Brian Fisher; Adem Kiliçman; Blagovest Damyanov; J. C. Ault

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On the non-commutative neutrix product $\ln x_+ \circ x_+^{-s}$

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Abstract. The non-commutative neutrix product of the distributions $\ln x_+$ and x_+^{-s} is proved to exist for $s = 1, 2, \dots$ and is evaluated for $s = 1, 2$. The existence of the non-commutative neutrix product of the distributions x_+^{-r} and x_+^{-s} is then deduced for $r, s = 1, 2, \dots$ and evaluated for $r = s = 1$.

Keywords: distribution, delta-function, neutrix, neutrix limit, neutrix product

Classification: 46F10

In the following, we let N be the neutrix, see van der Corput [1], having domain $N' = \{1, 2, \dots, n, \dots\}$ and range the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n : \qquad \lambda > 0, \quad r = 1, 2, \dots$$

and all functions which converge to zero in the normal sense as n tends to infinity.

We now let $\varrho(x)$ be any infinitely differentiable function having the following properties:

(i) $\varrho(x) = 0$ for $|x| \geq 1$,
(ii) $\varrho(x) \geq 0$,
(iii) $\varrho(x) = \varrho(-x)$,
(iv) $\displaystyle\int_{-1}^{1} \varrho(x)\, dx = 1$.

Putting $\delta_n(x) = n\varrho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

Now let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. Then if f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t) \rangle$$

for $n = 1, 2, \dots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

A first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2] or [3].

Definition 1. *Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a,b), f is the k-th derivative of a locally summable function F in $L^p(a,b)$ and $g^{(k)}$ is a locally summable function in $L^q(a,b)$ with $1/p+1/q=1$. Then the product $fg=gf$ of f and g is defined on the interval (a,b) by*

$$fg=\sum_{i=0}^{k}\binom{k}{i}(-1)^i[Fg^{(i)}]^{(k-i)}.$$

The following definition for the neutrix product of two distributions was given in [4] and generalizes Definition 1.

Definition 2. *Let f and g be distributions in $\mathcal{D}'$ and let $g_n(x)=(g*\delta_n)(x)$. We say that the neutrix product $f\circ g$ of f and g exists and is equal to the distribution h on the interval (a,b) if*

$$\operatorname*{N-lim}_{n\to\infty}\langle f(x)g_n(x),\phi(x)\rangle=\langle h(x),\phi(x)\rangle$$

for all functions ϕ in $\mathcal{D}$ with support contained in the interval (a,b).

Note that if

$$\lim_{n\to\infty}\langle f(x)g_n(x),\phi(x)\rangle=\langle h(x),\phi(x)\rangle,$$

we simply say that the *product* $f.g$ exists and equals h, see [3].

It is obvious that if the product $f.g$ exists then the neutrix product $f\circ g$ exists and the two are equal. Further, it was proved in [3] that if the product fg exists by Definition 1, then the product $f.g$ exists by Definition 2 and the two are equal. Note also that although the product defined in Definition 1 is always commutative, the neutrix product defined in Definition 2 is in general non-commutative.

The following theorem holds, see [7].

Theorem 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix products $f\circ g^{(i)}$ (or $f^{(i)}\circ g$) exist on the interval (a,b) for $i=0,1,2,\ldots,r$. Then the neutrix products $f^{(k)}\circ g$ (or $f\circ g^{(k)}$) exist on the interval (a,b) for $k=1,2,\ldots,r$ and*

$$f^{(k)}\circ g=\sum_{i=0}^{k}\binom{k}{i}(-1)^i[f\circ g^{(i)}]^{(k-i)}$$

or

$$f\circ g^{(k)}=\sum_{i=0}^{k}\binom{k}{i}(-1)^i[f^{(i)}\circ g]^{(k-i)}$$

on the interval (a,b) for $k=1,2,\ldots,r$.

