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HALFNETS AND PARTIAL $\mathfrak{Z}$-LOOPS

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In this paper there are generalized some results given in [3] concerning 3-nets and loops for k-nets and their corresponding algebras of loops. The first part deals with problems of halfnets and their extensions, and with a construction of a maximal extension chain. There are made notions of a free net and $P_{\xi,\eta}$—centered free net. In the second part there is defined first the partial $\mathfrak{Z}$-loop for coordinatization of $P_{\xi,\eta}$—centered halfnet. This structure has been built as a generalization of $\mathfrak{Z}$-loops from [5]. Secondly there is studied a homomorphism of partial $\mathfrak{Z}$-loops and free $\mathfrak{Z}$-loops in connection with a homomorphism of halfnets and free nets.

1 Halfnets

Definition 1.1. A *halfnet* is an ordered tetrad $(\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ where $\mathscr{P}$ is a set of elements called *points*, $\mathscr{L}$ is a set of elements called *lines*, $(\mathscr{L}_\iota)_{\iota\in\mathscr{I}}$ is a system of mutually disjoint subsets of $\mathscr{L}$, the union of which is $\mathscr{L}$, $\mathscr{I}$ is a set of indices; $\# \mathscr{I} \geqq 3$, $\mathrm{I} \subset \mathscr{P}\times\mathscr{L}$ is an *incidence relation* and the following conditions are satisfied:

(i) $\forall\, \mathrm{P}\in\mathscr{P}\ \forall\, \iota\in\mathscr{I}\quad \#\{p \mid \mathrm{PI}p, p\in\mathscr{L}_\iota\} \leqq 1,$
(ii) $\forall\, \alpha, \beta\in\mathscr{I};\ \alpha \neq \beta\ \forall\, k\in\mathscr{L}_\alpha, \forall\, h\in\mathscr{L}_\beta\quad \#\{\mathrm{P} \mid \mathrm{PI}k, \mathrm{PI}h\} \leqq 1.$

Note. From (i) it follows:

$$\forall\, \iota\in\mathscr{I}\ \forall\, k, h\in\mathscr{L}_\iota;\ k \neq h\ \{\mathrm{X} \mid \mathrm{XI}k, \mathrm{XI}h\} = \emptyset.$$

Definition 1.2. A net is a halfnet $(\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ such that $\mathscr{P} \neq \emptyset$ and

(i) $\forall\, \mathrm{P}\in\mathscr{P}\ \forall\, \iota\in\mathscr{I}\ \exists\, !h\in\mathscr{L}_\iota\ \mathrm{PI}h,$
(ii) $\forall\, \alpha, \beta\in\mathscr{I};\ \alpha \neq \beta\ \forall\, h\in\mathscr{L}_\alpha, \forall\, k\in\mathscr{L}_\beta\ \exists\, !\mathrm{P}\in\mathscr{P}\ \mathrm{PI}h, \mathrm{PI}k.$

The set $\mathscr{L}_\alpha$ is called the α^{th} *pencil*, its lines the α-*lines*. Lines of the same pencil (distinct pencils) are called *parallel* (*non-parallel*). Points A, B are termed *joinable* if there is a line p such that $\mathrm{AI}p$, $\mathrm{BI}p$; if moreover $\mathrm{A} \neq \mathrm{B}$, then this line is called a *join* of A, B and is written as AB. A point P, for which $\mathrm{PI}h$, $\mathrm{PI}k$ with $h, k\in\mathscr{L}$; $h \neq k$, is called the *point of intersection* and is written as $h \sqcap k$. As customary we say B is "on" p or p "passes through" B if B$\mathrm{I}p$. A line from the α^{th} pencil passing through the point B is written as $\alpha(\mathrm{B})$. Let $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I})$. If $\mathrm{X} \in \mathscr{P}$ and $x \in \mathscr{L}$ we say that X and x are in $\mathscr{H}$, respectively. If moreover X$\mathrm{I}x$, we say that X is on x in $\mathscr{H}$.

Definition 1.3. Let $(\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I})$ be a halfnet. The cardinality of the set $\mathscr{I}$ is called the degree of this halfnet.

