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# A semifilter approach to selection principles

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*Abstract.* In this paper we develop the semifilter approach to the classical Menger and Hurewicz properties and show that the small cardinal $\mathfrak{g}$ is a lower bound of the additivity number of the $\sigma$-ideal generated by Menger subspaces of the Baire space, and under $\mathfrak{u} < \mathfrak{g}$ every subset $X$ of the real line with the property $\mathrm{Split}(\Lambda, \Lambda)$ is Hurewicz, and thus it is consistent with ZFC that the property $\mathrm{Split}(\Lambda, \Lambda)$ is preserved by unions of less than $\mathfrak{b}$ subsets of the real line.

*Keywords:* Menger property, Hurewicz property, property $\mathrm{Split}(\Lambda, \Lambda)$, semifilter, multifunction, small cardinals, additivity number

*Classification:* 03A, 03E17, 03E35, 54D20

## Introduction

In this paper we shall present two directions of applications of semifilters in selection principles on topological spaces. First, we shall consider preservation by unions of the Menger property.

Trying to describe the $\sigma$-compactness in terms of open covers, K. Menger introduced in [15] the following property, called *the Menger property*: a topological space $X$ is said to have this property if for every sequence $(u_n)_{n\in\omega}$ of open covers of $X$ there exists a sequence $(v_n)_{n\in\omega}$ such that each $v_n$ is a finite subfamily of $u_n$ and the collection $\{\bigcup v_n : n \in \omega\}$ is a cover of $X$. The class of Menger topological spaces, i.e. spaces having the Menger property appeared to be much wider than the class of $\sigma$-compact spaces (see [5], [7], [10] and many others), but it has interesting properties itself and poses a number of open questions. One of them, namely the question about the value of additivity of corresponding $\sigma$-ideal, will be discussed in this paper. Let us recall that a collection $\mathcal{I}$ of subsets of a set $X$ is called a *$\sigma$-ideal* if it is closed under taking subsets and countable unions. Therefore, the union $\bigcup \mathcal{J}$ belongs to $\mathcal{I}$ for every countable subfamily $\mathcal{J}$ of $\mathcal{I}$. In light of this property of $\sigma$-ideals it is interesting to find the smallest cardinality $\tau$ such that the union $\bigcup \mathcal{J}$ is not in $\mathcal{I}$ for some $\mathcal{J} \subset \mathcal{I}$ with $|\mathcal{J}| = \tau$. If $\bigcup \mathcal{I} = X$ and $X \notin \mathcal{I}$ such a cardinality obviously exists and we denote it by $\mathrm{add}(\mathcal{I})$. It is easy to prove (see, for example, [10]) that the collection $M(X)$ of subspaces of a topological space $X$ contained in subspaces with the Menger property form a $\sigma$-ideal, so one can ask about the value of $\mathrm{add}(M(X))$. According to [4], for the Baire space $\mathbb{N}^\omega$ this additivity number is situated between cardinals $\mathfrak{b}$ and $\mathrm{cf}(\mathfrak{d})$, where $\mathfrak{b}$ and $\mathfrak{d}$ are well-known bounding and dominating numbers respectively, see [25]. It was also asked in [4] whether $\mathrm{add}(M(X)) = \mathfrak{b}$, see Problem 2.4 there. We shall prove here that another small cardinal, namely $\mathfrak{g}$, is a lower bound of $\mathrm{add}(M(X))$ for each hereditarily Lindelöf topological space $X$. Since there are models of ZFC with $\mathfrak{b} < \mathfrak{g}$ (see [25]), this answers the above mentioned problem in negative[1]. Concerning topological spaces which contain non-Lindelöf subspaces, the straightforward proof of the fact that this additivity equals $\aleph_1$ is left to the reader.

Another direction is devoted to splittability of open covers. Following [8] and [10] we say that a family $u$ of subsets of a set $X$ is

- *a large cover* of $X$, if every $x \in X$ belongs to infinitely many $U \in u$;
- *an $\omega$-cover*, if for every finite subset $K$ of $X$ the family $\{U \in u : K \subset U\}$ is infinite;
- *a $\gamma$-cover*, if for every $x \in X$ the family $\{U \in u : x \notin U\}$ is finite.

From now on we denote by $\Lambda(X)$ (resp. $\Omega(X)$, $\Gamma(X)$) the family of all large (resp. $\omega$-, $\gamma$-) covers of $X$. A topological space $X$ satisfies the selection hypothesis $\mathrm{Split}(\Lambda, \Lambda)$, if for every $u \in \Lambda(X)$ there are $v_1, v_2 \in \Lambda(X)$ such that $v_1 \cap v_2 = \emptyset$ and $v_1 \cup v_2 \subset u$. The class $\mathrm{Split}(\Lambda, \Lambda)$ contains all Hurewicz spaces and all spaces with the Rothberger property, see [17, Corollary 29, Theorem 15]. Recall, that a topological space $X$ has the Hurewicz property, if for every sequence $(u_n)_{n\in\omega}$ of open covers of $X$ there exists a $\gamma$-cover $\{B_n : n \in \omega\}$ of $X$ such that each $B_n$ is $u_n$-bounded, which means that $B_n \subset \bigcup v$ for some finite $v \subset u_n$. Substituting "$\gamma$" for "$\omega$" in the above sentence, we obtain the definition of the property $\bigcup_{\mathrm{fin}}(\Gamma, \Omega)$, which will be referred in this paper as the property of Scheepers. If, additionally, each $v_n$ in the definition of the Menger property contains only one element of $u_n$, we obtain the definition of the Rothberger property.

The following problem is open.

**Problem 1** ([20, Issue 9, Problem 4.1], [24, Problem 6.7])**.** *Is the property* $\mathrm{Split}(\Lambda, \Lambda)$ *preserved by unions of subsets of* $\mathbb{R}$*?*

We shall show that under additional strong set-theoretic assumption $\mathfrak{u} < \mathfrak{g}$ every Lindelöf paracompact topological space $X$ is Hurewicz provided it has the property $\mathrm{Split}(\Lambda, \Lambda)$, which implies that the positive answer to the above problem is consistent. In particular, this implies that under $\mathfrak{u} < \mathfrak{g}$ every Rothberger space is Hurewicz. It is worth to mention here, that under CH there are so called Luzin subsets of the Baire space $\mathbb{N}^\omega$, which have the Rothberger property but fail be Hurewicz, see [10] for details. Therefore the statement "the family of Hurewicz subspaces and the family of subspaces with the property $\mathrm{Split}(\Lambda, \Lambda)$ of the real line coincide" is independent of ZFC.

[1]See Remark 3 for further explanation

The reason why such different results of these two parts are unified in one paper is that both of them are proved with the use of semifilters.