In the following two theorems, which were proved in [6] and [9] respectively, the distributions x_+^{-r} and x_-^{-r} are defined by

$$x_+^{-r}=\frac{(-1)^{r-1}}{(r-1)!}(\ln x_+)^{(r)},\quad x_-^{-r}=-\frac{1}{(r-1)!}(\ln x_-)^{(r)},$$

for $r = 1, 2, \ldots$ and is distinct from the definition given by Gel'fand and Shilov [8]. Further, the distribution $F(x_+, -r)\ln x_+$ is defined for an arbitrary ϕ in $\mathcal{D}$ by

$$\langle F(x_+, -r)\ln x_+, \phi(x)\rangle = \int_0^\infty x^{-r}\ln x\Big[\phi(x) - \sum_{k=0}^{r-2}\frac{x^k}{k!}\phi^{(k)}(0) + \\ - \frac{x^{r-1}}{(r-1)!}\phi^{(r-1)}(0)H(1-x)\Big]\,dx,$$

for $r = 1, 2, \ldots$, where the sum is empty when $r = 1$, and H denotes Heaviside's function. The distribution $x_+^{-r}\ln x_+$ is then defined by

$$x_+^{-r}\ln x_+ = F(x_+, -r)\ln x_+ + \frac{(-1)^r}{(r-1)!}\psi_1(r-1)\delta^{(r-1)}(x), \tag{1}$$

for $r = 1, 2, \ldots$, where

$$\psi_1(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^r \frac{\psi(i)}{i}, & r \geq 1, \end{cases} \quad \psi(r) = \begin{cases} 0, & r = 0, \\ \sum_{i=1}^r \frac{1}{i}, & r \geq 1. \end{cases}$$

It follows that

$$(\ln^2 x_+)' = 2x_+^{-1}\ln x_+, \quad (x_+^{-r}\ln x_+)' = -rx_+^{-r-1}\ln x_+ + x_+^{-r-1},$$

see [10].

Theorem 2. *The neutrix products $x_+^{-r} \circ x_-^{-s}$ and $x_-^{-s} \circ x_+^{-r}$ exist and*

$$x_+^{-r} \circ x_-^{-s} = \frac{(-1)^r c_1}{(r+s-1)!}\delta^{(r+s-1)}(x),$$

$$x_-^{-s} \circ x_+^{-r} = \frac{(-1)^{r-1} c_1}{(r+s-1)!}\delta^{(r+s-1)}(x)$$

for $r, s = 1, 2, \ldots$, where

$$c_1(\varrho) = \int_0^1 \ln t\,\varrho(t)\,dt.$$

It was shown in [5] that with suitable choice of the function ϱ, $c_1(\varrho)$ can take any negative value.

Theorem 3. *The neutrix products $\ln x_+ \circ x_-^{-s}$ and $x_-^{-s} \circ \ln x_+$ exist and*

$$\ln x_+ \circ x_-^{-s} = \frac{1}{(s-1)!}\Big(c_2 - \frac{\pi^2}{12}\Big)\delta^{(s-1)}(x) + \\ - \sum_{i=1}^{s-1}\frac{(-1)^i c_1}{(s-i-1)!i!i}\delta^{(s-1)}(x),$$

$$= x_-^{-s} \circ \ln x_+$$

$$= (-1)^{s-1}\ln x_- \circ x_+^{-s} = (-1)^{s-1} x_+^{-s} \circ \ln x_-$$

for $s = 1, 2\dots$, *where*

$$c_2(\varrho) = \int_0^1 \ln^2 t\, \varrho(t)\, dt.$$

We now prove the following theorem:

Theorem 4. *The neutrix product* $\ln x_+ \circ x_+^{-s}$ *exists for* $s = 1, 2, \dots$. *In particular,*

$$\ln x_+ \circ x_+^{-1} = x_+^{-1} \ln x_+, \tag{2}$$

$$\ln x_+ \circ x_+^{-2} = x_+^{-2} \ln x_+ + (c_1 - 1)\delta'(x). \tag{3}$$

PROOF: We put

$$(x_+^{-s})_n = x_+^{-s} * \delta_n(x)$$

so that

$$(x_+^{-s})_n = \frac{(-1)^{s-1}}{(s-1)!} \int_{-1/n}^{x} \ln(x-t)\delta_n^{(s)}(t)\, dt$$

on the interval $[0, 1/n]$ and

$$(x_+^{-s})_n = \frac{(-1)^{s-1}}{(s-1)!} \int_{-1/n}^{1/n} \ln(x-t)\delta_n^{(s)}(t)\, dt = \int_{-1/n}^{1/n} (x-t)^{-s}\delta_n(t)\, dt$$

on the interval $[1/n, \infty)$.