Definition 1.4. Let $(\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I}) = \mathscr{H}$ and $(\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota \in \mathscr{I}}, \mathrm{I}') = \mathscr{H}'$ be halfnets. By a *homomorphism* of the halfnet $\mathscr{H}$ into the halfnet $\mathscr{H}'$ we mean a pair of mappings (φ, Φ) $\varphi : \mathscr{P} \to \mathscr{P}'$, $\Phi : \mathscr{L} \to \mathscr{L}'$ such that the following holds:

(a) $p \in \mathscr{L}_\alpha \Rightarrow p^\Phi \in \mathscr{L}'_\alpha$, $\alpha \in \mathscr{I}$,
(b) $\mathrm{BI}p \Rightarrow \mathrm{B}^\varphi \mathrm{I} p^\Phi$, $\mathrm{B} \in \mathscr{P}$, $p \in \mathscr{L}$.

The homomorphism (φ, Φ) of $\mathscr{H}$ into $\mathscr{H}'$ is called an *isomorphism* if φ and Φ are bijections and $(\varphi^{-1}, \Phi^{-1})$ is a homomorphism of $\mathscr{H}'$ into $\mathscr{H}$.

Definition 1.5. Say that the halfnet $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I})$ is a *subhalfnet* of the halfnet $\mathscr{H}' = (\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota \in \mathscr{I}}, \mathrm{I}')$ if $\mathscr{P} \subset \mathscr{P}'$, $\mathscr{L} \subset \mathscr{L}'$, $\mathscr{L}_\alpha \subset \mathscr{L}'_\alpha$ $\forall \alpha \in \mathscr{I}$, $\mathrm{I} \subset \mathrm{I}'$ and $(\mathrm{id}_\mathscr{P}, \mathrm{id}_\mathscr{L})$ is a homomorphism of $\mathscr{H}$ indo $\mathscr{H}'$. If $\mathscr{H}$ is a subhalfnet of the halfnet $\mathscr{H}'$, we say that $\mathscr{H}$ is in $\mathscr{H}'$ and we write $\mathscr{H} \subset \mathscr{H}'$.

Definition 1.6. If $\mathscr{H}$ is a subhalfnet of the halfnet $\mathscr{H}'$ and $\mathscr{H}$ is a net, we say that $\mathscr{H}$ is a *subnet* of the halfnet $\mathscr{H}'$.

Definition 1.7. A halfnet $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I})$ where $\mathscr{P} = \emptyset$, and there exists just one $\iota \in \mathscr{I}$ such that $\mathscr{L} = \mathscr{L}_\iota$ is called a *degenerate halfnet.*

Definition 1.8. By a *trivial halfnet* we understand a halfnet $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathscr{I})$ where

(1) $\# \mathscr{P} \leqq 1, \forall \iota \in \mathscr{I}, \# \mathscr{L}_i \leqq 1$,
(2) $\forall p \in \mathscr{L}, \forall \mathrm{B} \in \mathscr{P}$ $\mathrm{BI}p$.

Note. It is easy to verify that in every non-trivial net there exist at least two lines in any pencil and at least two points on any line. Throughout, the degenerate halfnets will be excluded from our consideration.

Definition 1.9. A subhalfnet $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I})$ of the halfnet $\mathscr{H}' = (\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota \in I}, \mathrm{I}')$ is termed *closed* in $\mathscr{H}'$, if the following implications hold:

$$(\mathrm{B} \in \mathscr{P} \wedge \mathrm{BI}'h, h \in \mathscr{L}') \Rightarrow h \in \mathscr{L}$$
$$(h, k \in \mathscr{L} \wedge h \sqcap k = \mathrm{Q} \in \mathscr{P}') \Rightarrow \mathrm{Q} \in \mathscr{P}$$

Clearly, any subnet $\mathscr{M}$ of a net $\mathscr{N}$ is closed in $\mathscr{N}$.

Definition 1.10. We say that the halfnet $\mathscr{H}'$ is *generated* by its subhalfnet $\mathscr{H}$ if $\mathscr{H}'$ is the only subhalfnet of $\mathscr{H}'$ containing H and being closed in $\mathscr{H}'$.

