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Abelian complex structures on 6-dimensional compact nilmanifolds

Luis A. Cordero, Marisa Fernández, Luis Ugarte

Dedicated to Professor Oldřich Kowalski on the occasion of his 65th birthday

Abstract. We classify the 6-dimensional compact nilmanifolds that admit abelian complex structures, and for any such complex structure J we describe the space of symplectic forms which are compatible with J.

Keywords: nilpotent Lie algebras, abelian complex structures, symplectic forms

Classification: 17B30, 53C15, 53C30, 53D05

1. Introduction

Let G be a simply-connected, connected and rational s-step nilpotent Lie group of real dimension $2n$. A compact nilmanifold $\Gamma\backslash G$, Γ being a lattice in G of maximal rank [10], has an *abelian complex structure* if $\Gamma\backslash G$ is a complex manifold whose complex structure is defined from a left invariant complex structure J on G by passing to the quotient $\Gamma\backslash G$ and J satisfies the condition:

$$[JX,Y] = -[X,JY], \tag{1}$$

for any $X, Y \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G. Abelian complex structures have been introduced and studied by Barberis-Dotti-Miatello in the context of Clifford structures [1]. On the other hand, if G is a complex Lie group with complex structure J, then $[JX,Y] = [X,JY]$ for any $X, Y \in \mathfrak{g}$, and $\Gamma\backslash G$ is a complex parallelizable manifold in the sense of Wang [15].

According to the Hodge Decomposition Theorem, on any compact Kähler manifold N, the Dolbeault cohomology groups $H^{p,q}_{\bar\partial}(N)$ and $H^{q,p}_{\bar\partial}(N)$ are isomorphic by conjugation and $H^k(N) \cong \oplus_{p+q=k} H^{p,q}_{\bar\partial}(N)$, where $H^k(N)$ denotes the de Rham cohomology group of order k. Therefore, if ω is the Kähler form on N, the Strong Lefschetz Theorem implies that the cup product

$$[\omega]^{n-1}\colon H^{1,0}_{\bar\partial}(N) \longrightarrow H^{n,n-1}_{\bar\partial}(N)$$

is an isomorphism, n being the complex dimension of N.

Recently, the authors of this paper have studied in [7] to what extent this property holds for compact nilmanifolds $\Gamma\backslash G$ endowed with an abelian complex structure J and a compatible symplectic form Ω, that is, Ω being of type $(1,1)$ with respect to J. Concretely, in [7] it is proved that such nilmanifolds $\Gamma\backslash G$ are characterized by the fact that they all have *Lefschetz complex type* $(1,0)$, that is, the cup product

$$[\Omega]^{n-1}\colon H^{1,0}_{\bar\partial}(\Gamma\backslash G) \longrightarrow H^{n,n-1}_{\bar\partial}(\Gamma\backslash G)$$

is an *injective homomorphism*. Therefore, using Serre duality, there is an injective homomorphism $H^{n-1,n}_{\bar\partial}(\Gamma\backslash G) \hookrightarrow H^{n,n-1}_{\bar\partial}(\Gamma\backslash G)$.

The goal of this paper is to classify the compact nilmanifolds $\Gamma\backslash G$ of real dimension 6 having Lefschetz complex type $(1,0)$. To this end, firstly we distinguish in Propositions 3.3 and 3.4 the 6-dimensional compact nilmanifolds that admit abelian complex structures. Section 4 is devoted to give the classification of 6-dimensional compact nilmanifolds with abelian complex structures that admit compatible symplectic forms. This classification problem is more subtle because the existence of such symplectic forms depends strongly on the abelian complex structure given on $\Gamma\backslash G$. In Theorems 4.1 and 4.2 we describe, for any abelian complex structure J, the space of symplectic forms on $\Gamma\backslash G$ that are compatible with J (see also Table 2 at the end of Section 4).

2. Abelian complex structures

Let G be a simply-connected, connected and rational s-step nilpotent Lie group of real dimension $2n$, and denote by $\mathfrak{g}$ the Lie algebra of G. Suppose that G carries a left invariant complex structure J, that is, J is identified with a linear mapping $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying that $J^2 = -\mathrm{id}$ and

$$[JX, JY] = [X,Y] + J[JX,Y] + J[X,JY], \tag{2}$$

for all $X, Y \in \mathfrak{g}$.

In general, given an arbitrary nilpotent Lie algebra $\mathfrak{g}$, a linear mapping $J\colon \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying $J^2 = -\mathrm{id}$ and (2) will be called a *complex structure* on $\mathfrak{g}$. If in addition J satisfies the condition (1) then we shall say that J is an *abelian complex structure* on $\mathfrak{g}$. Notice that (1) implies (2).

Recall that if $\mathfrak{g}$ is s-step nilpotent then the ascending central series $\{\mathfrak{g}_l; l \geq 0\}$ of $\mathfrak{g}$ increases strictly until the term $\mathfrak{g}_s = \mathfrak{g}$, that is, $\mathfrak{g}_0 = \{0\} \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_s = \mathfrak{g}$, where the terms $\mathfrak{g}_l$ are defined inductively by

$$\mathfrak{g}_l = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{g}_{l-1}\}, \quad l \geq 1.$$

Suppose that $\mathfrak{g}$ has a complex structure J. Each $\mathfrak{g}_l$ is an ideal of $\mathfrak{g}$ but, in general, $\mathfrak{g}_l$ is not a complex subspace of $\mathfrak{g}$ because $J(\mathfrak{g}_l) \not\subset \mathfrak{g}_l$. However, there is an *ascending series* $\{\mathfrak{a}_l(J); l \geq 0\}$ *associated to* J whose terms are complex subspaces of $\mathfrak{g}$. It is defined inductively by

$$\mathfrak{a}_0(J) = 0, \quad \mathfrak{a}_l(J) = \{X \in \mathfrak{g} \mid [X, \mathfrak{g}] \subseteq \mathfrak{a}_{l-1}(J) \text{ and } [JX, \mathfrak{g}] \subseteq \mathfrak{a}_{l-1}(J)\}, \quad l \geq 1.$$

A *nilpotent complex structure* on $\mathfrak{g}$ is a complex structure J for which $\mathfrak{a}_t(J) = \mathfrak{g}$ for some integer $t > 0$.

Lemma 2.1. *Any abelian complex structure J on $\mathfrak{g}$ is nilpotent.*

PROOF: Condition (1) for J implies that the ascending central series $\{\mathfrak{g}_l; l \geq 0\}$ of $\mathfrak{g}$ and the ascending series $\{\mathfrak{a}_l(J); l \geq 0\}$ associated to J are the same. Thus, $\mathfrak{a}_s(J) = \mathfrak{g}_s = \mathfrak{g}$ if the Lie algebra $\mathfrak{g}$ is nilpotent in the step s, i.e.J is nilpotent. □

Next, let us suppose that $\Gamma\backslash G$ is a compact nilmanifold, Γ being a discrete subgroup of G such that the quotient space $\Gamma\backslash G$ is compact; such a subgroup Γ always exists if G is rational [10].