Then

$$\begin{aligned}
(-1)^{s-1}(s-1)! &\int_0^1 x^k \ln x (x_+^{-s})_n\, dx \\
&= \int_0^1 x^k \ln x \int_{-1/n}^{1/n} \ln(x-t)_+ \delta_n^{(s)}(t)\, dt\, dx \\
&= \int_{-1/n}^{0} \delta_n^{(s)}(t) \int_0^1 x^k \ln x \ln(x-t)\, dx\, dt + \\
&\qquad + \int_0^{1/n} \delta_n^{(s)}(t) \int_t^1 x^k \ln x \ln(x-t)\, dx\, dt \\
&= (-1)^s \int_0^{1/n} \delta_n^{(s)}(t) \int_0^t x^k \ln x \ln(x+t)\, dx\, dt + \\
&\qquad + (-1)^s \int_0^{1/n} \delta_n^{(s)}(t) \int_t^1 x^k \ln x \ln(x+t)\, dx\, dt + \\
&\qquad + \int_0^{1/n} \delta_n^{(s)}(t) \int_t^1 x^k \ln x \ln(x-t)\, dx\, dt.
\end{aligned} \tag{4}$$

Now

$$\begin{aligned}
\int_0^t x^k \ln x \ln(x+t)\,dx &= \int_0^t x^k \ln x[\ln t + \ln(1+x/t)]\,dx \\
&= \frac{t^{k+1}\ln^2 t}{k+1} - \frac{t^{k+1}\ln t}{(k+1)^2} - \sum_{i=1}^{\infty} \frac{(-1)^i}{it^i} \int_0^t x^{k+i} \ln x\,dx \\
&= \frac{t^{k+1}\ln^2 t}{k+1} - \frac{t^{k+1}\ln t}{(k+1)^2} - \sum_{i=1}^{\infty} \Big[\frac{(-1)^i t^{k+1}\ln t}{i(k+i+1)} - \frac{(-1)^i t^{k+1}}{i(k+i+1)^2}\Big] \\
&= \alpha_{k1} t^{k+1}\ln^2 t + \beta_{k1} t^{k+1}\ln t + \gamma_{k1} t^{k+1}, \qquad (5)
\end{aligned}$$

$$\begin{aligned}
\int_t^1 x^k \ln x \ln(x+t)\,dx &= \int_t^1 x^k \ln x[\ln x + \ln(1+t/x)]\,dx \\
&= \frac{2}{(k+1)^3} - \frac{t^{k+1}\ln^2 t}{k+1} + \frac{2t^{k+1}\ln t}{(k+1)^2} - \frac{2t^{k+1}}{(k+1)^3} + \\
&\qquad - \sum_{i=1}^{\infty} \frac{(-t)^i}{i} \int_t^1 x^{k-i}\ln x\,dx \\
&= \frac{2}{(k+1)^3} - \frac{t^{k+1}\ln^2 t}{k+1} + \frac{2t^{k+1}\ln t}{(k+1)^2} - \frac{2t^{k+1}}{(k+1)^3} + \\
&\quad + \sum_{\substack{i=1\\ i\neq k+1}}^{\infty} \Big[\frac{(-1)^i t^{k+1}\ln t}{i(k-i+1)} - \frac{(-1)^i t^{k+1}}{i(k-i+1)^2} + \frac{(-t)^i}{i(k-i+1)^2}\Big] + \\
&\qquad + \frac{(-t)^{k+1}}{2(k+1)} \ln^2 t \\
&= 2(k+1)^{-3} + \alpha_{k2} t^{k+1}\ln^2 t + \beta_{k2} t^{k+1}\ln t + \gamma_{k2} t^{k+1} + \\
&\qquad + \sum_{\substack{i=1\\ i\neq k+1}}^{\infty} \frac{(-t)^i}{i(k-i+1)^2}, \qquad (6)
\end{aligned}$$