## Semifilters

To begin with, let us recall from [25] the definition of the small cardinal $\mathfrak{g}$. Let $C$ be a countable set. A family $\mathcal{D} \subset [C]^{\aleph_0}$ is said to be *open*, if $X \in \mathcal{D}$ provided $X \subset^* Y$ for some $Y \in \mathcal{D}$ (here and subsequently $X \subset^* Y$ means that the complement $X \setminus Y$ is finite, and $[A]^{\aleph_0}$ ($A^{<\aleph_0}$) denotes the set of all countable infinite (finite) subsets of a set $A$). A family $\mathcal{D}$ is called *groupwise dense*, if for every infinite collection $\Pi$ of finite pairwise disjoint subsets of $C$ there exists an infinite $\mathcal{H} \subset \Pi$ such that $\bigcup \mathcal{H} \in \mathcal{D}$. By definition, $\mathfrak{g}$ equals the smallest cardinality of a collection of groupwise dense families with empty intersection. Given an arbitrary groupwise dense family $\mathcal{D}$, consider the family $\mathcal{F} = \{C \setminus D : D \in \mathcal{D}\} \cup \mathfrak{Fr}(C)$, where $\mathfrak{Fr}(C)$ denotes the Fréchet filter on $C$ consisting of cofinite subsets. From the above it follows that $\mathcal{F}$ satisfies the following conditions:

(1) $G \in \mathcal{F}$ provided $F \subset^* G$ for some $F \in \mathcal{F}$;

(2) every collection $\Pi$ of pairwise disjoint finite subsets of $C$ contains an infinite subset $\mathcal{H}$ such that $C \setminus \bigcup \mathcal{H}$ belongs to $\mathcal{F}$.

Following [6], we call a family $\mathcal{F}$ of infinite subsets of $C$ a *semifilter*, if it satisfies the above mentioned condition (1).

However, another approach to the definition of groupwise dense families is not the purpose of introduction of semifilters. Quite the contrary, semifilters seem to constitute some rather interesting area of Set Theory, see [6]. In particular, they inherited many useful properties of filters, for example the following classical theorem due to Talagrand holds, see [18] or [6].

**Theorem 1.** *Let $\mathcal{F}$ be a semifilter on a countable set $C$. Then $\mathcal{F}$ fails to be meager if and only if it satisfies the above mentioned condition* (2).

(Since every semifilter $\mathcal{F}$ on a countable set $C$ is a subset of the powerset $\mathcal{P}(C)$, which can be identified with the product $\{0,1\}^C$, we can speak about topological properties of semifilters. Since $C$ is countable, $\mathcal{P}(C)$ and $[C]^{\aleph_0}$ are nothing else but homeomorphic copies of the Cantor and Baire space respectively. For example, the base of the topology on $[C]^{\aleph_0}$ consists of subsets of the form $G(s,t) = \{A \in [C]^{\aleph_0} : A \cap s = t\}$, where $s$ and $t$ are finite subset of $C$.)

Theorem 1 implies the following characterization of groupwise dense families: a family $\mathcal{D} \subset [C]^{\aleph_0}$ is groupwise dense if and only if the family $\{C \setminus D : D \in \mathcal{D}\} \cup \mathfrak{Fr}(C)$ is a nonmeager semifilter. Therefore $\mathfrak{g}$ is equal to the smallest cardinality of a collection $\mathsf{F}$ of nonmeager semifilters such that $\bigcap \mathsf{F} = \mathfrak{Fr}(C)$. We shall prove a bit more.

**Observation 1.** *The cardinal* $\mathfrak{g}$ *is equal to the smallest cardinality of a family* $\mathsf{F}$ *of semifilters on a countable set* $C$ *such that* $\bigcap \mathsf{F}$ *is meager.*

Proof: Let $\mathsf{F}$ be a family of semifilters such that $\bigcap \mathsf{F}$ is meager. The only thing to be proved is that $|\mathsf{F}| \geq \mathfrak{g}$. For this aim let us fix a sequence $(I_n)_{n\in\omega}$ of pairwise disjoint finite subsets of $C$ such that each member of $\bigcap \mathsf{F}$ meets all but finitely many $I_n$. Without loss of generality, $\bigcup_{n\in\omega} I_n = C$. For each $\mathcal{F} \in \mathsf{F}$ let us make the notation $\mathcal{G}_{\mathcal{F}} = \{G \subset \omega : \bigcup_{n\in G} I_n \in \mathcal{F}\}$. Now, it suffices to observe that each $\mathcal{G}_{\mathcal{F}}$ is a nonmeager semifilter on $\omega$ and $\bigcap_{\mathcal{F}\in\mathsf{F}} \mathcal{G}_{\mathcal{F}} = \mathfrak{Fr}(\omega)$. □

The family of all semifilters on a set $C$ is evidently closed under taking unions and intersections of arbitrary subfamilies. In addition to these operations there is another unary one. Given any semifilter $\mathcal{F}$, let $\mathcal{F}^\perp = \{G \subset C : \forall F \in \mathcal{F}(F \cap G \neq \emptyset)\}$. (For a filter $\mathcal{F}$ the family $\mathcal{F}^\perp$ is nothing else but $\mathcal{F}^+$ in notations of C. Laflamme, see [14]). It is clear that $\mathcal{F}^\perp$ is a semifilter too. In other words, $\mathcal{F}^\perp = \mathcal{P}(C) \setminus \{C \setminus F : F \in \mathcal{F}\}$. Consequently $(\mathcal{F}^\perp)^\perp = \mathcal{F}$ and $\mathcal{F}^\perp$ is comeager if and only if $\mathcal{F}$ is meager. Let us also observe that $(\bigcap \mathsf{F})^\perp = \bigcup_{\mathcal{F}\in\mathsf{F}} \mathcal{F}^\perp$ for an arbitrary collection of semifilters $\mathsf{F}$. Thus we obtain another characterization of the cardinal $\mathfrak{g}$: it is the smallest size of a family $\mathsf{F}$ of non comeager semifilters on a countable set $C$ such that $\bigcup \mathsf{F}$ is comeager. In what follows we shall simply write $\mathfrak{Fr}$ in place of $\mathfrak{Fr}(\omega)$.

Next, similarly to [3], for every semifilter $\mathcal{F}$ on $\omega$ we shall define a cardinal characteristic $\mathfrak{b}(\mathcal{F})$. Its definition involves a special relation $\leq_{\mathcal{F}}$ on $\mathbb{N}^\omega$:

$$(x_n)_{n\in\omega} \leq_{\mathcal{F}} (y_n)_{n\in\omega} \text{ iff } \{n \in \omega : x_n \leq y_n\} \in \mathcal{F}.$$

Now, $\mathfrak{b}(\mathcal{F})$ stands for the smallest size of unbounded subset of $\mathbb{N}^\omega$ with respect to $\leq_{\mathcal{F}}$. When $\mathcal{F} = \mathfrak{Fr}$, then $\leq_{\mathcal{F}}$ is nothing else but the well-known eventual dominance preorder $\leq^*$. For example, $\mathfrak{b}(\mathfrak{Fr}^\perp) = \mathfrak{d}$ and $\mathfrak{b}(\mathfrak{Fr}) = \mathfrak{b}$. Almost literal repetition of the proof of Theorem 16 from [3] gives us the following

**Proposition 1.** $\mathfrak{b}(\mathcal{F}) \geq \mathfrak{g}$ *for each nonmeager semifilter* $\mathcal{F}$ *on* $\omega$.