Theorem 1.11. (The theorem generalizes the theorem 1.1 in [3]). Let (φ, Φ) and (ψ, Ψ) be two homomorphisms of the halfnet $\mathscr{H}$ into a halfnet $\mathscr{H}'$ and let $\mathscr{H}$ be generated by its subhalfnet $\mathscr{K}$. If $X^\varphi = X^\psi$ and $x^\Phi = x^\Psi$ for every point X and every line x in $\mathscr{K}$, then $\varphi = \psi$ and $\Phi = \Psi$.

Proof. Let $\mathscr{T}$ be the subhalfnet of $\mathscr{H}$ consisting of those points and lines of $\mathscr{H}$ for which $X^\varphi = X^\psi$ and $x^\Phi = x^\Psi$. Clearly, $\mathscr{K} \subset \mathscr{T}$. Let besides $\mathscr{T}$ be closed in $\mathscr{H}$. Namely, if B is in $\mathscr{T}$, h is in $\mathscr{H}$ and B$\mathrm{I}h$ in $\mathscr{H}$, then $B^\varphi \mathrm{I}' h^\Phi$ and $B^\psi \mathrm{I}' h^\Psi$ in $\mathscr{H}'$. However $B^\varphi = B^\psi$, and there is at most one line in every pencil passing through the point $B^\varphi = B^\psi$ in $\mathscr{H}'$. Hence $h^\Phi = h^\Psi$ and h is in $\mathscr{T}$. Similarly, if h, k are two lines of distinct pencils in $\mathscr{T}$ and if $h \sqcap k = Q$ is in $\mathscr{H}$, then $h^\Phi = h^\Psi$, $k^\Phi = k^\Psi$ and Q^φ and Q^ψ are on h^Φ and on k^Φ in $\mathscr{H}'$. Since $h^\Phi \neq k^\Phi$ (by definition 1.4.a) necessarily $Q^\varphi = Q^\psi$. Hence Q is in $\mathscr{T}$. Since $\mathscr{H}$ generated by $\mathscr{K}$ and $\mathscr{K} \subset \mathscr{T}$, it holds $\mathscr{H} = \mathscr{T}$; that is $(\varphi, \Phi) = (\psi, \Psi)$.

Definition 1.12. We say that a halfnet $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I})$ is an *$\mathscr{L}$-extension* of its subhalfnet $\mathscr{H}' = (\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota \in \mathscr{I}}, \mathrm{I}')$ if:

(1) $\mathscr{P} = \mathscr{P}'$,

(2) $\forall p \in \mathscr{L} \setminus \mathscr{L}' \quad \exists \mathrm{B} \in \mathscr{P}' \quad \mathrm{BI}p$,

$\mathscr{H}$ is a *complete $\mathscr{L}$-extension* of $\mathscr{H}'$ if besides:

(3) $\forall \mathrm{B} \in \mathscr{P} \quad \forall \iota \in \mathscr{I} \quad \exists p \in \mathscr{L}_\iota \quad \mathrm{BI}p$,

$\mathscr{H}$ is a *free $\mathscr{L}$-extension* of $\mathscr{H}'$ if (1), (2) and the following condition are satisfied:

(2') $\forall p \in \mathscr{L} \setminus \mathscr{L}' \quad \# \{\mathrm{B} \mid \mathrm{B} \in \mathscr{P}' \ \mathrm{BI}p\} \leqq 1$.

Definition 1.13. We say that a halfnet $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I})$ is a *$\mathscr{P}$-extension* of its subhalf $\mathscr{H}' = (\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota \in \mathscr{I}}, \mathrm{I}')$ if:

(1) $\mathscr{L} = \mathscr{L}'$,

(2) $\forall \mathrm{B} \in \mathscr{P} \setminus \mathscr{P}' \quad \# \{p \mid p \in \mathscr{L}', \mathrm{BI}p\} \geqq 2$,

$\mathscr{H}$ is a *complete $\mathscr{P}$-extension* of $\mathscr{H}'$ if moreover:

(3) $\forall p \in \mathscr{L}'_\alpha \quad \forall h \in \mathscr{L}'_\beta \quad \forall \alpha, \beta \in \mathscr{I}; \alpha \neq \beta \quad \exists \mathrm{B} \in \mathscr{P} \mathrm{BI}p, \mathrm{BI}h$,

$\mathscr{H}$ is a *free $\mathscr{P}$-extension* of $\mathscr{H}'$ if (1), (2) and the following condition holds:

(2) $\forall \mathrm{B} \in \mathscr{P} \setminus \mathscr{P}' \quad \# \{p \mid p \in \mathscr{L}', \mathrm{BI}p\} \leqq 2$.

