminus \beta\})$; the other points of $X$ are isolated. It is clear that $X$ is a $P$-space, hence, $X \subset \omega^*$, let $x$ be $*$ under this embedding. Let us note that $\cap\{\overline{B}_\beta : \beta \in \omega_1\} = \{x\}$ (the closure is taken in $\omega^*$), because for every point $y \in \omega^* \setminus \{x\}$ there exist $\beta \in \omega_1$ and a neighbourhood $Oy$ so that $B_\beta \cap Oy = \emptyset$.

On every $A_\beta$ let us take, according to Lemma 2.3 a countable family $\mathcal{E}^\beta = \{E_\alpha^\beta : \alpha < \beta\}$, which is countably independent and let $K_\alpha = \cup\{E_\alpha^\beta : \alpha < \beta < \omega_1\}$ for every $\alpha \in \omega_1$. It is easy to see that $\{K_\alpha : \alpha \in \omega_1\}$ is a countably independent family of closed subsets with empty intersection.                                         $\square$

**4.2 Theorem** [CH]. *For every $x \in \omega^*$ the space $\omega^* \setminus \{x\}$ is not inversely Lindelöf.*

PROOF: Under CH for every $x \in \omega^*$ there exists an uncountable subset $\{z_\alpha : \alpha \in \omega_1\}$ such that $|\{z_\beta : z_\beta \notin Ox\}| \leq \aleph_0$ for every neighbourhood $Ox$ of $x$. Now we use Lemma 2.8 for $m = \mathfrak{C} = \aleph_1$. Let $E_\alpha = \{z_\beta : \beta \in A_\alpha\}$ and $F_\alpha = \overline{E}_\alpha$ (the closure is taken in $\omega^* \setminus \{x\}$). It is easy to check that $\{F_\alpha : \alpha \in \omega_1\}$ is the countably independent family of closed subsets in $\omega^* \setminus \{x\}$ with empty intersection.                                         $\square$

## 5. Examples of inversely Lindelöf non Lindelöf spaces under CH and in ZFC

**5.1 Example** [CH]. *On $\omega_1$ there exists a Hausdorff locally countable topology $\theta$, such that for every infinite subset $A \subset \omega_1$ there exists an $\alpha$, such that $[\alpha, \omega_1) \subset \overline{A}$.*

**Construction.** Let $\{A_\alpha : \alpha \in \omega_1 \setminus \omega\}$ be an enumeration of the family $[\omega_1]^\omega$ such that $A_\alpha \subset \alpha$ for every $\alpha \in \omega_1 \setminus \alpha$. Let all points $n \in \omega$ be isolated. Let us describe a general step of the transfinite induction.

Let us suppose that on some $\alpha \in \omega_1 \setminus \omega$ a Hausdorff topology $\tau$ with countable base has been defined and for every $\beta \in \alpha \setminus \omega$ a countable family $\mathcal{A}_\beta \subset [\beta]^\omega$ has been fixed such that $\beta \in \overline{A}$ for every $A \in \mathcal{A}_\beta$. Let $\mathcal{A}_\alpha = \{A_\beta : \beta \in \alpha\}$, so $\mathcal{A}_\alpha$ is countable. Now our task is to define on $\alpha + 1$ a Hausdorff topology $\mu$ with countable base, $\tau \subset \mu$ and in that $\beta \in \overline{A}$ for every $A \in \mathcal{A}^\beta$ and $\beta \leq \alpha$.

**5.2 Lemma.** *Let on a set $Z$ a countable family $\mathcal{E}$ of infinite subsets be given, then there exist a disjoint family $\mathcal{P}$ of infinite subsets and one-to-one function $\phi : \mathcal{E} \to \mathcal{P}$ such that $\phi(E) \subset E$ for each $E \in \mathcal{E}$.*

This lemma belongs to the set-theoretic folklore.

So, according to this lemma if we take the family $(\cup\{\mathcal{A}^\beta : \beta \leq \alpha\})$ as $\mathcal{E}$ we will obtain families $\mathcal{B}^\beta$, $\beta \leq \alpha$, elements of $\mathcal{B}^\beta$ are refined into elements of $\mathcal{A}^\alpha$, $\beta \leq \alpha$, and $\cup\{\mathcal{B}^\beta : \beta \leq \alpha\}$ is a disjoint family.