Next, in what follows we shall intensively use set-valued maps. By a *set-valued map* $\Phi$ from a set $X$ into a set $Y$ we understand a map from $X$ into $\mathcal{P}(Y)$ and write $\Phi : X \Rightarrow Y$ (here $\mathcal{P}(Y)$ denotes the set of all subsets of $Y$). For a subset $A$ of $X$ we put $\Phi(A) = \bigcup_{x\in A} \Phi(x) \subset Y$. When the sets $X$ and $Y$ are endowed with some topologies, it is interesting to consider set-valued maps with certain topological properties. The set-valued map $\Phi$ between topological spaces $X$ and $Y$ is said to be

- *compact-valued*, if $\Phi(x)$ is compact for every $x \in X$;
- *upper semicontinuous*, if for every open subset $V$ of $Y$ the set $\Phi^{-1}_{\subset}(V) = \{x \in X : \Phi(x) \subset V\}$ is open in $X$.

**Lemma 1.** *Let $\Phi : X \Rightarrow Y$ be a compact-valued upper semicontinuous map between topological spaces $X$ and $Y$ such that $\Phi(X) = Y$. Then $Y$ is Menger (Hurewicz) provided so is $X$.*

PROOF: Let us fix an arbitrary sequence $(w_n)_{n\in\omega}$ of open covers of $Y$. For every $n \in \omega$ consider the family $u_n = \{\Phi^{-1}_{\subset}(\bigcup v) : v \in [w_n]^{<\aleph_0}\}$. Since $\Phi$ is upper semicontinuous and compact-valued, each $u_n$ is an open cover of $X$. The Menger property of $X$ implies the existence of a sequence $(c_n)_{n\in\omega}$, where each $c_n$ is a finite subset of $u_n$, such that $\{\bigcup c_n : n \in \omega\}$ is a ($\gamma$-) cover of $X$. From the above it follows that for every $n \in \omega$ we can find a finite subset $v_n$ of $w_n$ with $\Phi(\bigcup c_n) \subset \bigcup v_n$. Therefore $\{\bigcup v_n : n \in \omega\}$ is a ($\gamma$-) cover of $Y$, consequently $Y$ is Menger (Hurewicz). □

The main idea of this paper is to assign to a topological space $X$ the collection $\mathsf{U}(X) = \{\mathcal{U}(u, X) : u \in \Lambda_\omega(X)\}$ of semifilters on countable sets, where $\Lambda_\omega(X)$ denotes the family of all countable large open covers of $X$ and $\mathcal{U}(u, X)$ is the smallest semifilter on $u$ containing the family $\{I(x, u, X) = \{U \in u : x \in U\} : x \in X\}$. It is clear that $\mathcal{U}(u, X)$ can be represented in the form $\bigcup_{v\in[u]^{<\aleph_0}} \bigcup_{x\in X} \uparrow_v I(x, u, X)$, where for a subsets $A$ and $B$ of a set $Z$ we denote by $\uparrow_A B$ the family $\{C \subset Z : C \supset B \setminus A\}$. When $A = \emptyset$, we shall simply write $\uparrow$ in place of $\uparrow_A$. When $X$ (and $u$) are clear from the context, we shall write $\mathcal{U}(u)$ and $I(x, u)$ ($I(x)$) instead of $\mathcal{U}(u, X)$ and $I(x, u, X)$.

We are in a position now to present a characterization of the properties of Menger and Hurewicz in terms of topological properties of semifilters, which implies the results mentioned in Introduction.

**Theorem 2.** *Let $X$ be a Lindelöf topological space. Then $X$ is Menger (Hurewicz) if and only if so is each $\mathcal{U}(u) \in \mathsf{U}(X)$. Moreover, if $X$ is paracompact, then it is Hurewicz provided each semifilter $\mathcal{U}(u) \in \mathsf{U}(X)$ is meager.*

**Remark 1.** 1. Every Hurewicz semifilter on a countable set $C$ is meager. Indeed, [10, Theorem 5.7] implies that each Hurewicz semifilter $\mathcal{F}$ on $C$ is contained in a $\sigma$-compact subset of $[C]^{\aleph_0}$, and each $\sigma$-compact subset of the Baire space is meager.

2. The "meager" part of the characterization of the Hurewicz property from Theorem 2 was independently proven by B. Tsaban for zero-dimensional metrizable spaces, see [22, Theorem 4]. □

We shall divide the proof of Theorem 2 into a sequence of lemmas.

**Lemma 2.** *Let $X$ be a topological space and $u \in \Lambda_\omega(X)$. Then the set-valued map $\Phi : X \Rightarrow \mathcal{P}(\omega)$, $\Phi : x \mapsto \uparrow I(x)$, is compact-valued and upper semicontinuous.*

PROOF: It is clear that $\Phi$ is compact-valued, because $\Phi(x) = \uparrow I(x)$ is a closed and precompact subspace of $\mathcal{P}(u)$. Let us show that $\Phi$ is upper semicontinuous. For this aim let us consider arbitrary $x \in X$ and an open subset $G$ of $\mathcal{P}(u)$

containing $\Phi(x)$. For every $v \in \Phi(x)$ we can find $s_v \in [u]^{<\aleph_0}$ such that $G(s_v, s_v \cap v) \subset G$. Since $\Phi(x)$ is compact, we can find a finite family $\mathbf{v} \subset \Phi(x)$ such that $\Phi(x) \subset \bigcup\{G(s_v, s_v \cap v) : v \in \mathbf{v}\}$. Put $s = \bigcup\{s_v : v \in \mathbf{v}\}$, $c = s \cap I(x)$ and $U = \bigcap c$. It is clear that $x \in U$ and $U$ is open. Thus the upper semicontinuity of $\Phi$ will be proven as soon as we show that $\Phi(U) \subset G$. For this purpose let us fix an arbitrary $x_1 \in U$ and observe that $I(x_1) \cap s \supset c = I(x) \cap s$, consequently $\Phi(x_1) \subset \bigcup\{G(s, v \cap s) : v \in \Phi(x)\} = G$, and finally $\Phi(U) \subset G$, which implies the upper semicontinuity of $\Phi$. □

**Corollary 1.** *Let $X$ be a Menger (Hurewicz) topological space and $u \in \Lambda_\omega(X)$. Then the semifilter $\mathcal{U}(u, X)$ is Menger (Hurewicz).*

Proof: Given any $v \in [u]^{<\aleph_0}$, consider the set-valued map $\Phi_v : X \Rightarrow \mathcal{P}(u)$, $\Phi_v : x \mapsto \uparrow_v I(x, u)$. Let us observe, that $\Phi_v$ is a composition $\Psi_2 \circ \Psi_1$, where $\Psi_1 : X \Rightarrow \mathcal{P}(u \setminus v)$, $\Psi_1 : x \mapsto \uparrow I(x, u \setminus v)$, and $\Psi_2 : \mathcal{P}(u \setminus v) \Rightarrow \mathcal{P}(u)$, $\Psi_2 : w \mapsto \uparrow w$. It is clear that $\Psi_2$ is compact-valued upper semicontinuous, while $\Psi_1$ is so by Lemma 2. Now, Lemma 1 implies that $\bigcup_{x \in X} \uparrow_v I(x, u) = \Phi_v(X)$ is Menger (Hurewicz). Since the property of Menger (Hurewicz) is preserved by countable unions, the semifilter $\mathcal{U}(u) = \bigcup_{v \in [u]^{<\aleph_0}} \Phi_v(X)$ is Menger (Hurewicz). □

**Lemma 3.** *Let $X$ be a Lindelöf topological space which fails to be Menger (Hurewicz). Then there exists $u \in \Lambda_\omega(X)$ such that the semifilter $\mathcal{U}(u)$ is not Menger (Hurewicz).*

Proof: Assuming that $X$ is not Menger (Hurewicz), we can find a sequence $(u_n)_{n \in \omega}$ of countable open large covers of $X$ such that there is no sequence $(v_n)_{n \in \omega}$ such that each $v_n$ is a finite subset of $u_n$ and the family $\{\bigcup v_n : n \in \omega\}$ is a ($\gamma$-)cover of $X$. Let us denote by $u$ the union $\bigcup\{u_n : n \in \omega\}$.