Definition 1.14. The sequence of halfnets $(\mathscr{H}^0, \mathscr{H}^1, \ldots)$

$$\mathscr{H}^0 \subset \mathscr{H}^1 \subset \mathscr{H}^2 \subset \ldots \subset \mathscr{H}^{2n} \subset \mathscr{H}^{2n+1} \subset \mathscr{H}^{2n+2} \subset \ldots$$

where $\mathscr{H}^{2n+1}$ is an $\mathscr{L}$-extension of $\mathscr{H}^{2n}$ and $\mathscr{H}^{2n+2}$ is a $\mathscr{P}$-extension of $\mathscr{H}^{2n+1}$ is called an *extension chain* of $\mathscr{H}^0$. If all $\mathscr{H}^k$ are complete extension of $\mathscr{H}^{k-1}$, it is called a *maximal extension chain* of $\mathscr{H}^0$. If moreover all $\mathscr{L}$- and $\mathscr{P}$-extensions are free, then this chain is called a *maximal chain of free extensions* of $\mathscr{H}^0$ (or a maximal free extension chain).

Note. Let

$$\mathscr{H}^0 \subset \mathscr{H}^1 \subset \ldots \subset \mathscr{H}^k \subset \ldots \tag{1}$$

be an extension chain of $\mathscr{H}^0$, where $\mathscr{H}^k = (\mathscr{P}^k, \mathscr{L}^k, (\mathscr{L}^k_\iota)_{\iota\in\mathscr{I}}, \mathrm{I}^k)$. Let us put $\mathscr{P} := \bigcup_{n=0}^{\infty} \mathscr{P}^n$, $\mathscr{L} := \bigcup_{n=0}^{\infty} \mathscr{L}^n$, $\mathscr{L}_\alpha := \bigcup_{n=0}^{\infty} \mathscr{L}^n_\alpha \ \forall \alpha \in \mathscr{I}$, $\mathrm{I} := \bigcup_{n=0}^{\infty} \mathrm{I}^n$. Then, obviously, $\mathscr{M} := (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ is a halfnet. Let us write $\mathscr{M} = \cup\mathscr{H}^n$.

If chain (1) is maximal, then $\cup\mathscr{H}^n$ is a net because every point $\mathrm{B} \in \mathrm{P}$ is in some halfnet $\mathscr{H}^{2k+1}$ in which exactly one line p from any pensil exists such that $\mathrm{BI}^{2k+1}p$, and each two lines h, k of distinct pencils in $\cup\mathscr{H}^n$ are both in some halfnet $\mathscr{H}^{2k+2}$ with exactly one point B for any pair of non-parallel lines such that B is on h and on k, as well.

If the chain (1) is a maximal chain of free extensions of $\mathscr{H}^0$, then $\cup\mathscr{H}^n$ is a net with infinite number of points on any line and infinitely many lines passing through any point.

Proposition 1.15. Let $\mathscr{H}^0$ be a subhalfnet of a halfnet $\mathscr{K}$. Then $\mathscr{K}$ is generated by the halfnet $\mathscr{H}^0$ if and only if there exists an extension chain $\mathscr{H} \subset \mathscr{H}^1 \subset \ldots \subset \mathscr{H}^k \subset \ldots$ of $\mathscr{H}^0$ such that $\mathscr{K} = \cup\mathscr{H}^n$.

Proof is given in [3] p. 503 for a halfnet of degree three. The proposition nor its proof are independent of the degree of halfnets.

With regard to the definition of a net and to that of the complete extension we have:

Corollary 1.16. Let $\mathscr{H}^0$ be a subhalfnet of a net $\mathscr{N}$. Then N is generated by the subhalfnet $\mathscr{H}^0$ if and only if there exists a maximal chain $\mathscr{H}^0 \subset \mathscr{H}^1 \subset \ldots \subset \mathscr{H}^k \ldots$ of an extension of $\mathscr{H}_0$ such that $\mathscr{N} = \cup\mathscr{H}^n$.