**5.3 Lemma.** *Let on a countable infinite set $X$ for every point $x \in X$ an infinite subset $E_x$ be given, so that if $x_1 \neq x_2$ then $E_{x_1} \cap E_{x_2} = \emptyset$. Further on each subset $E_x$ a free filter $\mathcal{F}_x$ is given. Let us define a topology $\nu$ on $X$ in the following manner: a set $V$ on $X$ is open iff for each $x \in V$ there exists $F_x \in \mathcal{F}_x$, $F_x \subset V$. Then this topology $\nu$ is normal and $T_1$ (moreover it is Hausdorff).*

PROOF: For every point $x \in X$ the subset $X \setminus \{x\}$ is open, as all filters $\mathcal{F}_y$, $y \in X$, are free. So, $\nu$ is a $T_1$-topology. Let us prove its normality. Let $A$, $B$ be two closed disjoint subsets. Let us find their disjoint neighbourhoods. As it is easy to see, a set $\{y\} \cup F$ is closed for every $y \in X$ and $F \in \mathcal{F}_y$. As $B$ is closed, $X \setminus B$ is open, hence for an arbitrary point $a \in A$ there exists $F_a \in \mathcal{F}_a$ such that $(A \cup F_a) \cap B = \emptyset$. Let us note that $A \cup F_a$ is closed as it is the sum of two closed subsets: $A$ and $\{a\} \cup F_a$. Having been performing countably many similar operations let us find subsets $W$ and $T$, $A \subset W$, $B \subset T$, $W \cap T = \emptyset$ and such that for every $x \in W$ some $F_x \in \mathcal{F}_x$ is contained in $W$, analogously for the subset $T$. But this means that $W$ and $T$ are open.

Let us continue in the construction of our example. Recall that the sets $T_\beta = \cup\mathcal{B}_\beta$, $\beta \leq \alpha$, are disjoint. On each $T_\beta$, $\beta < \alpha$, let us consider a filter $\mathcal{F}_\beta$ formed by the family $\{(T_\beta \cap V) \setminus \Delta : V$ is a neighbourhood of $\beta$ and $\Delta$ is an arbitrary finite subset of $\alpha\}$. On $T_\alpha$ let us consider a filter $\mathcal{F}_\alpha$ formed by the family $\{T_\alpha \setminus \Delta : \Delta$ is an arbitrary finite subset of $\alpha\}$, each filter $\mathcal{F}_\beta$, $\beta \leq \alpha$, is free, of course. Let us denote by $\delta$ a topology formed on $\alpha + 1$ by system filters $\{\mathcal{F}_\beta : \beta \leq \alpha\}$ according to Lemma 5.3. Let us note that $\beta \in \overline{A}$ for every $A \in \mathcal{A}_\beta$ and $\beta \leq \alpha$ with respect to topology $\delta$. It is easy to see that $\tau \subset \delta$. Indeed, if $M$ is open in $\tau$, then $M \subset \alpha$ and for each point $\beta \in M$  $M$ is the neighbourhood for $\beta$. However then $M \cap T_\beta \subset M$, hence $M \in \delta$.

Let us note that $\tau$ has a countable base, denote it by $\mathcal{S}$. As $\delta$ is Hausdorff there exists a countable subfamily $\mathcal{W} \subset \delta$ such that for every $x, y \in \alpha + 1$, $x \neq y$, there are $U, V \in \mathcal{W}$ are disjoint neigbourhoods of $x, y$. The family $\mathcal{W} \cup \mathcal{S}$ is countable, let $\mu$ be a smallest topology, containing $\mathcal{W} \cup \mathcal{S}$. It is clear that $\mu$ has a countable base and $\beta \in \overline{A}$ for each $A \in \mathcal{A}_\beta$ and $\beta \leq \alpha$ concerning $\mu$. The general

step of transfinite induction has been described. Now the join of all intermediate topologies will form the base of $\theta$.                                                       $\square$

**5.4 Theorem** [CH]. *The space $(\omega_1, \theta)$ is inversely Lindelöf, but non Lindelöf.*

Indeed, as this space is locally countable, so it is not Lindelöf. Later, each closed subset is either countable or its complement is countable. According to Theorem 3.1 it implies inverse Lindelöfness.

**5.5 Example.** *On $\omega_1$ there exists a locally countable topology $\Delta$ that is inversely Lindelöf under CH. Hence the space $(\omega_1, \Delta)$ is inversely Lindelöf non Lindelöf space.*

**Construction.** Let $\{A_\alpha : \alpha \in \mathfrak{C} \setminus \omega\}$ be an enumeration of the family $[\omega_1]^\omega$ by smallest ordinal (it is evidently, $\mathfrak{C}$) such that $A_\alpha \subset \alpha$ for every $\alpha \in \mathfrak{C} \setminus \omega$. Now carry out the transfinite induction exactly as in Example 5.1, but only making the first $\omega_1$ steps. Denote the topology so constructed $\Delta$. It is clear that $\Delta$ is Hausdorff and locally countable. Hence it is not Lindelöf. Under CH $\mathfrak{C} = \omega_1$, hence we will have made all necessary steps of the transfinite induction of Example 5.1 and therefore $\Delta$ will coincide with $\theta$, so under CH $(\omega_1, \Delta)$ is inversely Lindelöf. If CH does not hold then $(\omega_1, \Delta)$ is still inversely Lindelöf by Theorem 2.1, because it has the power $\aleph_1 < \mathfrak{C}$.

## References

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