Definition 2.2 ([5])**.** We say that a compact nilmanifold $\Gamma\backslash G$ has a *nilpotent* (resp. *abelian*) complex structure provided that $\Gamma\backslash G$ is a complex manifold whose complex structure is induced by a *nilpotent* (resp.*abelian*) complex structure on the Lie algebra $\mathfrak{g}$ of G by passing to the quotient.

Nilpotent complex structures are characterized as follows:

Proposition 2.3 ([5])**.** *Let $M = \Gamma\backslash G$ be a compact nilmanifold of dimension $2n$. Then, M has a nilpotent complex structure J if and only if there exists a basis $\{\omega_1, \dots, \omega_n, \bar\omega_1, \dots, \bar\omega_n\}$ of left invariant complex 1-forms on G satisfying*

$$d\omega_i = \sum_{j<k\leq i} A_{ijk}\, \omega_j \wedge \omega_k + \sum_{j,k\leq i} B_{ijk}\, \omega_j \wedge \bar\omega_k \quad (1 \leq i \leq n).$$

It is easy to see that condition (1) is equivalent to say that $[Z, W] = 0$ for all left invariant (complex) vector fields Z, W of type $(1, 0)$ on G. Thus, from Lemma 2.1 and Proposition 2.3 we conclude:

Corollary 2.4. *Let $M = \Gamma\backslash G$ be a compact nilmanifold with a nilpotent complex structure J. Then, J is abelian if and only if there exists a basis $\{\omega_1, \dots, \omega_n, \bar\omega_1, \dots, \bar\omega_n\}$ of left invariant complex 1-forms on G such that*

$$d\omega_i = \sum_{j,k\leq i} B_{ijk}\, \omega_j \wedge \bar\omega_k \quad (1 \leq i \leq n).$$

To finish this section, we recall some properties about *Lefschetz complex conditions* for complex manifolds (see [7] for details).

Let $M = \Gamma\backslash G$ be a compact nilmanifold of real dimension $2n$ endowed with a nilpotent complex structure J, and denote by $\mathfrak{X}(M)$ the Lie algebra of vector fields on M. Suppose that M carries a compatible symplectic form Ω, that is,

$$\Omega(JX, JY) = \Omega(X, Y),$$

for all $X, Y \in \mathfrak{X}(M)$ or, equivalently, Ω is of bidegree $(1,1)$ with respect to the bigraduation induced by the complex structure J. Then, the metric g on M given by

$$g(X, Y) = \Omega(X, JY),$$

for $X, Y \in \mathfrak{X}(M)$ is pseudo-Hermitian with respect to J. Hence, g is an indefinite Kähler metric on M. (It is well-known that M does not admit a positive definite Kähler metric, unless it is a torus [3], [2], [8], [14].)

Under these conditions, we say that M has *Lefschetz complex type* $(1,0)$ (resp.$(0,1)$) if the cup product $[\Omega]^{n-1}\colon H^{1,0}_{\bar\partial}(M) \longrightarrow H^{n,n-1}_{\bar\partial}(M)$ (resp.the cup product $[\Omega]^{n-1}\colon H^{0,1}_{\bar\partial}(M) \longrightarrow H^{n-1,n}_{\bar\partial}(M)$) is injective, where $H^{p,q}_{\bar\partial}(M)$ denotes the Dolbeault cohomology group of M of bidegree (p,q).

Compact nilmanifolds of Lefschetz complex type $(1,0)$ or $(0,1)$ are characterized as follows:

Theorem 2.5 [7]. *Let $M = \Gamma\backslash G$ be a compact nilmanifold carrying a nilpotent complex structure J and a compatible symplectic form Ω. Then,*

(i) *M has Lefschetz complex type $(1,0)$ if and only if J is abelian;*
(ii) *M has Lefschetz complex type $(0,1)$ if and only if it is a complex torus.*

3. Six-dimensional compact nilmanifolds with abelian complex structures

In this section we determine the compact nilmanifolds of (real) dimension 6 that admit an abelian complex structure and those that do not. Firstly, we remind the classification, given in [4], of 6-dimensional compact nilmanifolds $\Gamma\backslash G$ that possess nilpotent complex structures. They are the compact nilmanifolds corresponding to the sixteen (nonisomorphic) classes of nilpotent Lie algebras given in the following table:

Table 1

Six-dimensional Lie Algebras admitting Nilpotent Complex Structures

Algebra	Defining bracket relations	$b_1(\mathfrak{g}), b_2(\mathfrak{g})$	$\dim(\mathfrak{g}_l)$
$\mathfrak{h}_1 = \mathfrak{a}^6$	$[\ ,\ ] \equiv 0$	$6, 15$	(6)
$\mathfrak{h}_2$	$[X_1, X_2] = X_5,\ [X_3, X_4] = X_6$	$4, 8$	$(2, 6)$
$\mathfrak{h}_3$	$[X_1, X_2] = [X_3, X_4] = X_6$	$5, 9$	$(2, 6)$
$\mathfrak{h}_4$	$[X_1, X_2] = X_5,\ [X_1, X_3] = [X_2, X_4] = X_6$	$4, 8$	$(2, 6)$
$\mathfrak{h}_5$	$-[X_1, X_3] = [X_2, X_4] = X_5,$ $[X_1, X_4] = [X_2, X_3] = -X_6$	$4, 8$	$(2, 6)$
$\mathfrak{h}_6$	$[X_1, X_2] = X_5,\ [X_1, X_3] = X_6$	$4, 9$	$(3, 6)$
$\mathfrak{h}_7$	$[X_1, X_3] = X_4,\ [X_2, X_3] = X_5,$ $[X_1, X_2] = X_6$	$3, 8$	$(3, 6)$
$\mathfrak{h}_8 = \mathfrak{K} \times \mathfrak{a}^2$	$[X_1, X_2] = X_3$	$5, 11$	$(4, 6)$
$\mathfrak{h}_9$	$[X_1, X_2] = X_3,\ [X_1, X_3] = [X_2, X_4] = X_6$	$4, 7$	$(2, 4, 6)$
$\mathfrak{h}_{10}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_5,$ $[X_1, X_4] = X_6$	$3, 6$	$(2, 4, 6)$
$\mathfrak{h}_{11}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = -[X_2, X_4] = X_5,$ $[X_1, X_4] = X_6$	$3, 6$	$(2, 4, 6)$
$\mathfrak{h}_{12}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_5,$ $[X_2, X_4] = -X_6$	$3, 6$	$(2, 4, 6)$
$\mathfrak{h}_{13}$	$[X_1, X_2] = X_3 + X_4,\ [X_1, X_3] = X_5,$ $[X_2, X_4] = X_6$	$3, 5$	$(2, 4, 6)$
$\mathfrak{h}_{14}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_5,$ $[X_1, X_4] = [X_2, X_3] = X_6$	$3, 5$	$(2, 4, 6)$
$\mathfrak{h}_{15}$	$[X_1, X_2] = -X_4,\ -[X_1, X_3] = [X_2, X_4] = X_5,$ $[X_1, X_4] = [X_2, X_3] = -X_6$	$3, 5$	$(2, 4, 6)$
$\mathfrak{h}_{16}$	$[X_1, X_2] = X_3,\ [X_1, X_3] = X_5,$ $[X_2, X_3] = X_6$	$3, 5$	$(3, 4, 6)$