Definition 1.17. Let $\mathscr{K} = \cup\mathscr{H}^n$, where $\mathscr{H}^0 = \mathscr{H}$ and $\mathscr{H}^{k+1}$ is a free ($\mathscr{L}$- or $\mathscr{P}$-) extension of $\mathscr{H}^k$ for every integer k. Then $\mathscr{K}$ is said to be *freely generated* by the halfnet $\mathscr{H}$.

Note. The net $\mathscr{N}$ is freely generated by a halfnet $\mathscr{H}$ if $\mathscr{N} = \cup\mathscr{H}^n$, where $\mathscr{H}^0 = \mathscr{H}$ and the chain (1) of a free extension of $\mathscr{H}^0$ is maximal.

Definition 1.18. Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ be a net freely generated by its subhalfnet $(\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota\in\mathscr{I}}, \mathrm{I}')$. The net $\mathscr{N}$ is called a *free net* if $\mathscr{P}' = \emptyset$.

Construction 1.19. Let $\mathscr{H}^0 = (\mathscr{P}^0, \mathscr{L}^0, (\mathscr{L}^0_\iota)_{\iota\in\mathscr{I}}, I^0)$ be a halfnet. Let us define:

$$\mathscr{P}^{2k+1} := \mathscr{P}^{2k}$$
$$\mathscr{L}^{2k+1}_\iota := \mathscr{L}^{2k}_\iota \cup \{(\iota, P) \mid P \in \mathscr{P}^{2k}, \iota(P) \text{ is not in } \mathscr{H}^{2k}\}, \iota \in \mathscr{I}$$
$$\mathscr{L}^{2k+1} := \bigcup_{\iota\in I} \mathscr{L}^{2k+1}_\iota$$
$$\mathscr{I}^{2k+1} := I^{2k} \cup \{(P, (\iota, P)) \mid P \in \mathscr{P}^{2k}, (\iota, P) \in \mathscr{L}^{2k+1} \setminus \mathscr{L}^{2k}\}.$$

This evidently implies that $\mathscr{H}^{2k+1} = (\mathscr{P}^{2k+1}, \mathscr{L}^{2k+1}, (\mathscr{L}^{2k+1}_\iota)_{\iota\in\mathscr{I}}, I^{2k+1})$ is a halfnet. Since the conditions (1), (2), (2′), and (3) of definition 1.12. are satisfied, the halfnet $\mathscr{H}^{2k+1}$ is a complete free $\mathscr{L}$-extension of the halfnet $\mathscr{H}^{2k}$.

Now, let us define:

$$\mathscr{P}^{2k+2} := \mathscr{P}^{2k+1} \cup \{\{k, h\}\ k \in \mathscr{L}^{2k+1}_\alpha, h \in \mathscr{L}^{2k+1}_\beta; \alpha \neq \beta, k \sqcap h \text{ is not in } \mathscr{H}^{2k+1}\},$$
$$\mathscr{L}^{2k+2} := \mathscr{L}^{2k+1},$$
$$\mathscr{L}^{2k+2}_\iota := \mathscr{L}^{2k+1}_\iota \quad \forall \iota \in \mathscr{I},$$
$$\mathscr{I}^{2k+2} := \mathscr{I}^{2k+1} \cup \{(\{h, k\}, h) \mid \{h, k\} \in \mathscr{P}^{2k+2} \setminus \mathscr{P}^{2k+1}, h \in \mathscr{L}^{2k+1}\} \cup \cup \{(\{h, k\}, k) \mid \{h, k\} \in \mathscr{P}^{2k+2} \setminus \mathscr{P}^{2k+1}, k \in \mathscr{L}^{2k+1}\}.$$

Again, it is obvious that $\mathscr{H}^{2k+2} = (\mathscr{P}^{2k+2}, \mathscr{L}^{2k+2}, (\mathscr{L}^{2k+2}_\iota)_{\iota\in\mathscr{I}}, I^{2k+2})$ is a halfnet and a complete free $\mathscr{P}$-extension of the halfnet $\mathscr{H}^{2k+1}$.