We claim that the semifilter $\mathcal{U}(u)$ is not Menger (Hurewicz). Indeed, consider the sequence $(o_n)_{n \in \omega}$ of countable families of open subsets of $\mathcal{P}(u)$, where $o_n = \{\{w \in \mathcal{P}(u) : U \in w\} : U \in u_n\}$. Since each $u_n$ is a large cover of $X$, every $o_n$ covers $\mathcal{U}(u)$. It suffices to show that there is no sequence $(c_n)_{n \in \omega}$ such that every $c_n$ is a finite subset of $o_n$ and $\{\bigcup c_n : n \in \omega\}$ is a large ($\gamma$-) cover of $\mathcal{U}(u)$, see [17, Corollary 5]. Assume, to the contrary, that such a sequence $(c_n)_{n \in \omega}$ exists. Then for every $n \in \omega$ we can find a finite subset $v_n$ of $u_n$ such that $c_n = \{\{w \in \mathcal{P}(u) : w \ni U\} : U \in v_n\}$. For every $w \in \mathcal{U}(u)$ set $J_w = \{n \in \omega : w \in \bigcup c_n\} = \{n \in \omega : w \cap v_n \neq \emptyset\}$. From the above it follows that $\mathcal{J} = \{J_w : w \in \mathcal{U}(u)\}$ consists of infinite (cofinite) subsets of $\omega$. From the above it follows that the family $\{J_{I(x,u)} : x \in X\}$ consists of infinite (cofinite) subsets of $\omega$ too. But

$$J_{I(x,u)} = \{n \in \omega : I(x, u) \cap v_n \neq \emptyset\} = \{n \in \omega : x \in \bigcup v_n\},$$

consequently $\{\bigcup v_n : n \in \omega\}$ is a ($\gamma$-) cover of $X$, which contradicts our choice of the sequence $(u_n)_{n \in \omega}$. □

Let $u$ be a family of a set $X$ and $B \subset X$. From now on $\mathcal{S}t(B,u)$ denotes the set $\bigcup\{U \in u : U \cap B \neq \emptyset\}$.

**Lemma 4.** *Let $X$ be a paracompact Lindelöf topological space. Then $X$ is Hurewicz provided each semifilter $\mathcal{U}(u) \in \mathsf{U}(X)$ is meager.*

PROOF: Assuming that $X$ is not Hurewicz, we shall show that $X$ possess a countable large open cover $u$ such that the semifilter $\mathcal{U}(u)$ is not meager. Let $(u_n)_{n\in\omega}$ be a sequence of open covers of $X$ such that $\{\bigcup v_n : n \in \omega\}$ is a $\gamma$-cover of $X$ for no sequence $(v_n)_{n\in\omega}$ such that each $v_n$ is a finite subcollection of $u_n$. Now, it is a simple exercise to construct a sequence $(w_n)_{n\in\omega}$ of open covers of $X$, where $w_n = \{U_{n,k} : k \in \mathbb{N}\}$ is a refinement of $u_n$, such that $U_{n_2,k} \subset \bigcup\{U_{n_1,l} : l \leq k\}$ for all $n_2 \geq n_1$ and $\mathcal{S}t(B, w_n)$ is $w_n$-bounded for every $w_n$-bounded subset $B$ of $X$. From the above it follows that for every $n \in \omega$ there exists a sequence $(p(n,k))_{k\in\omega}$ of natural numbers such that $\bigcup\{U_{m,i} : i \leq k, n \leq m\} \cap \bigcup\{U_{n,i} : i \geq p(n,k)\} = \emptyset$. Without loss of generality, $u_n = w_n$.

Denote by $u$ the union $\bigcup\{u_n : n \in \omega\}$. Let $\nu : u \to \omega$, $\nu : U_{n,k} \mapsto m_{n,k}$ be a bijective enumeration of $u$. Let us write $\omega$ in the form $\omega = \sqcup_{n\in\omega}\Omega_n$, where $\Omega_n = \{m_{n,k} : k \in \mathbb{N}\}$. We claim that the semifilter $\mathcal{U}(u)$ fails to be meager. For this aim we shall show that the image $\mathcal{F} = \nu(\mathcal{U}(u))$ of $\mathcal{U}(u)$ is not meager in $\mathcal{P}(\omega)$. Otherwise, by Theorem 1 there exists a sequence $(m_l)_{l\in\omega}$ of natural numbers such that every $F \in \mathcal{F}$ (and, in particular, $\nu(I(x,u))$ for every $x \in X$) meets all but finitely many half-intervals $[m_l, m_{l+1})$. Passing to a subsequence, if necessary, we may assume that $m_{l+1} > \max\{m_{n_1,p(n_2,k)} : m_{n_2,k} \leq m_l, [0, m_l] \cap \Omega_{n_1} \neq \emptyset\}$. Consider a function $\varphi : \omega \to \omega$ such that $\varphi^{-1}(l) = [m_l, m_{l+1})$ for all $l \in \omega$ and denote by $B_l$ the union $\bigcup\{\nu^{-1}(m) : m \in \varphi^{-1}(l)\}$. Then for every $x \in X$ there exists $n \in \omega$ such that $(\varphi \circ \nu)(I(x,u)) \supset [n, +\infty)$. From the above it follows that

$$X = \bigcup_{x\in X} \bigcap_{m\in\nu(I(x,u))} \nu^{-1}(m) \subset \subset \bigcup_{x\in X} \bigcap_{l\in(\varphi\circ\nu)(I(x,u))} \bigcup_{m\in\varphi^{-1}(l)} \nu^{-1}(m) \subset \bigcup_{n\in\omega} \bigcap_{i\geq n} B_i.$$

A crucial observation here is that the intersection $\bigcap_{l\in A} B_l$ is $u_n$-bounded for every $n \in \omega$ and all infinite subsets $A$ of $\omega$. Before proving this observation, let us note, that we can limit ourselves to subsets $A$ such that $|l_1 - l_2| > 1$ for all $l_1, l_2 \in A$. Given arbitrary $l \in \omega$, denote by $K_{n,l}$ the intersection $\Omega_n \cap [m_l, m_{l+1})$. Equipped with these notations, we can write

$$\bigcap_{l\in A} B_l = \bigcap_{l\in A} \bigcup_{n\in\omega} \bigcup_{m\in[m_l,m_{l+1})\cap\Omega_n} \nu^{-1}(m) = \bigcap_{l\in A} \bigcup_{n\in\omega} \bigcup_{m\in K_{n,l}} \nu^{-1}(m).$$