Remark 3.1. This table was obtained using the classification of nilpotent Lie algebras of dimension 6 ([11], [9]). There it is proved that there exist 34 non-isomorphic nilpotent Lie algebras. Recently, Salamon has proved in [13] that, in addition to the 16 Lie algebras in the table above, there exist two more 6-dimensional Lie algebras admitting non-nilpotent complex structures.

In Table 1 by $\mathfrak{a}^k$ we mean an Abelian Lie algebra of dimension k. The Lie algebra $\mathfrak{K}$ that appears in the eighth row is the 4-dimensional nilpotent Lie algebra underlying the well-known Kodaira-Thurston manifold, and any complex structure on $\mathfrak{K}$ is abelian ([4], [13]).

Each Lie algebra $\mathfrak{h}_r$ in the table above has a basis $\{X_1, \dots, X_6\}$ such that the nonzero Lie brackets $[X_i, X_j]$ are given in the second column. Taking into account that $d\gamma(X,Y) = -\gamma([X,Y])$, for all $\gamma \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$, the Lie algebra $\mathfrak{h}_r$ can be defined alternatively by $(d\gamma_1, \dots, d\gamma_6)$, where $\{\gamma_i\}$ is the basis of $\mathfrak{g}^*$ dual to $\{X_i\}$.

In the third column of Table 1 we indicate for $k = 1, 2$ the dimension of the cohomology space $H^k(\mathfrak{h}_r) = Z^k(\mathfrak{h}_r)/d(\Lambda^{k-1}(\mathfrak{h}_r))$, where $Z^k(\mathfrak{h}_r) = \ker\{d : \Lambda^k(\mathfrak{h}_r) \longrightarrow \Lambda^{k+1}(\mathfrak{h}_r)\}$. By Nomizu's theorem [12], this dimension equals the kth Betti number b_k of any compact nilmanifold (if it exists) with underlying Lie algebra $\mathfrak{h}_r$. Therefore, $b_{6-k} = b_k$ and $b_3 = 2(1 - b_1 + b_2)$, so it is sufficient to compute b_1 and b_2.

Let $\mathfrak{g}$ be a 6-dimensional nilpotent Lie algebra endowed with an abelian complex structure J. In the following result we show that, up to a complex transformation, there are two types of such structures J on $\mathfrak{g}$. Here the expression *complex transformation* is understood in the usual sense, that is a mapping $\alpha\colon (\mathfrak{g}, J) \longrightarrow (\mathfrak{g}', J')$ which is an isomorphism of Lie algebras and commutes with the complex structures, i.e. $J' \circ \alpha = \alpha \circ J$. Notice that the latter condition is equivalent to say that $\alpha\colon \mathfrak{g}^{\mathbb{C}} \longrightarrow (\mathfrak{g}')^{\mathbb{C}}$, extended to the complexification, preserves the bigraduation induced by the complex structures J and J'.

Lemma 3.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension* 6, *and let J be an abelian complex structure on $\mathfrak{g}$. Then, the complex structure equations of* $(\mathfrak{g}, J)$ *are* (*up to a complex transformation*) *of the following types:*

$$\text{(I)}: \begin{cases} d\omega_1 = 0, \\ d\omega_2 = \omega_1 \wedge \bar\omega_1, \\ d\omega_3 = B\,\omega_1 \wedge \bar\omega_2 + C\,\omega_2 \wedge \bar\omega_1, \end{cases} \quad \text{or} \quad \text{(II)}: \begin{cases} d\omega_1 = d\omega_2 = 0, \\ d\omega_3 = A\,\omega_1 \wedge \bar\omega_1 + B\,\omega_1 \wedge \bar\omega_2 \\ \qquad\quad + C\,\omega_2 \wedge \bar\omega_1 + D\,\omega_2 \wedge \bar\omega_2, \end{cases}$$

where $A, B, C, D \in \mathbb{C}$. Moreover, $\mathfrak{g}$ is nilpotent in step $s \leq 2$ if and only if J is of type (II).

Proof: Let us consider $n = 3$ in Corollary 2.4. Then, we have that ω_1 must be closed, $d\omega_2 \in \bigwedge^2\langle \omega_1, \bar\omega_1\rangle$, and $d\omega_3 \in \bigwedge^2\langle \omega_1, \bar\omega_1, \omega_2, \bar\omega_2\rangle$. Therefore, there exist $\lambda, A, B, C, D \in \mathbb{C}$ such that $d\omega_2 = \lambda\,\omega_1 \wedge \bar\omega_1$ and $d\omega_3 = A\,\omega_1 \wedge \bar\omega_1 + B\,\omega_1 \wedge \bar\omega_2 + C\,\omega_2 \wedge \bar\omega_1 + D\,\omega_2 \wedge \bar\omega_2$. So, if $\lambda = 0$ then we obtain (II).

Suppose now that $\lambda \neq 0$. Taking the complex transformation $\omega_1' = \omega_1$, $\omega_2' = (1/\lambda)\omega_2$ and $\omega_3' = \omega_3 - (A/\lambda)\omega_2$, we get

$$\begin{cases} d\omega_1' = 0, \\ d\omega_2' = \omega_1' \wedge \bar\omega_1', \\ d\omega_3 = B'\,\omega_1' \wedge \bar\omega_2' + C'\,\omega_2' \wedge \bar\omega_1' + D'\,\omega_2' \wedge \bar\omega_2'. \end{cases}$$

Thus, we can consider that $\omega_1 \wedge \bar\omega_1$ does not appear in the expression of $d\omega_3$. Moreover, the Jacobi identity for the Lie algebra structure implies that $d(d\omega_3) =$

$D'\,\omega'_1 \wedge \bar\omega'_1 \wedge \bar\omega'_2 - D'\,\omega'_1 \wedge \omega'_2 \wedge \bar\omega'_1 = 0$, i.e.$D' = 0$. So, if $\lambda \neq 0$ then we obtain complex structure equations of the form (I).