Hence, the halfnet $\mathscr{H}^{2k+1}$ satisfies the condition (i′) of definition 1.2 i.e. exactly one line of every pencil passes through every point and the halfnet $\mathscr{H}^{2k+2}$ satisfies the condition (ii′) i.e. every two lines of distinct pencils intersect in a single point. Hence the chain $\mathscr{H}^0 \subset \mathscr{H}^1 \subset \ldots \subset \mathscr{H}^{2k} \subset \mathscr{H}^{2k+1} \subset \mathscr{H}^{2k+2} \ldots$ is the maximal chain of free extensions of $\mathscr{H}^0$ and $\mathscr{N} = \cup\mathscr{H}^n$ is a net freely generated by $\mathscr{H}^0$.

Note. 1. By the construction, any halfnet may be embedded in a net at least in one way.

2. The net $\mathscr{N}$ of the construction 1.19. is a free net if $\mathscr{P}^0 = \emptyset$, $\mathscr{L}^0 \neq \emptyset$ holds for the halfnet $\mathscr{H}^0 = (\mathscr{P}^0, \mathscr{L}^0, (\mathscr{L}^0_\iota)_{\iota\in\mathscr{I}}, I^0)$.

Theorem 1.20. Let $\mathscr{H}' = (\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota\in\mathscr{I}}, I')$ be a free ($\mathscr{P}$- or $\mathscr{L}$-) extension of the halfnet $\mathscr{H}^0 = (\mathscr{P}^0, \mathscr{L}^0, (\mathscr{L}^0_\iota)_{\iota\in\mathscr{I}}, I^0)$. Then any homomorphism of $\mathscr{H}^0$ into a net $\mathscr{N} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_i)_{i\in I}, I)$ may be explicitly extended to a homomorphism of $\mathscr{H}'$ into $\mathscr{N}$.

Proof. Let (φ, Φ) be a homomorphism of $\mathscr{H}^0$ into $\mathscr{N}$.

1. Let $\mathscr{H}$ be a free $\mathscr{L}$-extension of the halfnet $\mathscr{H}^0$. Let us define a pair of mappings $(\psi, \Psi)\colon \mathscr{H}' \to \mathscr{N}$ as follows:

$$\forall X \in \mathscr{P}^0 \subset \mathscr{P}', \qquad X^\psi = X^\varphi$$
$$\forall h \in \mathscr{L}^0 \subset \mathscr{L}', \qquad h^\Psi = h^\Phi$$

since

$$\forall\, h \in \mathcal{L}'_\alpha \setminus \mathcal{L}^0_\alpha \quad \exists!\, \mathrm{B} \in \mathcal{P}^0 \quad \mathrm{BI}'h$$

and

$$\forall\, \mathrm{B} \in \mathcal{P}^0 \quad \exists!\, k \in \mathcal{L}_\alpha,\ \mathrm{B}^\varphi = \mathrm{B}^\psi \mathrm{I}k$$

we may put

$$h^\Psi = k \qquad \text{for } h \in \mathcal{L}' \setminus \mathcal{L}^0$$

Hence the mapping (ψ, Ψ) is single-valued and it is a homomorphism of $\mathcal{H}'$ into $\mathcal{N}$ and $(\psi, \Psi)/\mathcal{H}^0 = (\varphi, \Phi)$. Hence (φ, Φ) is extended to the homomorphism (ψ, Ψ) of $\mathcal{H}'$ into $\mathcal{N}$.

2. Let $\mathcal{H}'$ be a free $\mathcal{P}$-extension of the halfnet $\mathcal{H}^0$. Let us define a pair of mappings $(\pi, \Pi) : \mathcal{H}' \to \mathcal{N}$ as follows:

$$\forall\, \mathrm{X} \in \mathcal{P}^0 \subset \mathcal{P}', \qquad \mathrm{X}^\pi = \mathrm{X}^\varphi$$
$$\forall\, h \in \mathcal{L}^0 \subset \mathcal{L}', \qquad h^\Pi = h^\Phi$$