Set $B_{n,l} = \bigcup\{\nu^{-1}(m) : m \in K_{n,l}\}$. Then $\bigcap_{l\in A} B_l = \bigcap_{l\in A} \bigcup_{n\in\omega} B_{n,l} = \bigcup_{z\in\mathbb{N}^A} \bigcap_{l\in A} B_{z(l),l}$. By our choice of the sequence $(m_l)_{l\in\omega}$ and the subset $A$

of $\omega$ we have $B_{z(l_1),l_1} \cap B_{z(l_2),l_2} = \emptyset$ provided $z(l_2) \le z(l_1)$ for some $l_1, l_2 \in A$ with $l_2 > l_1$. Consequently

$$\bigcap_{l\in A} B_l = \bigcup_{z\in\mathbb{N}^{\uparrow A}} \bigcap_{l\in A} B_{z(l),l} = \bigcap_{l\in A} \bigcup_{n\ge|A\cap[0,l)|} B_{n,l},$$

where $\mathbb{N}^{\uparrow A} = \{z \in \mathbb{N}^A : z(l_2) > z(l_1)$ for all $l_2 > l_1\}$. Since the union $\bigcup_{n\ge|A\cap[0,l)|} B_{n,l}$ is $u_{|A\cap[0,l)|}$-bounded and $|A\cap[0,l)| \to +\infty$, $l \to +\infty$, the above intersection is $u_n$-bounded for all $n \in \omega$ (recall, that each $u_{n+1}$-bounded subset of $X$ is $u_n$-bounded). Therefore there exists a sequence $(v_n)_{n\in\omega}$, where $v_n$ is a finite subset of $u_n$, such that $\bigcup v_n \supset \bigcup_{k\le n}\bigcap_{i\ge k} B_i$, consequently the family $\{\bigcup v_n : n \in \omega\}$ is a $\gamma$-cover of $X$, which contradicts our choice of the sequence $(u_n)_{n\in\omega}$. □

Proof of Theorem 2: Follows from Lemmas 1, 3, 4, Corollary 1, and the remark after the formulation of Theorem 2. □

The following statement is of great importance in evaluation of additivity of the family of subspaces with the Menger property of a topological space $X$.

**Proposition 2.** *No comeager semifilter $\mathcal{F}$ on $\omega$ is Menger.*

Proof: If $\mathcal{F}$ is comeager in the space $[\omega]^{\aleph_0}$, which is homeomorphic to the Baire space $\mathbb{N}^\omega$, there exists a dense $G_\delta$ subset $G$ of $[\omega]^{\aleph_0}$ such that $G \subset \mathcal{F}$. Thus $G$ is an analytic and not $\sigma$-compact subset of $[\omega]^{\aleph_0}$, consequently it contains a closed in $[\omega]^{\aleph_0}$ subset $D$ homeomorphic to $\mathbb{N}^\omega$, see [12, Theorem 29.3]. But $\mathbb{N}^\omega$ simply fails to be Menger, consequently so is $\mathcal{F}$, a contradiction. □

Theorem 2 and Proposition 2 enable us to introduce a new class of topological spaces. A topological space $X$ is defined to be *almost Menger*, if the semifilter $\mathcal{U}(u)$ is not comeager for every $u \in \Lambda_\omega(X)$. Theorem 2 implies that every Lindelöf Menger space is almost Menger.

**Problem 2.** *Is every (metrizable separable) Lindelöf almost Menger space Menger?*

Sometimes it is more convenient to use a modification of Theorem 2. Let $X \subset Y$ and $u = (U_n)_{n\in\omega}$ be a sequence of subsets of $Y$. For every $x \in X$ let $I_s(x, u, X) = \{n \in \omega : x \in U_n\}$. If every $I_s(x, u, X)$ is infinite, then we shall denote by $\mathcal{U}_s(u, X)$ the smallest semifilter on $\omega$ containing all $I_s(x, u, X)$ (the letter $s$ comes from "sequence"). In what follows we shall denote by $\Lambda_s(X)$ the set of all sequences $u = (U_n)_{n\in\omega}$ of open nonempty subsets of a topological space $X$ such that all $I_s(x, uX)$ are infinite. Again, we shall often simplify these notations by writing $\mathcal{U}_s(u)$ and $I_s(x, u)$ or $I_s(x)$ in place of $\mathcal{U}_s(u, X)$ and $I_s(x, u, X)$.

**Theorem 3.** *Let $X$ be a Lindelöf topological space. Then $X$ is Menger (Hurewicz) if and only if for every sequence $u = (U_n)_{n\in\omega} \in \Lambda_s(X)$ the semifilter $\mathcal{U}_s(x,u)$ is Menger (Hurewicz). In addition, if $X$ is paracompact, then it is Hurewicz provided $\mathcal{U}_s(u)$ is meager for every $u \in \Lambda_s(X)$.*

Proof: Assuming that $X$ is Menger (Hurewicz), let us fix any sequence $u = (U_n)_{n\in\omega} \in \Lambda_s(X)$ and denote by $Y$ the product $X \times \mathbb{N}$. The space $Y$ is Menger (Hurewicz) being a countable union of its Menger (Hurewicz) subspaces. Consider the cover $w = \{W_n : n \in \omega\}$ of $Y$, where $W_n = U_n \times \{1, \dots, n\}$, and observe that $w \in \Lambda_\omega(Y)$. Applying Theorem 2, we conclude that $\mathcal{U}(w)$ is a Menger (Hurewicz) subspace of $\mathcal{P}(w)$. Now, it suffices to observe that $\mathcal{U}_s(u)$ is a continuous image of $\mathcal{U}(w)$ under the map $f : w \to \omega$, $f : W_n \mapsto n$.

Next, assume that the semifilter $\mathcal{U}_s(u)$ is Menger (resp. Hurewicz, meager) for all $u \in \Lambda_s(X)$ and fix any $w \in \Lambda_w(X)$. Let $w = \{W_n : n \in \omega\}$ be a bijective enumeration of $w$ and $u$ be a sequence $(W_n)_{n\in\omega}$. Then $f(\mathcal{U}_s(u)) = \mathcal{U}(w)$, where $f : n \mapsto W_n$ is a bijection. Therefore the semifilter $\mathcal{U}(w)$ is Menger (resp. Hurewicz, meager). Now, it suffices to apply Theorem 2. $\square$

## Additivity of the Menger property

As we have already said in Introduction, one of the main result of this paper is the following

**Theorem 4.** *Let $X$ a hereditarily Lindelöf space. Then*

$$\mathrm{add}(M(X)) \geq \mathrm{add}(M(\mathbb{N}^\omega)) \geq \mathfrak{g}.$$

Proof: Let $\mathcal{Y}$ be a subfamily of $M(X)$ of size $|\mathcal{Y}| < \mathrm{add}(M(\mathbb{N}^\omega))$ and $u = (U_n)_{n\in\omega} \in \Lambda_s(\bigcup \mathcal{Y})$. Then the semifilter $\mathcal{U}_s(u, \bigcup \mathcal{Y})$ is equal to the union $\bigcup_{Y\in\mathcal{Y}} \mathcal{U}_s(u, Y)$. Theorem 3 implies that every $\mathcal{U}_s(u, Y)$ is Menger, consequently so is their union $\mathcal{U}_s(u, \bigcup \mathcal{Y})$ by our choice of $\mathcal{Y}$. Applying Theorem 3 once again, we conclude that $\bigcup \mathcal{Y}$ is Menger, which implies the inequality $\mathrm{add}(M(X)) \geq \mathrm{add}(M(\mathbb{N}^\omega))$.