$$\begin{aligned}
\int_t^1 x^k \ln x \ln(x-t)\,dx &= \int_t^1 x^k \ln x[\ln x + \ln(1-t/x)]\,dx \\
&= \frac{2}{(k+1)^3} - \frac{t^{k+1}\ln^2 t}{k+1} + \frac{2t^{k+1}\ln t}{(k+1)^2} - \frac{2t^{k+1}}{(k+1)^3} + \\
&\quad + \sum_{\substack{i=1\\ i\neq k+1}}^{\infty} \Big[\frac{t^{k+1}\ln t}{i(k-i+1)} - \frac{t^{k+1}}{i(k-i+1)^2} + \frac{t^i}{i(k-i+1)^2}\Big] \\
&= 2(k+1)^{-3} + \alpha_{k3} t^{k+1}\ln^2 t + \beta_{k3} t^{k+1}\ln t + \gamma_{k3} t^{k+1} + \\
&\qquad + \sum_{\substack{i=1\\ i\neq k+1}}^{\infty} \frac{t^i}{i(k-i+1)^2}, \qquad (7)
\end{aligned}$$

for $k = 0, 1, 2, \dots$.

Putting $nt = u$, we have

$$\int_0^{1/n} t^k \ln^i t \delta_n^{(s)}(t)\, dt = n^{s-k} \int_0^1 u^k (\ln u - \ln n)^i \varrho^{(s)}(u)\, du.$$

It follows that

$$\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^k \ln^i t\, \delta_n^{(s)}(t)\, dt = 0,$$

for $i = 0, 1, 2;\ \ k = 0, 1, 2 \dots, s-1$ and $s = 1, 2, \dots$ and

$$\lim_{n\to\infty} \int_0^{1/n} t^k \delta_n^{(s)}(t)\, dt = 0,$$

for $k = s+1, s+2, \dots$ and $s = 1, 2, \dots$.

Further,

$$\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^s \ln^i t\, \delta_n^{(s)}(t)\, dt = \int_0^1 u^s \ln^i u\, \varrho^{(s)}(u)\, du$$

and it follows easily by induction that

$$\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^s \delta_n^{(s)}(t)\, dt = (-1^s s! \int_0^1 \varrho(u)\, du = \tfrac{1}{2}(-1)^s s!, \tag{8}$$

$$\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^s \ln t\, \delta_n^{(s)}(t)\, dt = (-1)^s s! c_1 + \tfrac{1}{2}(-1)^s s! \psi(s), \tag{9}$$

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty} \int_0^{1/n} t^s \ln^2 t\, \delta_n^{(s)}(t)\, dt = (-1)^s s! c_2 + 2(-1)^s s! \psi(s) c_1 + \\ + (-1)^s s! \sum_{i=1}^{s-1} \frac{\psi(i)}{i+1},\end{aligned} \tag{10}$$

the sum being empty when $s = 1$.

It follows that

$$\operatorname*{N-lim}_{n\to\infty} \int_0^1 x^k \ln x (x_+^{-s})_n\, dx = -(s-k-1)^{-2} \tag{11}$$

for $k = 0, 1, 2, \dots s-2$ and $s = 1, 2 \dots$ and with

$$\alpha_s = \alpha_{s-1,1} + \alpha_{s-1,2} + (-1)^s \alpha_{s-1,3}, \quad \beta_s = \beta_{s-1,1} + \beta_{s-1,2} + (-1)^s \beta_{s-1,3},$$
$$\gamma_s = \gamma_{s-1,1} + \gamma_{s-1,2} + (-1)^s \gamma_{s-1,3},$$

it follows from equations (4) to (10) that

$$
\begin{aligned}
\operatorname*{N-lim}_{n\to\infty}\int_0^1 x^{s-1}\ln x (x_+^{-s})_n\,dx = &-(-1)^s s\alpha_s\Big[c_2+2\psi(s)c_1+\sum_{i=1}^{s-1}\frac{\psi(i)}{i+1}\Big]+ \\
&-(-1)^s s\beta_s[c_1+\tfrac{1}{2}\,\psi(s)]-\tfrac{1}{2}(-1)^s s\gamma_s \\
= &\ \Lambda_s,
\end{aligned} \tag{12}
$$

for $s=1,2,\ldots$.