Finally, if $\mathfrak{g}$ has a complex structure of type (II) then the complexification $\mathfrak{g}_1^{\mathbb{C}}$ of the center $\mathfrak{g}_1$ of $\mathfrak{g}$ contains at least the elements Z_3 and $\bar Z_3$, where $\{Z_i\}$ denotes the dual basis of $\{\omega_i\}$, and thus $\mathfrak{g}_2^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$, i.e.$\mathfrak{g}$ is nilpotent in step $s \leq 2$. To prove the converse, let us suppose that J is not of type (II), that is, the structure equations have the form (I) with B or C nonzero; then $[Z_1, \bar Z_1] = -Z_2 + \bar Z_2$ and $[Z_2, \bar Z_1] = -\overline{[Z_1, \bar Z_2]} = -C\,Z_3 + \bar B\,\bar Z_3$ is nonzero, which implies that $\mathfrak{g}_1^{\mathbb{C}} = \langle Z_3, \bar Z_3\rangle$, $\mathfrak{g}_2^{\mathbb{C}} = \langle Z_2, \bar Z_2, Z_3, \bar Z_3\rangle$ and $\mathfrak{g}_3^{\mathbb{C}} = \mathfrak{g}^{\mathbb{C}}$, i.e. the Lie algebra $\mathfrak{g}$ is 3-step nilpotent. $\square$

From now on, $\mathfrak{h}_r$ denotes the 6-dimensional nilpotent Lie algebra given in Table 1, for $1 \leq r \leq 16$.

Proposition 3.3. *The Lie algebras $\mathfrak{h}_6$, $\mathfrak{h}_7$, $\mathfrak{h}_{16}$ and $\mathfrak{h}_r$, for $10 \leq r \leq 14$, do not admit abelian complex structures.*

Proof: We know that a necessary condition for a Lie algebra $\mathfrak{g}$ to have an abelian complex structure is that the dimension of its center $\mathfrak{g}_1$ be even. In Table 1 we see that the dimension of the center of $\mathfrak{h}_6$, $\mathfrak{h}_7$ and $\mathfrak{h}_{16}$ is equal to 3, which proves the proposition for these Lie algebras.

Let us prove next that $\mathfrak{h}_r$ does not admit abelian complex structure for $10 \leq r \leq 14$. From Table 1, it is sufficient to prove that any 3-step nilpotent Lie algebra $\mathfrak{g}$ with $b_1(\mathfrak{g}) = 3$ admitting an abelian complex structure must be isomorphic to $\mathfrak{h}_{15}$.

From Lemma 3.2, if $\mathfrak{g}$ is 3-step nilpotent then the complex structure equations of $(\mathfrak{g}, J)$ are of the form (I), with complex coefficients $B = b_1 + ib_2$ and $C = c_1 + ic_2$ nonzero simultaneously. Then, taking the basis $\{\gamma_1, \dots, \gamma_6\}$ for $\mathfrak{g}^*$ given by $\gamma_1 - i\gamma_2 = \omega_1$, $2\gamma_3 + i(2\gamma_4) = \omega_2$ and $2\gamma_5 + i(2\gamma_6) = \omega_3$, we get from (I) the following (real) structure equations for $\mathfrak{g}$:

$$(3)\qquad \begin{cases} d\gamma_1 = d\gamma_2 = d\gamma_3 = 0, \\ d\gamma_4 = \gamma_1 \wedge \gamma_2, \\ d\gamma_5 = (b_1 - c_1)(\gamma_1 \wedge \gamma_3 - \gamma_2 \wedge \gamma_4) + (b_2 + c_2)(\gamma_1 \wedge \gamma_4 + \gamma_2 \wedge \gamma_3), \\ d\gamma_6 = (b_2 - c_2)(\gamma_1 \wedge \gamma_3 - \gamma_2 \wedge \gamma_4) - (b_1 + c_1)(\gamma_1 \wedge \gamma_4 + \gamma_2 \wedge \gamma_3). \end{cases}$$

Now, there are three possible cases:

(i) If $B \neq 0$ and $C = 0$, then multiplying ω_3 by $1/B$ we can suppose that $B = b_1 = 1$, and changing the sign of γ_6 we have from (3) that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{15}$.

(ii) If $B = 0$ and $C \neq 0$, then we can multiply ω_3 by $-1/C$ and suppose that $C = c_1 = 1$, and from (3) we see that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_{15}$.

(iii) Finally, if $B, C \neq 0$ then multiplying ω_3 by $1/B$ we can suppose that $B = b_1 = 1$. Moreover, if $b_1(\mathfrak{g}) = 3$ then we must have $|C| \neq 1$, because otherwise $H^1(\mathfrak{g}^{\mathbb{C}})$ would be generated by ω_1, $\bar\omega_1$, $\omega_2+\bar\omega_2$ and $\bar C\omega_3+\bar\omega_3$. Now, let us consider the transformation

$$(4)\qquad \gamma'_j=\gamma_j\ (1\le j\le 4),\ \gamma'_5=-\frac{(1+c_1)\gamma_5+c_2\gamma_6}{|C|^2-1},\ \gamma'_6=\frac{c_2\gamma_5+(1-c_1)\gamma_6}{|C|^2-1}\,.$$

From (3) it follows that $\mathfrak{g}$ is again isomorphic to $\mathfrak{h}_{15}$.

On the other hand, if $|C|=1$ then $b_1(\mathfrak{g})=4$ and a straightforward calculation using (I) shows that $b_2(\mathfrak{g})=7$. Therefore, it follows from Table 1 that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_9$.

□

Proposition 3.4. *There are abelian complex structures on the Lie algebras $\mathfrak{h}_r$, for $1\le r\le 5$ and $r=8,9,15$.*

PROOF: We give explicitly an abelian complex structure J_r on each $\mathfrak{h}_r$. Let $\{X_1,\dots,X_6\}$ be the basis of $\mathfrak{h}_r$ given in Table 1. We define the almost complex structure J_r for $r=1,2,3,4,8$ and 9 by

$$J_r(X_1)=X_2,\qquad J_r(X_3)=X_4,\qquad J_r(X_5)=X_6;$$

for $r=5$ we define J_5 by

$$J_5(X_1)=-X_2,\qquad J_5(X_3)=X_4,\qquad J_5(X_5)=X_6;$$

and for $r=15$ we define J_{15} by

$$J_{15}(X_1)=X_2,\qquad J_{15}(X_3)=-X_4,\qquad J_{15}(X_5)=X_6.$$

It is straightforward to check that J_r satisfies $[J_r(X_i),X_j]=-[X_i,J_r(X_j)]$, for $1\le i,j\le 6$, and so condition (1) holds for J_r ($1\le r\le 5$ and $r=8,9,15$), i.e.it is abelian. □

Let us denote by H_r the simply-connected nilpotent Lie group whose Lie algebra is $\mathfrak{h}_r$, $1\le r\le 16$. Since in the basis given in Table 1 for each Lie algebra $\mathfrak{h}_r$ all the structure constants are rational numbers, Mal'cev theorem [10] implies that there is a compact nilmanifold $M_r=\Gamma_r\backslash H_r$ corresponding to the Lie algebra $\mathfrak{h}_r$, $1\le r\le 16$. From Propositions 3.3 and 3.4 we get the following

Corollary 3.5. *Let $M=\Gamma\backslash G$ be a six-dimensional compact nilmanifold and $\mathfrak{g}$ the Lie algebra of G. Then, $M=\Gamma\backslash G$ has an abelian complex structure if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_r$, for some $1\le r\le 5$ or $r=8,9,15$.*

4. Six-dimensional compact nilmanifolds of Lefschetz complex type $(1,0)$

In this section we classify 6-dimensional compact nilmanifolds with abelian complex structures admitting compatible symplectic forms. From Theorem 2.5, such a compact nilmanifold has Lefschetz complex type $(1,0)$.