Next, we shall show that $\mathrm{add}(M(X)) \geq \mathfrak{g}$. Let $(w_n)_{n\in\omega}$ be a sequence of open covers of $\bigcup \mathcal{Y}$. Since $X$ is hereditarily Lindelöf, we can assume that every $w_n$ is a countable cover of $\bigcup \mathcal{Y}$ of the form $w_n = \{W_{n,k} : k \in \omega\}$. For every $Y \in \mathcal{Y}$ let us find a sequence $(k_n(Y))_{n\in\omega}$ of natural numbers such that the sequence $u_Y = (B_n(Y))_{n\in\omega}$, where $B_n(Y) = \bigcup_{k\leq k_n(Y)} W_{n,k}$, belongs to $\Lambda_s(Y)$. Now, Theorem 3 and Proposition 2 imply that the semifilter $\mathcal{U}_s(u_Y, Y)$ fails to be comeager. Since $|\mathcal{Y}| < \mathfrak{g}$, the semifilter $\mathcal{F} = \bigcup_{Y\in\mathcal{Y}} \mathcal{U}_s(u_Y, Y)$ is not comeager too, consequently $\mathcal{F}^\perp$ fails to be meager. Using $|\mathcal{Y}| < \mathfrak{g} \leq \mathfrak{b}(\mathcal{F}^\perp)$, we can find a sequence $(k_n)_{n\in\omega}$ such that $(k_n(Y))_{n\in\omega} \leq_{\mathcal{F}^\perp} (k_n)_{n\in\omega}$ for every $Y \in \mathcal{Y}$. Let us make the following notation: $B_n = \bigcup\{U_{n,k} : k \leq k_n\}$. We claim that $\{B_n : n \in \omega\}$ is a cover of $\bigcup \mathcal{Y}$. Indeed, let us fix arbitrary $Y \in \mathcal{Y}$ and $y \in Y$. Since

$(k_n(Y))_{n\in\omega} \leq_{\mathcal{F}^\perp} (n_k)_{k\in\omega}$, the set $A = \{n \in \omega : k_n \geq k_n(Y)\}$ belongs to $\mathcal{F}^\perp$ and thus there exists $m \in A \cap I_s(y, u_Y, Y)$. It suffices to observe that $y \in B_m$. $\square$

**Problem 3.** *Is the equation* $\mathrm{add}(M(X)) = \mathrm{add}(M(\mathbb{N}^\omega))$ *true for every hereditarily Lindelöf topological space?*

**Additivity of the property** $\mathrm{Split}(\Lambda, \Lambda)$

Throughout this paragraph, which is devoted to the property $\mathrm{Split}(\Lambda, \Lambda)$, every topological space is hereditarily Lindelöf. Since every large open cover of a space $X$ contains a countable large subcover (see [21, Proposition 1.1]), we can restrict ourselves to countable ones.

**Theorem 5.** *Under* $\mathfrak{u} < \mathfrak{g}$ *every paracompact space* $X$ *with the property* $\mathrm{Split}(\Lambda, \Lambda)$ *is Hurewicz.*

In our proof of Theorem 5 we shall use the following straightforward consequence of a fundamental result of C. Laflamme. A semifilter $\mathcal{F}$ on a countable set $C$ is said to be *bi-Baire*, if it is neither meager nor comeager.

**Theorem 6** ([1, Theorem 9.22], [13])**.** *Let $C$ be a countable set and $\mathcal{F}$ be a semifilter on $C$. If $\mathcal{F}$ is comeager* ($\mathfrak{u} < \mathfrak{g}$ *and $\mathcal{F}$ is bi-Baire*), *then there exists a sequence $(K_n)_{n\in\omega}$ of pairwise disjoint finite subsets of $C$ such that the set $\mathcal{U} = \{\{n \in \omega : F \cap K_n \neq \emptyset\} : F \in \mathcal{F}\}$ equals $\mathfrak{Fr}^\perp$ (is an ultrafilter on $\omega$).*

**Remark 2.** Let us observe, that if a sequence $(K_n)_{n\in\omega}$ is such as in Theorem 6, then for every increasing sequence $(m_n)_{n\in\omega}$ of natural numbers the sequence $(K'_n = \bigcup_{m\in[m_n, m_{n+1})} K_m)_{n\in\omega}$ satisfies the condition of this theorem too. $\square$

Proof of Theorem 5: In light of Corollary 29 from [17] asserting that each Hurewicz space has the property $\mathrm{Split}(\Lambda, \Lambda)$, the only step to be proven is the inverse implication under $\mathfrak{u} < \mathfrak{g}$. Suppose that the paracompact space $X$ is not Hurewicz. Then Theorem 2 supplies us with a cover $u \in \Lambda_\omega(X)$ such that the semifilter $\mathcal{U}(u)$ is not meager. Therefore there exists a finite subset $v$ of $u$ such that no finite subset $v_1$ of $u \setminus v$ is a cover of $X$, because otherwise we can simply construct by induction a sequence $(v_n)_{n\in\omega}$ of pairwise disjoint finite subsets of $u$ such that each $w \in \mathcal{U}(u)$ meets all but finitely many $v_n$, and thus $\mathcal{U}(u)$ is meager by Theorem 1. Two cases are possible.

1. $\mathcal{U}(u)$ is comeager. Then we can find a sequence $(v_n)_{n\in\omega}$ of finite subsets of $u$ as in Theorem 6. Let $n_0 \in \omega$ be such that no finite subset of $\bigcup_{n\geq n_0} v_n$ covers $X$. Since every $w \in \mathcal{U}(u)$ meets $u_0 = \bigcup_{n\geq n_0} v_n$, we conclude that $u_0 \in \Lambda_\omega(X)$. From the above it follows that there exists an increasing sequence $(m_n)_{n\in\omega}$ of natural numbers such that $m_0 \geq n_0$ and $U_{n_1} \neq U_{n_2}$ for all $n_1 \neq n_2$, where $U_n = \bigcup_{m\in[m_n, m_{n+1})} \bigcup v_m$, $n \in \omega$. Let us denote by $v'_n$ the union $\bigcup_{m\in[m_n, m_{n+1})} v_m$ and observe that the sequence $(v'_n)_{n\in\omega}$ satisfies the condition of Theorem 6.