Now let ϕ be an arbitrary function in $\mathcal{D}$. Then with $2n^{-1}<\eta<1$,

$$
\begin{aligned}
\langle \ln x_+(x_+^{-s})_n,\phi(x)\rangle &= \int_0^\infty \ln x(x_+^{-s})_n\phi(x)\,dx \\
&= \int_0^\eta \ln x(x_+^{-s})_n\Big[\phi(x)-\sum_{k=0}^{s-1}\frac{x^k}{k!}\phi^{(k)}(0)\Big]\,dx+ \\
&\quad+\int_\eta^\infty \ln x(x_+^{-s})_n\Big[\phi(x)-\sum_{k=0}^{s-2}\frac{x^k}{k!}\phi^{(k)}(0)-\frac{x^{s-1}}{(s-1)!}\phi^{(s-1)}(0)H(1-x)\Big]\,dx+ \\
&\quad+\sum_{k=0}^{s-1}\frac{\phi^{(k)}(0)}{k!}\int_0^1 x^k\ln x(x_+^{-s})_n\,dx+\sum_{k=0}^{s-2}\frac{\phi^{(k)}(0)}{k!}\int_1^\infty x^k\ln x(x_+^{-s})_n\,dx.
\end{aligned}
$$

Since $(x_+^{-s})_n$ converges uniformly to the function x^{-s} on the interval $[\eta,\infty)$, it follows that

$$
\begin{aligned}
\lim_{n\to\infty}\int_\eta^\infty \ln x(x_+^{-s})_n&\Big[\phi(x)-\sum_{k=0}^{s-2}\frac{x^k}{k!}\phi^{(k)}(0)-\frac{x^{s-1}}{(s-1)!}\phi^{s-1}(0)H(1-x)\Big]\,dx \\
&= \int_\eta^\infty x^{-s}\ln x\Big[\phi(x)-\sum_{k=0}^{s-2}\frac{x^k}{k!}\phi^{(k)}(0)-\frac{x^{s-1}}{(s-1)!}\phi^{s-1}(0)H(1-x)\Big]\,dx \\
&= \int_0^\infty x^{-s}\ln x\Big[\phi(x)-\sum_{k=0}^{s-2}\frac{x^k}{k!}\phi^{(k)}(0)-\frac{x^{s-1}}{(s-1)!}\phi^{s-1}(0)H(1-x)\Big]\,dx+ \\
&\qquad\qquad+O(\eta\ln\eta),
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{k=0}^{s-2}\frac{\phi^{(k)}(0)}{k!}\int_1^\infty x^k\ln x(x_+^{-s})_n\,dx &= \sum_{k=0}^{s-2}\frac{\phi^{(k)}(0)}{k!}\int_1^\infty x^{k-s}\ln x\,dx \\
&= \sum_{k=0}^{s-2}\frac{\phi^{(k)}(0)}{k!(s-k-1)^2},
\end{aligned}
$$

and on using equations (11) and (12), we have

$$
\operatorname*{N-lim}_{n\to\infty}\sum_{k=0}^{s-1}\frac{\phi^{(k)}(0)}{k!}\int_0^1 x^k\ln x(x_+^{-s})_n\,dx = \sum_{k=0}^{s-2}\frac{\phi^{(k)}(0)}{k!(s-k-1)^2}+\frac{\Lambda_s\phi^{(s-1)}(0)}{(s-1)!}.
$$

Further,

$$\int_0^\eta \ln x(x_+^{-s})_n\Big[\phi(x) - \sum_{k=0}^{s-1}\frac{x^k}{k!}\phi^{(k)}(0)\Big]\,dx = \int_0^{2/n} x^s \ln x(x_+^{-s})_n\phi^{(s)}(\xi x)\,dx+$$
$$+\int_{2/n}^{\eta} x^s \ln x(x_+^{-s})_n\phi^{(s)}(\xi x)\,dx,$$

where $0 < \xi < 1$. Now on the interval $[0, 2/n]$, it is easily seen that

$$(x_+^{-s})_n = \frac{(-1)^{s-1}}{(s-1)!}\int_{-1/n}^{1/n} \ln(x-t)_+\delta_n^{(s)}(t)\,dt = O(n^s \ln n)$$

and so

$$\lim_{n\to\infty}\int_0^{2/n} x^s \ln x(x_+^{-s})_n\,dx = 0.$$

Putting $K = \sup\{|\phi^{(s)}(x)|\}$, we have

$$\left|\int_{2/n}^{\eta} x^s \ln x(x_+^{-s})_n\phi(\xi x)\,dx\right| \le -K\int_{-1/n}^{1/n}\delta_n(t)\int_{2/n}^{\eta} x^s(x-t)^{-s}\ln x\,dx\,dt,$$