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension 6 admitting abelian complex structures. We want to know if there exist a *compatible pair* (J,Ω), that is, a symplectic form Ω which is compatible with an abelian complex structure J on $\mathfrak{g}$. The problem of classifying these structures is more subtle, because the existence of such a form Ω depends on the given structure J. For example, in order to prove that there is no compatible pair (J,Ω) on a Lie algebra $\mathfrak{g}$ (as it happens in Theorem 4.1 below), we must take into account the whole space of abelian complex structures J on $\mathfrak{g}$ and prove the nonexistence of Ω's for any such J.

On the other hand, notice that this classification problem is up to a complex transformation in the following sense: if $\alpha\colon(\mathfrak{g},J)\longrightarrow(\mathfrak{g}',J')$ is a complex transformation then, Ω' is a symplectic form on $\mathfrak{g}'$ compatible with J' if and only if its pullback $\Omega=\alpha^*(\Omega')$ is a symplectic form on $\mathfrak{g}$ compatible with J.

Moreover, as we shall see below in Theorem 4.2, it turns out that there are Lie algebras having abelian complex structures J_1 and J_2, such that there exist compatible symplectic forms for J_1 but there is no symplectic form compatible with J_2.

Let $\mathfrak{g}$ be a nilpotent Lie algebra having an abelian complex structure J. We introduce the following notation:

$$\mathcal{S}_{\mathrm{c}}(\mathfrak{g},J)=\{\,\text{symplectic forms } \Omega \text{ on } \mathfrak{g} \text{ compatible with } J\,\}.$$

When J is fixed on $\mathfrak{g}$, the complex structure J induces a bigraduation on the spaces $\Lambda^k_{\mathbb{C}}(\mathfrak{g}^*)=\oplus_{p+q=k}\Lambda^{p,q}(\mathfrak{g}^*)$, where $\Lambda^k_{\mathbb{C}}(\mathfrak{g}^*)$ denotes the complexification of $\Lambda^k(\mathfrak{g}^*)$. Let us also denote by $d\colon\Lambda^k_{\mathbb{C}}(\mathfrak{g}^*)\longrightarrow\Lambda^{k+1}_{\mathbb{C}}(\mathfrak{g}^*)$ the extension to $\Lambda^k_{\mathbb{C}}(\mathfrak{g}^*)$ of the Chevalley-Eilenberg differential d of the Lie algebra. Then, $\mathcal{S}_{\mathrm{c}}(\mathfrak{g},J)$ can be identified with

$$\mathcal{S}_{\mathrm{c}}(\mathfrak{g},J)=Z^{1,1}(\mathfrak{g},J)\cap\mathcal{S}(\mathfrak{g}) \tag{5}$$

where $Z^{1,1}(\mathfrak{g},J)=\ker\{d|_{\Lambda^{1,1}(\mathfrak{g}^*)}\colon\Lambda^{1,1}(\mathfrak{g}^*)\longrightarrow\Lambda^3_{\mathbb{C}}(\mathfrak{g}^*)\}$ is the vector space of all closed (1,1)-forms on $\mathfrak{g}^{\mathbb{C}}$ and $\mathcal{S}(\mathfrak{g})$ is the set of real 2-forms on $\mathfrak{g}$ which are non-degenerate.

Theorem 4.1. *The Lie algebras $\mathfrak{h}_2,\mathfrak{h}_3$ and $\mathfrak{h}_4$ admit no symplectic form compatible with any abelian complex structure. Therefore, $\mathcal{S}_{\mathrm{c}}(\mathfrak{h}_r,J)=\emptyset$ for any abelian complex structure J on $\mathfrak{h}_r$, $2\leq r\leq 4$.*

Proof: As we see in Table 1, the Lie algebras $\mathfrak{h}_2,\mathfrak{h}_3$ and $\mathfrak{h}_4$ are nilpotent in step 2, so from Lemma 3.2 an abelian complex structure on any of these Lie algebras would be defined by equations of the form (II). Thus, it is sufficient to prove that any Lie algebra $\mathfrak{g}$ having a compatible pair (J,Ω), where J is an abelian complex structure defined by (II) and Ω a symplectic form compatible with J, is isomorphic to the Lie algebras $\mathfrak{h}_5$ or $\mathfrak{h}_8$.

Let $A=a_1+ia_2$, $B=b_1+ib_2$, $C=c_1+ic_2$ and $D=d_1+id_2$ be the complex coefficients in the equations (II). Thus, taking the basis $\{\gamma_1,\dots,\gamma_6\}$ for $\mathfrak{g}^*$ given by $\gamma_1-i\gamma_2=\omega_1$, $\gamma_3+i\gamma_4=\omega_2$ and $\gamma_5+i\gamma_6=\omega_3$, we get from (II) the following (real) structure equations for $\mathfrak{g}$:

$$
(6)\qquad \begin{cases} d\gamma_1=d\gamma_2=d\gamma_3=d\gamma_4=0,\\ d\gamma_5=-2a_2\,\gamma_1\wedge\gamma_2+2d_2\,\gamma_3\wedge\gamma_4\\ \qquad\quad +(b_1-c_1)(\gamma_1\wedge\gamma_3-\gamma_2\wedge\gamma_4)+(b_2+c_2)(\gamma_1\wedge\gamma_4+\gamma_2\wedge\gamma_3),\\ d\gamma_6=2a_1\,\gamma_1\wedge\gamma_2-2d_1\,\gamma_3\wedge\gamma_4\\ \qquad\quad +(b_2-c_2)(\gamma_1\wedge\gamma_3-\gamma_2\wedge\gamma_4)-(b_1+c_1)(\gamma_1\wedge\gamma_4+\gamma_2\wedge\gamma_3). \end{cases}
$$

Let $\mathfrak{g}$ be a nilpotent Lie algebra admitting a compatible pair (J,Ω) with J defined by (II). In order to prove that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_5$ or $\mathfrak{h}_8$, we divide the proof in several cases depending on the number of coefficients, among A,B,C and D in the equations (II), that vanish:

(i) Only one coefficient is nonzero. There are four possibilities:

- (i.1) If $A\neq 0$ and $B=C=D=0$, then multiplying ω_3 by $1/A$ we can suppose that $A=a_1=1$ and from (6) it follows that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_8$.
- (i.2) If $B\neq 0$ and $A=C=D=0$, then we can multiply ω_3 by $1/B$ and suppose that $B=b_1=1$. Now, changing the sign of γ_6 we have from (6) that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_5$.
- (i.3) If $C\neq 0$ and $A=B=D=0$, then interchanging ω_1 with ω_2 we lie in the previous case.
- (i.4) Finally, if $D\neq 0$ and $A=B=C=0$, then this case is reduced to (i.1) by interchanging ω_1 with ω_2.