It is clear that the family $u' = \{U_n = \bigcup v'_n : n \in \omega\}$ is a large cover of $X$. We claim that $u'$ is not splittable. Assuming the converse, we would find two disjoint infinite subsets $A$ and $B$ of $\omega$ such that both of $u_A = \{U_n : n \in A\}$ and $u_B = \{U_n : n \in B\}$ are large covers of $X$. Since the sequence $(v'_n)_{n\in\omega}$ satisfies the condition of Theorem 6, there exists $w \in \mathcal{U}(u)$ such that $\{n \in \omega : w \cap v'_n \neq \emptyset\} = A$. By the definition of the semifilter $\mathcal{U}(u)$, $I(x,u) \subset^* w$ for some $x \in X$. Therefore the set $\{n \in B : x \in U_n\}$ is a subset of a finite set $\{n \in B : (I(x,u) \setminus w) \cap v'_n \neq \emptyset\}$, and thus $u_B$ is not a large cover of $X$, a contradiction.

2. $\mathcal{U}(u)$ is not comeager. Then the same argument as in the first case implies that there exists a sequence $(v_n)_{n\in\omega}$ of finite subsets of $u$ with the following properties:

(1) it satisfies the conditions of Theorem 6;

(2) no finite subset of $\bigcup_{n\in\omega} v_n$ covers $X$;

(3) $\bigcup v_n \neq \bigcup v_m$, $n \neq m$, and the family $u' = \{\bigcup v_n : n \in \omega\}$ is a large cover of $X$.

We claim that $u'$ is not splittable. Assume, to the contrary, that there are infinite disjoint subsets $A$ and $B$ of $\omega$ such that $u_A, u_B \in \Lambda_\omega(X)$, where $u_A$ and $u_B$ are defined as above. Enlarging $A$, if necessary, we can additionally assume that $A \cup B = \omega$. Since the family $\mathcal{F} = \{\{n \in \omega : w \cap v_n \neq \emptyset\} : w \in \mathcal{U}(u)\}$ is an ultrafilter, either $A \in \mathcal{F}$ or $B \in \mathcal{F}$. Without loss of generality, $A \in \mathcal{F}$, which means that there exists $x \in X$ such that $\{n \in \omega : I(x,u) \cap v_n \neq \emptyset\} \subset^* A$, and thus $\{n \in B : I(x,u) \cap v_n \neq \emptyset\} = \{n \in B : x \in \bigcup v_n\}$ is finite, a contradiction.

□

## Other applications of Theorem 2

Here we shall show that we cannot restrict ourselves to $\omega$-covers in Theorem 2, and thus the family $\mathsf{U}(X)$ of semifilters cannot be reduced to the family $\{\mathcal{U}(u) : u \in \Omega(X)\}$ of filters corresponding to $\omega$-covers of a space $X$. Thus Theorems 2 and 3 are "purely semifilter statements".

**Proposition 3.** *Let $u$ be an $\omega$-cover of $\mathbb{N}^\omega$. Then $\mathcal{U}(u)$ is meager. Moreover, the smallest filter $\mathcal{V}$ containing $\mathcal{U}(u)$ is meager.*

Proof: Since $u$ is an $\omega$-cover, the semifilter $\mathcal{U}(u)$ is centered and the filter $\mathcal{V}$ is free. As it was shown in the proof of Corollary 1, there exists a sequence $(\Phi_n)_{n\in\omega}$ of compact-valued upper semicontinuos multifunctions from $\mathbb{N}^\omega$ into $\mathcal{P}(u)$ such that $\mathcal{U}(u) = \bigcup_{n\in\omega} \Phi_n(\mathbb{N}^\omega)$. Each $\Phi_n(\mathbb{N}^\omega)$ is an image of $\mathbb{N}^\omega$ under a compact-valued upper semicontinuous set-valued map ($\equiv \Phi_n(\mathbb{N}^\omega)$ is $K$-analytic). Since every $K$-analytic metrizable space $X$ is analytic (see [11]), so is the semifilter $\mathcal{U}(u)$ being a countable union of analytic spaces, see [12, 25.A]. Let us note, that $\mathcal{V} = \bigcup_{n\in\omega} \mathcal{U}_n$, where $\mathcal{U}_0 = \mathcal{U}(u)$ and $\mathcal{U}_{n+1}$ is a continuous image of $\mathcal{U}_n^2$ under the map $(U_1, U_2) \mapsto U_1 \cap U_2$. From the above it follows that $\mathcal{V}$ is analytic too, consequently by [19, Theorem 1, p. 30] it has the Baire property in $\mathcal{P}(u)$, and thus is meager by [19, Theorem 1, p. 32]. $\square$

Reformulating the above proposition in other terms, we obtain the subsequent result proved in [16].

**Theorem 7.** *Every $\omega$-cover of $\mathbb{N}^\omega$ is $\omega$-groupable.*

Proposition 2 and Theorem 6 imply the subsequent

**Corollary 2.** *Under $\mathfrak{u} < \mathfrak{g}$ every Menger space is Scheepers ($\equiv$ has the property $\bigcup_{\mathrm{fin}}(\Gamma, \Omega)$).*

Proof: Let $(u_n)_{n\in\omega}$ be a sequence of open covers of $X$ such that $u_{n+1}$ is a refinement of $u_n$ for all $n \in \omega$. Since $X$ is Menger, there exists a sequence $(v_n)_{n\in\omega}$ such that each $v_n$ is a finite subset of $u_n$ and $w = (\bigcup v_n)_{n\in\omega}$ belongs to $\Lambda_s(X)$. Applying Theorem 3 and Proposition 2, we conclude that the semifilter $\mathcal{U}_s(w)$ is not comeager.

If $\mathcal{U}_s(w)$ is meager, then Theorem 1 gives us an increasing sequence $(m_n)_{n\in\omega}$ of natural numbers such that each $A \in \mathcal{U}_s(w)$ meets all but finitely many half-intervals $[m_n, m_{n+1})$. Let $B_n = \bigcup_{m\in[m_n,m_{n+1})} \bigcup v_m$, $n \in \omega$. From the above it follows that each $B_n$ is $u_n$-bounded and the family $\{B_n : n \in \omega\}$ is a $\gamma$-cover of $X$.

If $\mathcal{U}_s(w)$ is bi-Baire, then by Theorem 6 there exists a sequence $(K_n)_{n\in\omega}$ of finite subsets of $\omega$ such that the family $\mathcal{F} = \{\{n \in \omega : A \cap K_n \neq \emptyset\} : A \in \mathcal{U}_s(w)\}$ is an ultrafilter on $\omega$. Let $(m_n)_{n\in\omega}$ be an increasing sequence of natural numbers with the property $\min \bigcup\{K_m : m \in [m_{n+2}, m_{n+3})\} > \max \bigcup\{K_m : m \in [m_n, m_{n+1})\}$. Since $\mathcal{F}$ is an ultrafilter, either $F_{\mathrm{even}} = \bigcup\{[m_n, m_{n+1}) : n \text{ is even}\}$ or $F_{\mathrm{odd}} = \bigcup\{[m_n, m_{n+1}) : n \text{ is odd}\}$ belongs to $\mathcal{F}$. Without loss of generality, $F = F_{\mathrm{even}} \in \mathcal{F}$. For every $n \in \omega$ denote by $B_n$ the set $\bigcup_{m\in[m_{2n},m_{2n+1})} \bigcup_{k\in K_m} \bigcup v_k$ and note that $B_n$ is $u_n$-bounded. We claim that $\{B_n : n \in \omega\}$ is an $\omega$-cover of $X$. Indeed, given any finite subset $S$ of $X$, for every $x \in S$ denote by $F_x$ the set $\{n \in \omega : I_s(x, w) \cap K_n \neq \emptyset\}$ and note that $F_x \in \mathcal{F}$. Since $\mathcal{F}$ is centered, there exists $m \in F \cap \bigcap_{x\in S} F_s$. Let $n \in \omega$ be such that $m \in [m_{2n}, m_{2n+1})$. We claim that $S \subset B_n$. Indeed, for every $x \in S$ there exists $k \in K_m \cap I_s(x, w)$, and thus $x \in \bigcup v_k \subset B_n$, which finishes our proof. $\square$