where

$$\begin{aligned}\int_{2/n}^{\eta} x^s(x-t)^{-s}\ln x\,dx &= \sum_{k=0}^{\infty}\int_{2/n}^{\eta}\binom{-s}{k}\frac{(-t)^k}{x^k}\ln x\,dx\\ &= \eta\ln\eta - \eta - 2n^{-1}\ln(2/n) + 2n^{-1} + \tfrac{1}{2}\,st[\ln^2\eta - \ln^2(2\nu)]+\\ &\quad+\sum_{k=2}^{\infty}(-t)^k\binom{-s}{k}\left[\frac{x^{1-k}\ln x}{1-k} - \frac{x^{1-k}}{(1-k)^2}\right]_{2/n}^{\eta}.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\int_{2/n}^{\eta} x^s \ln x(x_+^{-s})_n\phi(\xi x)\,dx = O(\eta\ln\eta).$$

Since we also have

$$\int_0^\eta x^{-s}\Big[\phi(x) - \sum_{k=0}^{s-1}\frac{x^k}{k!}\phi^{(k)}(0)\Big]\,dx = O(\eta\ln\eta),$$

we see that

$$\operatorname*{N-lim}_{n\to\infty}\langle \ln x_+(x_+^{-s})_n, \phi(x)\rangle = \langle F(x_+, -s)\ln x_+, \phi(x)\rangle + \frac{\Lambda_s\phi^{(s-1)}(0)}{(s-1)!}.$$

This proves the existence of the neutrix product $\ln x_+ \circ x_+^{-s}$ and in fact

$$\begin{aligned}\ln x_+ \circ x_+^{-s} &= F(x_+,-s)\ln x_+ - \frac{(-1)^s\Lambda_s}{(s-1)!}\delta^{(s-1)}(x) \\ &= x_+^{-s}\ln x_+ - \frac{\Lambda_s+\psi_1(s-1)}{(s-1)!}(-1)^s\delta^{(s-1)}(x),\end{aligned}$$

on using equation (1). It can be shown that

$$\alpha_s = 0, \quad \beta_s = \frac{(-1)^s}{s}\psi(s-1), \quad \gamma_s = \frac{(-1)^{s+1}}{s^2}[\psi(s-1)+s\chi(s-1)],$$
$$\Lambda_s = -c_1\psi(s-1) + \frac{1}{2}\big[\chi(s-1)-\psi^2(s-1)\big]$$

for $s = 1, 2, \dots$, where

$$\chi(s) = \begin{cases} 0, & s = 0, \\ \sum_{i=1}^{s} 1/i^2, & s \geq 1 \end{cases}$$

so that in particular,

$$\Lambda_1 = 0, \quad \Lambda_2 = -c_1$$

and equations (2) and (3) follow. This completes the proof of the theorem. □

Corollary 1. *The neutrix products $x_+^{-s} \circ \ln x_+$ and $x_+^{-r} \circ x_+^{-s}$ exist for $r, s = 1, 2, \dots$. In particular,*

$$x_+^{-1} \circ \ln x_+ = x_+^{-1}\ln x_+ + (c_2 + 2c_1)\delta(x), \tag{13}$$

$$x_+^{-2} \circ \ln x_+ = x_+^{-2}\ln x_+ - (c_2 + \tfrac{1}{2})\delta'(x), \tag{14}$$

$$x_+^{-1} \circ x_+^{-1} = x_+^{-2} + (2c_1 - \tfrac{1}{2})\delta'(x). \tag{15}$$

Proof: The existence of the product $x_+^{-r} \circ x_+^{-s}$ follows immediately from Theorems 1 and 4 for $r, s = 1, 2, \dots$.

The product of the locally summable function $\ln x_+$ by itself exists by Definition 1 and is equal to the locally summable function $\ln^2 x_+$. Differentiating the equation

$$\ln x_+ \ln x_+ = \ln x_+ \circ \ln x_+ = \ln^2 x_+,$$

we get

$$x_+^{-1} \circ \ln x_+ + \ln x_+ \circ x_+^{-1} = 2x_+^{-1}\ln x_+.$$

The existence of the neutrix product $x_+^{-1} \circ \ln x_+$ and equation (13) follows from equation (2). The existence of $x_+^{-s} \circ \ln x_+$ now follows from this result, the existence of $x_+^{-r} \circ x_+^{-s}$ and Theorem 1.