(ii) Two coefficients are nonzero. We have the following six possibilities:

- (ii.1) If $A,B\neq 0$ and $C=D=0$, then taking the complex transformation $\omega_1'=\omega_1$, $\omega_2'=\bar{A}\,\omega_1+\bar{B}\,\omega_2$ and $\omega_3'=\omega_3$, we reduce this case to (i.2).
- (ii.2) If $A,C\neq 0$ and $B=D=0$, then taking the transformation $\omega_2'=A\,\omega_1+C\,\omega_2$, this case is reduced to (i.3), and therefore to (i.2).
- (ii.3) Suppose $A,D\neq 0$ and $B=C=0$, and let us see that any abelian complex structure J with these coefficients has no compatible symplectic form. In fact, an easy computation using (II) shows that the class $[\Omega]$ of any closed form Ω of type (1,1) with respect to such a J must be the class of a linear combination of the forms $\omega_1\wedge\bar{\omega}_1$, $\omega_1\wedge\bar{\omega}_2$, $\omega_2\wedge\bar{\omega}_1$ and $\omega_2\wedge\bar{\omega}_2$; that is,

$$
(7)\qquad [\Omega]=[p\,\omega_1\wedge\bar{\omega}_1+q\,\omega_1\wedge\bar{\omega}_2+r\,\omega_2\wedge\bar{\omega}_1+s\,\omega_2\wedge\bar{\omega}_2],
$$

where p, q, r, s are complex numbers. Therefore, $[\Omega]^3 = 0$, i.e.Ω is degenerate, and we conclude that there do not exist symplectic forms compatible with J.

(ii.4) If $B, C \neq 0$ and $A = D = 0$, then we have the same situation as in the preceding case: the class $[\Omega]$ of a closed form Ω of type (1,1) with respect to any abelian complex structure defined by these coefficients must also satisfy (7) and, therefore, any such Ω is again degenerate, so there are no compatible symplectic forms.

(ii.5) If $B, D \neq 0$ and $A = C = 0$, then interchanging ω_1 with ω_2 we lie in the previous case (ii.2), and thus in (i.2).

(ii.6) If $C, D \neq 0$ and $A = B = 0$, then interchanging ω_1 with ω_2 this case is reduced to (ii.1), and therefore to (i.2).

(iii) Only one coefficient vanishes. There are four possibilities:

(iii.1) Suppose that $A, B, C \neq 0$ and $D = 0$. This case is analogous to (ii.3) because $[\Omega]$ must satisfy (7) and therefore there are no symplectic forms Ω being compatible with an abelian complex structure defined by these coefficients.

(iii.2) If $A, B, D \neq 0$ and $C = 0$, then we get again condition (7) for the class of any closed (1,1)-form Ω. Thus, it is necessarily degenerate.

(iii.3) If $A, C, D \neq 0$ and $B = 0$, then interchanging ω_1 with ω_2 we lie in the preceding case.

(iii.4) Finally, if $B, C, D \neq 0$ and $A = 0$, then interchanging again ω_1 with ω_2 this case is reduced to (iii.1).

(iv) No coefficient vanishes: taking the complex transformation given by $\omega_1 = \omega_1' - \omega_2'$, $\omega_2 = (A/C)\omega_2'$ and $\omega_3 = A\,\omega_3'$, the equations (II) are reduced to

$$\begin{cases} d\omega_1' = d\omega_2' = 0, \\ d\omega_3' = \ \omega_1' \wedge \bar{\omega}_1' + P\,\omega_1' \wedge \bar{\omega}_2' + Q\,\omega_2' \wedge \bar{\omega}_2', \end{cases}$$

where $P = \bar{A}B/A\bar{C} - 1$ and $Q = \bar{A}D/|C|^2 - \bar{A}B/A\bar{C}$. That is, we can always suppose that the coefficient of $\omega_2 \wedge \bar{\omega}_1$ in the expression of $d\omega_3$ in (II) is zero. Therefore, case (iv) has been studied previously in (i), (ii) and (iii). □

Theorem 4.2. *Any abelian complex structure J on $\mathfrak{h}_1$, $\mathfrak{h}_8$ and $\mathfrak{h}_9$ satisfies:*

$$\dim \mathcal{S}_c(\mathfrak{h}_1, J) = 9, \qquad \dim \mathcal{S}_c(\mathfrak{h}_8, J) = 6, \qquad \dim \mathcal{S}_c(\mathfrak{h}_9, J) = 4.$$

For any abelian complex structure J on the Lie algebra $\mathfrak{h}_5$ we have that,

$$\textit{either} \quad \mathcal{S}_c(\mathfrak{h}_5, J) = \emptyset \quad \textit{or} \quad \dim \mathcal{S}_c(\mathfrak{h}_5, J) = 6.$$

Finally, any abelian complex structure J on the Lie algebra $\mathfrak{h}_{15}$ satisfies

$$\textit{either} \quad \mathcal{S}_c(\mathfrak{h}_{15}, J) = \emptyset \quad \textit{or} \quad \dim \mathcal{S}_c(\mathfrak{h}_{15}, J) = 4.$$

Proof: Since $\mathfrak{h}_1$ is an Abelian Lie algebra the result is clear. So, let us consider first the Lie algebras $\mathfrak{h}_5$ and $\mathfrak{h}_8$. These algebras are 2-step nilpotent, so Lemma 3.2 implies that their complex equations for any abelian complex structure J must have the form (II). But in the proof of Theorem 4.1 we have studied all the possibilities for such a J, and one can find that (up to a complex transformation) if J has a compatible symplectic form Ω then J necessarily lies in cases (i.1) or (i.2).