Another application of Theorem 2 takes its origin in the classical paper [9] of W. Hurewicz, where it was shown that a metrizable separable space $X$ is Menger if and only if for every continuous function $f : X \to \mathbb{R}^\omega$ the image $f(X)$ is not dominating with respect to the eventual dominance preorder. When $X$ is zero dimensional, the same assertion holds for continuous functions $f : X \to \mathbb{N}^\omega$. Trying to generalize the above result outside of metrizable separable spaces, all one can hope is the realm of Lindelöf spaces (every Menger topological space $X$ is obviously Lindelöf). However, this obstacle may be overcome by restriction to countable covers in the definitions of the Menger property.

**Definition 1.** A topological space $X$ has the property $E^*_\omega$, if for every sequence $(u_n)_{n\in\omega}$ of countable open covers of $X$ there exists a sequence $(v_n)_{n\in\omega}$ such that every $v_n$ is a finite subset of $u_n$ and $\bigcup_{n\in\omega}\bigcup v_n = X$.

It is clear that a topological space $X$ is Menger if and only if it has the property $E^*_\omega$ and is Lindelöf, and every countably compact noncompact space has the property $E^*_\omega$ but fails to be Menger. The ideas of Hurewicz still work for perfectly normal spaces: a perfectly normal space $X$ has the property $E^*_\omega$ if and only if there is no continuous function $f : X \to \mathbb{R}^\omega$ such that $f(X)$ is dominating, see [2]. However, because of topological spaces $X$ such that every continuous function $f : X \to \mathbb{R}$ is constant the above characterization of the property $E^*_\omega$ is not true for all topological spaces. For example, consider the topological space $Z = (\mathbb{R}, \tau)$, where $\tau = \{(-\infty, a) : a \in \mathbb{R}\}$. It is a simple exercise to show that $Z^\omega$ is Lindelöf and not Menger, but every continuous function $f : Z^\omega \to \mathbb{R}$ is constant.

Theorem 2 enables us to prove a general characterization of the property $E^*_\omega$ involving compact-valued upper semicontinuous maps.

**Theorem 8.** *A topological space $X$ has the property $E^*_\omega$ if and only if $\Phi(X) \neq \mathbb{N}^\omega$ for every compact-valued upper semicontinuous function $\Phi : X \Rightarrow \mathbb{N}^\omega$.*

Proof: The "only if" part follows from Lemma 1, which remains valid for the property $E^*_\omega$. To prove the "if" part, we have to find an upper semicontinuous compact-valued surjective map $\Phi : X \Rightarrow \mathbb{N}^\omega$ provided $X$ does not have the property $E^*_\omega$. A literal repetition of the proof of Lemma 3 gives us a semifilter $\mathcal{U} \in \mathsf{U}(X)$ which fails to be Menger. As it was shown in the proof of Corollary 1, there exists a sequence $(\Phi_n)_{n\in\omega}$ of compact-valued upper semicontinuous maps from $X$ into $[\omega]^{\aleph_0}$ such that $\mathcal{U} = \bigcup_{n\in\omega}\Phi_n(X)$. Since the union of countably many spaces with the Menger property is Menger, there exists $\Phi \in \{\Phi_n : n \in \omega\}$ such that the topological space $\Phi(X)$ does not have the Menger property. Using the already mentioned result of Hurewicz, we can find a continuous map $f : \Phi(X) \to \mathbb{N}^\omega$ such that $T = f(\Phi(X))$ is dominating in $\mathbb{N}^\omega$ with respect to $\leq^*$. Next, we shall find a continuous map $g : T \to \mathbb{N}^\omega$ such that $g(T)$ is dominating in the following stronger sense: for every $x \in \mathbb{N}^\omega$ there exists $y \in g(T)$ such that $y_n \geq x_n$ for all $n \in \omega$. To find such a map $g$ it suffices to note, that if none of the maps $g_i : T \ni (x_n)_{n\in\omega} \mapsto (x_{n+i})_{n\in\omega}$ has this property, then $T$ fails to be dominating. And finally, consider the set-valued map $\Psi : g(T) \Rightarrow \mathbb{N}^\omega$, $\Psi : g(T) \ni x \mapsto \{y \in \mathbb{N}^\omega : \forall n \in \omega (y_n \leq x_n)\}$. A direct verification shows that $\Psi$ is compact-valued and upper semicontinuous and $(\Psi \circ g \circ f \circ \Phi)(X) = \mathbb{N}^\omega$. $\square$

**Remark 3.** The additivity number $\mathrm{add}(\mathcal{I})$ is well-defined for an arbitrary family $\mathcal{I}$ of subsets of a set $X$ such that $\bigcup\mathcal{I} \notin \mathcal{I}$ and stands for the smallest size $\tau$ of a subfamily $\mathcal{J}$ of $\mathcal{I}$ with the property $\bigcup\mathcal{J} \notin \mathcal{I}$. B. Tsaban noted that the authors considered in [4] the family $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$ of all Menger subspaces of the Baire space, while $M(\mathbb{N}^\omega)$ is the smallest $\sigma$-ideal generated by $\bigcup_{\mathrm{fin}}(\mathcal{O}, \mathcal{O})$. In light of this the following question naturally arises: does Theorem 4 really answer Problem 2.4 from [4] in negative? The answer is "yes". In order to show this we shall simply prove that $\mathrm{add}(M(\mathbb{N}^\omega)) = \mathrm{add}(\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O}))$. It is a simple exercise to prove that $\mathrm{add}(\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})) \leq \mathrm{add}(M(\mathbb{N}^\omega))$. To show the inverse inequality, consider a family $\mathcal{J} \subset \bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ of size $\mathrm{add}(\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O}))$ with $\bigcup \mathcal{J} \notin \bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. Then Theorem 8 supplies us with a compact-valued upper semicontinuous surjective map $\Phi : \bigcup \mathcal{J} \Rightarrow \mathbb{N}^\omega$. Applying Lemma 1, we conclude that $\Phi(J) \in \bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O}) \subset M(\mathbb{N}^\omega)$ for all $J \in \mathcal{J}$, consequently $\mathbb{N}^\omega \notin M(\mathbb{N}^\omega)$ is a union of $\mathrm{add}(\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O}))$-many Menger subspaces, which means $\mathrm{add}(M(\mathbb{N}^\omega)) \leq \mathrm{add}(\bigcup_{\mathrm{fin}}(\mathcal{O},\mathcal{O}))$. □

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