Differentiating equation (2), we get

$$x_+^{-1} \circ x_+^{-1} - \ln x_+ \circ x_+^{-2} = x_+^{-2} - x_+^{-2} \ln x_+$$

and equation (15) follows on using equation (3).

Differentiating equation (13), we get

$$x_+^{-1} \circ x_+^{-1} - x_+^{-2} \circ \ln x_+ = x_+^{-2} - x_+^{-2} \ln x_+$$

and equation (14) follows on using equation (15). □

Corollary 2. *The neutrix products* $\ln x_- \circ x_-^{-s}$, $x_-^{-s} \circ \ln x_-$ *and* $x_-^{-r} \circ x_-^{-s}$ *exist for* $r, s = 1, 2, \dots$ *. In particular,*

$$\begin{aligned}
\ln x_- \circ x_-^{-1} &= x_-^{-1} \ln x_-, \\
x_-^{-1} \circ \ln x_- &= x_-^{-1} \ln x_-, \\
\ln x_- \circ x_-^{-2} &= x_-^{-2} \ln x_- - (c_1 - 1)\delta'(x), \\
x_-^{-2} \circ \ln x_- &= x_-^{-2} \ln x_- - (c_1 - 1)\delta'(x), \\
x_-^{-1} \circ x_-^{-1} &= x_-^{-2} - (c_1 - 1)\delta'(x).
\end{aligned}$$

PROOF: Replacing x by $-x$ in $\ln x_+$, x_+^{-s} and $\delta^{(s)}(x)$ gives us $\ln x_-$, x_-^{-s} and $(-1)^s \delta^{(s)}(x)$ respectively. The results now follow immediately from Theorem 4 and Corollary 1. □

Corollary 3. *The neutrix products* $\ln |x| \circ x^{-s}$, $x^{-s} \circ \ln |x|$ *and* $x^{-r} \circ x^{-s}$ *exist and*

$$\begin{aligned}
\ln |x| \circ x^{-s} &= x^{-s} \ln |x| = x^{-s} \circ \ln |x|, \\
x^{-r} \circ x^{-s} &= x^{-r-s},
\end{aligned}$$

for $r, s = 1, 2, \dots$ *.*

PROOF: Since the products $\ln x_+ \circ x_+^{-s}$, $x_+^{-s} \circ \ln x_+$ and $x_+^{-r} \circ x_+^{-s}$ are of the form

$$\begin{aligned}
\ln x_+ \circ x_+^{-s} &= x_+^{-s} \ln x_+ + M_s \delta^{(s-1)}(x), \\
x_+^{-s} \circ \ln x_+ &= x_+^{-s} \ln x_+ + M_s' \delta^{(s-1)}(x), \\
x_+^{-r} \circ x_+^{-s} &= x_+^{-r-s} + M_{rs} \delta^{(r+s-1)}(x),
\end{aligned}$$

for some constants M_s, M_s' and M_{rs}, it follows that we then have

$$\begin{aligned}
\ln x_- \circ x_-^{-s} &= x_-^{-s} \ln x_- - (-1)^s M_s \delta^{(s-1)}(x), \\
x_-^{-s} \circ \ln x_- &= x_-^{-s} \ln x_- - (-1)^s M_s' \delta^{(s-1)}(x), \\
x_-^{-r} \circ x_-^{-s} &= x_-^{-r-s} - (-1)^{r+s} M_{rs} \delta^{(r+s-1)}(x).
\end{aligned}$$

Noting that the neutrix product is clearly distributive with respect to addition and that

$$x^{-s} \ln |x| = x_+^{-s} \ln x_+ + (-1)^s x_-^{-s} \ln x_-, \quad x^{-s} = x_+^{-s} + (-1)^s x_-^{-s},$$

for $s = 1, 2, \dots$, the results follow from these equations and Theorems 2 and 3.

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(B. Fisher, A. Kiliçman and J.C. Ault)
Department of Mathematics and Computer Science, Leicester University, Leicester, LE1 7RH, England

(B.P. Damyanov)
Institute of Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, 1784 Sofia, Bulgaria

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