Let us consider a structure J on $\mathfrak{h}_8$ fitting in case (i.1). A straightforward computation shows that the vector space $Z^{1,1}(\mathfrak{h}_8, J)$ given in (5) is generated by $\omega_1 \wedge \bar{\omega}_1$, $\omega_1 \wedge \bar{\omega}_2$, $\omega_1 \wedge \bar{\omega}_3$, $\omega_2 \wedge \bar{\omega}_1$, $\omega_2 \wedge \bar{\omega}_2$ and $\omega_3 \wedge \bar{\omega}_1$. Therefore, any closed (1,1)-form Ω must be a linear combination

$$\Omega = p\,\omega_1 \wedge \bar{\omega}_1 + q\,\omega_1 \wedge \bar{\omega}_2 + r\,\omega_1 \wedge \bar{\omega}_3 + s\,\omega_2 \wedge \bar{\omega}_1 + u\,\omega_2 \wedge \bar{\omega}_2 + v\,\omega_3 \wedge \bar{\omega}_1,$$

where $p, q, r, s, u, v \in \mathbb{C}$. If we impose that Ω be real, i.e.$\Omega = \bar{\Omega}$, then the complex coefficients must satisfy $p = -\bar{p}$, $s = -\bar{q}$, $v = -\bar{r}$ and $u = -\bar{u}$. Finally, since $\Omega^3 = -6u|r|^2\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 \wedge \bar{\omega}_3$, we get that Ω is non-degenerate if and only if $u|r| \neq 0$. Therefore, from (5) we conclude that $\dim \mathcal{S}_{\mathrm{c}}(\mathfrak{h}_8, J) = 6$ for any J in case (i.1).

Analogously, consider a structure J on $\mathfrak{h}_5$ in case (i.2). An easy computation shows that $Z^{1,1}(\mathfrak{h}_5, J) = \langle \omega_1 \wedge \bar{\omega}_1, \omega_1 \wedge \bar{\omega}_2, \omega_2 \wedge \bar{\omega}_1, \omega_2 \wedge \bar{\omega}_2, \omega_2 \wedge \bar{\omega}_3, \omega_3 \wedge \bar{\omega}_2 \rangle$. Thus, any closed (1,1)-form Ω must be a linear combination

$$\Omega = p\,\omega_1 \wedge \bar{\omega}_1 + q\,\omega_1 \wedge \bar{\omega}_2 + r\,\omega_2 \wedge \bar{\omega}_1 + s\,\omega_2 \wedge \bar{\omega}_2 + u\,\omega_2 \wedge \bar{\omega}_3 + v\,\omega_3 \wedge \bar{\omega}_2,$$

where $p, q, r, s, u, v \in \mathbb{C}$. Now, Ω is real if and only if $p = -\bar{p}$, $r = -\bar{q}$, $v = -\bar{u}$ and $s = -\bar{s}$. Moreover, $\Omega^3 = -6p|u|^2\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 \wedge \bar{\omega}_3 \neq 0$ if and only if $p|u| \neq 0$. Thus, it follows from (5) that $\dim \mathcal{S}_{\mathrm{c}}(\mathfrak{h}_5, J) = 6$ for any J in case (i.2).

In order to complete our study for $\mathfrak{h}_8$, we must show that (i.1) is the only case (up to complex isomorphism) in which the underlying Lie algebra is isomorphic to $\mathfrak{h}_8$. Suppose $\mathfrak{g}$ is a 2-step nilpotent Lie algebra having an abelian complex structure J and such that $b_1(\mathfrak{g}) = 5$. Apart from case (i.1), J must correspond (up to a complex transformation) to the cases (ii.3), (ii.4) or (iii.1), with coefficients A, B, C and D satisfying the following relations: in case (ii.3) multiplying ω_3 by $1/A$ we can suppose $A = 1$ and therefore $D = \bar{D}$; in case (ii.4) multiplying ω_3 by $1/B$ we can suppose $B = 1$ and thus $|C| = 1$; finally, taking the transformation $\omega_1' = \omega_1$, $\omega_2' = (C/A)\omega_2$ and $\omega_3' = (1/A)\omega_3$ in case (iii.1) we can suppose that $A = C = 1$ and, therefore, $B = 1$. Now, it is easy to check that under these conditions we always have $b_2(\mathfrak{g}) = 9$, i.e.from Table 1 it follows that $\mathfrak{g}$ is not isomorphic to $\mathfrak{h}_8$ (indeed it is isomorphic to $\mathfrak{h}_3$).

Now, to complete the study of $\mathfrak{h}_5$ it remains to prove that there exists an abelian complex structure J on $\mathfrak{h}_5$ for which $\mathcal{S}_{\mathrm{c}}(\mathfrak{h}_5, J) = \emptyset$. Notice that it is sufficient to see that $\mathfrak{h}_5$ can occur as the underlying Lie algebra of a complex structure J in the case (ii.4), because from the proof of Theorem 4.1 we know that there is no symplectic forms compatible with J. In fact, let J be defined by coefficients $A = D = 0$, $B = 1$ and $C = c_1 + ic_2$ such that $|C| \neq 1$, which ensures that J lies in case (ii.4). Let $\{\gamma_1, \dots, \gamma_6\}$ be the basis for $\mathfrak{g}^*$ satisfying (6). Since $|C| \neq 1$ we can consider the transformation given by (4), and it follows from (6) that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_5$.

Let us consider next the nilpotent Lie algebras $\mathfrak{h}_9$ and $\mathfrak{h}_{15}$. From Table 1 we see that they are 3-step nilpotent, so Lemma 3.2 implies that any abelian complex structure J must be of type (I) with coefficients B and C non simultaneously zero. Notice that in the proof of Proposition 3.3 we have already considered the possible cases (i), (ii) and (iii) for any such J. Let us study now the existence of compatible symplectic forms in each one of these cases.

For any complex structure J in case (i) we have that $B \neq 0$ (we can suppose $B = 1$) and $C = 0$, and the underlying Lie algebra is isomorphic to $\mathfrak{h}_{15}$. An easy computation shows that $Z^{1,1}(\mathfrak{h}_{15}, J) = \langle \omega_1 \wedge \bar{\omega}_1,\ \omega_1 \wedge \bar{\omega}_2,\ \omega_2 \wedge \bar{\omega}_1,\ \omega_1 \wedge \bar{\omega}_3 + \omega_2 \wedge \bar{\omega}_2 + \omega_3 \wedge \bar{\omega}_1 \rangle$, so any closed (1,1)-form Ω is a linear combination

$$\Omega = p\,\omega_1 \wedge \bar{\omega}_1 + q\,\omega_1 \wedge \bar{\omega}_2 + r\,\omega_2 \wedge \bar{\omega}_1 + s(\omega_1 \wedge \bar{\omega}_3 + \omega_2 \wedge \bar{\omega}_2 + \omega_3 \wedge \bar{\omega}_1), \tag{8}$$

where $p, q, r, s \in \mathbb{C}$. Since Ω is real if and only if $p = -\bar{p}$, $r = -\bar{q}$ and $s = -\bar{s}$, and $\Omega^3 = 6s^3\,\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 \wedge \bar{\omega}_3 \neq 0$ if and only if $s \neq 0$, we conclude that $\dim \mathcal{S}_{\mathrm{c}}(\mathfrak{h}_{15}, J) = 4$.

A complex structure J in case (ii) has $B = 0$ and $C \neq 0$, and the underlying Lie algebra is again $\mathfrak{h}_{15}$. A direct calculation shows that $Z^{1,1}(\mathfrak{h}_{15}, J) = \langle \omega_1 \wedge \bar{\omega}_1,\ \omega_1 \wedge \bar{\omega}_2,\ \omega_2 \wedge \bar{\omega}_1,\ \omega_1 \wedge \bar{\omega}_3,\ \omega_3 \wedge \bar{\omega}_1 \rangle$, and therefore any closed (1,1)-form Ω is expressed as

$$\Omega = p\,\omega_1 \wedge \bar{\omega}_1 + q\,\omega_1 \wedge \bar{\omega}_2 + r\,\omega_2 \wedge \bar{\omega}_1 + s\,\omega_1 \wedge \bar{\omega}_3 + u\,\omega_3 \wedge \bar{\omega}_1,$$

where $p, q, r, s, u \in \mathbb{C}$. but any such Ω is degenerate, so $\mathcal{S}_{\mathrm{c}}(\mathfrak{h}_{15}, J) = \emptyset$.

Finally, consider an abelian complex structure J fitting in case (iii), that is, $B, C \neq 0$; moreover, we can suppose $B = 1$. The underlying Lie algebra is $\mathfrak{h}_9$ if $|C| = 1$, and $\mathfrak{h}_{15}$ if $|C| \neq 1$. In any case, it is easy to see that a closed form Ω of type (1,1) with respect to J must be a linear combination as in (8). Therefore, $\dim \mathcal{S}_{\mathrm{c}}(\mathfrak{h}_9, J) = \dim \mathcal{S}_{\mathrm{c}}(\mathfrak{h}_{15}, J) = 4$. Since any abelian complex structure J on $\mathfrak{h}_9$ must have coefficients $B, C \neq 0$ with $|C| = 1$, we conclude that $\dim \mathcal{S}_{\mathrm{c}}(\mathfrak{h}_9, J) = 4$ for any J. □

Remark 4.3. It is well-known that the Lie algebra $\mathfrak{h}_3$ has no symplectic forms ([6]). Also it is known that the Lie algebras $\mathfrak{h}_2$ and $\mathfrak{h}_4$ have symplectic forms (see for example [13]), thus using Theorem 4.2 we conclude that $\mathfrak{h}_2$ and $\mathfrak{h}_4$ have abelian complex structures J and symplectic forms Ω, but the pairs (J, Ω) are never compatible. Finally, $\mathfrak{h}_5$ is the Lie algebra underlying the Iwasawa manifold ([4]), that is, $\mathfrak{h}_5$ has a complex structure J such that $(\mathfrak{h}_5, J)$ becomes a complex Lie algebra, although there is no symplectic form compatible with J (in fact, this property is shared by all the complex Lie algebras [7]).

From Nomizu's theorem [12] and Theorems 4.1 and 4.2, it follows the following result for nilmanifolds.

Corollary 4.4. *Let $M = \Gamma\backslash G$ be a compact nilmanifold of dimension 6 and denote by $\mathfrak{g}$ the Lie algebra of G. Then, M does not admit symplectic forms (homogeneous or otherwise) compatible with any abelian complex structure if and only if $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_2, \mathfrak{h}_3$ or $\mathfrak{h}_4$.*

Table 2

Six-dimensional Lie Algebras admitting Abelian Complex Structures

Algebra	Defining bracket relations	$b_1(\mathfrak{g}), b_2(\mathfrak{g})$	$\dim(\mathfrak{g}_l)$	$\dim \mathcal{S}_{\mathrm{c}}(\mathfrak{g}, J)$
$\mathfrak{h}_1 = \mathfrak{a}^6$	$[\ ,\ ] \equiv 0$	$6, 15$	(6)	9
$\mathfrak{h}_2$	$[X_1, X_2] = X_5,\ [X_3, X_4] = X_6$	$4, 8$	$(2, 6)$	–
$\mathfrak{h}_3$	$[X_1, X_2] = [X_3, X_4] = X_6$	$5, 9$	$(2, 6)$	–
$\mathfrak{h}_4$	$[X_1, X_2] = X_5,$ $[X_1, X_3] = [X_2, X_4] = X_6$	$4, 8$	$(2, 6)$	–
$\mathfrak{h}_5$	$-[X_1, X_3] = [X_2, X_4] = X_5,$ $[X_1, X_4] = [X_2, X_3] = -X_6$	$4, 8$	$(2, 6)$	$-, 6$
$\mathfrak{h}_8 = \mathfrak{K} \times \mathfrak{a}^2$	$[X_1, X_2] = X_3$	$5, 11$	$(4, 6)$	6
$\mathfrak{h}_9$	$[X_1, X_2] = X_3,$ $[X_1, X_3] = [X_2, X_4] = X_6$	$4, 7$	$(2, 4, 6)$	4
$\mathfrak{h}_{15}$	$[X_1, X_2] = -X_4,$ $-[X_1, X_3] = [X_2, X_4] = X_5,$ $[X_1, X_4] = [X_2, X_3] = -X_6$	$3, 5$	$(2, 4, 6)$	$-, 4$

In Table 2 we have summarized the results of this paper. The last column in the table must be understood as follows: on the Lie algebras $\mathfrak{h}_2$, $\mathfrak{h}_3$ and $\mathfrak{h}_4$ there is no symplectic form compatible with any abelian complex structure; for $\mathfrak{h}_5$ and $\mathfrak{h}_{15}$ one has that $\mathcal{S}_{\mathrm{c}}(\mathfrak{h}_5, J)$ and $\mathcal{S}_{\mathrm{c}}(\mathfrak{h}_{15}, J)$ may be empty or may have dimension 6 and 4, respectively depending on the abelian complex structure J considered, i.e.there are abelian complex structures on $\mathfrak{h}_5$ (resp.$\mathfrak{h}_{15}$) with no compatible symplectic form, and there are abelian complex structures J on $\mathfrak{h}_5$ (resp.$\mathfrak{h}_{15}$) having compatible symplectic forms with $\dim \mathcal{S}_{\mathrm{c}}(\mathfrak{h}_5, J) = 6$ (resp.$\dim \mathcal{S}_{\mathrm{c}}(\mathfrak{h}_{15}, J) = 4$); finally, on $\mathfrak{h}_1$, $\mathfrak{h}_8$ and $\mathfrak{h}_9$ any abelian complex structure J has compatible symplectic forms.

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