furthermore, let $\mathrm{X} \in \mathcal{P}' \setminus \mathcal{P}^0$, then there exist lines h, k $h \in \mathcal{L}^0_\alpha$, $k \in \mathcal{L}^0_\beta$ $\alpha, \beta \in \mathcal{J}$; $\alpha \neq \beta$ such that both h and k pass through the point X. Since $h^\Phi = h^\Pi$ and $k^\Phi = k^\Pi$, there exists exactly one point $\mathrm{Q} \in \mathcal{P}'$, Q is on h^Φ and on k^Φ, as well. Then $\mathrm{X}^\Pi = \mathrm{Q}$ and hence (π, Π) is defined uniquely and is a homomorphism of $\mathcal{H}'$ into $\mathcal{N}$ and $(\pi, \Pi)/\mathcal{H}^0 = (\varphi, \Phi)$.

Theorem 1.21. Let $\mathcal{H}' = (\mathcal{P}', \mathcal{L}', (\mathcal{L}'_\iota)_{\iota \in \mathcal{J}}, \mathrm{I}')$ be a halfnet generated by a halfnet $\mathcal{H}^0 = (\mathcal{P}^0, \mathcal{L}^0, (\mathcal{L}^0_\iota)_{\iota \in \mathcal{J}}, \mathrm{I}^0)$ and let every homomorphism of $\mathcal{H}^0$ into a net $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota \in \mathcal{J}}, \mathrm{I})$ be extendable to a homomorphism of $\mathcal{H}'$ into $\mathcal{N}$. Then $\mathcal{H}'$ is a net freely generated by the halfnet $\mathcal{H}^0$.

Proof. By proposition 1.15. there exists an extension chain of a halfnet $\mathcal{H}^0$ such that $\mathcal{H}' = \cup \mathcal{H}^n$, where $\mathcal{H}^{i+1}$ is $\mathcal{L}$- or $\mathcal{P}$-extension of $\mathcal{H}^i$. We have to show that these extensions are free. By the presuppositions of the theorem it is clear that the homomorphism of $\mathcal{H}^k$ into a net $\mathcal{N}$ has already been extended to that of $\mathcal{H}^{k+h}$ into $\mathcal{N}$ where $\mathcal{H}^k$, $\mathcal{H}^{k+h}$ are members of the extension chain of the halfnet $\mathcal{H}^0$.

1. Let $\mathcal{H}^{2k+1}$ be an $\mathcal{L}$-extension of $\mathcal{H}^{2k}$. Suppose that this extension is not free. Then there is at least one line $h \in \mathcal{L}^{2k+1} \setminus \mathcal{L}^{2k}$ such that both distinct points B, $\mathrm{Q} \in \mathcal{P}^{2k}$ are on h. Consider a maximal chain of the free extension of $\mathcal{H}^{2k}$:

$$\mathcal{H}^{2k} = \tilde{\mathcal{H}}^0 \subset \tilde{\mathcal{H}}^1 \subset \ldots \subset \tilde{\mathcal{H}}^n \subset \ldots$$

Then $\cup \tilde{\mathcal{H}}^n = \mathcal{M}$ is a net $(\tilde{\mathcal{P}}, \tilde{\mathcal{L}}, (\tilde{\mathcal{L}}_\iota)_{\iota \in \mathcal{J}}, \tilde{\mathrm{I}})$. If we take the identity homomorphism (id, id) of $\mathcal{H}^{2k}$ into a net $\mathcal{M}$, it is extendable to a homomorphism (φ, Φ) of the halfnet $\mathcal{H}^{2k+1}$ into $\mathcal{M}$, where $\mathrm{B} = \mathrm{B}^\varphi \tilde{\mathrm{I}} h^\Phi$ and $\mathrm{Q} = \mathrm{Q}^\varphi \tilde{\mathrm{I}} h^\Phi$. Since $\mathrm{B}, \mathrm{Q} \in \mathcal{P}^{2k}$, then necessarily $h^\Phi \in \mathcal{L}^{2k}$. Since h, h^Φ belong to the same pencil and $\mathrm{BI}^{2k}h$, $\mathrm{B}\tilde{\mathrm{I}}h^\Phi$ then necessaril $h = h^\Phi$. Therefore $h \in \mathcal{L}^{2k}$ in contradiction to our assumption.

2. Let $\mathscr{H}^{2k+2}$ be a $\mathscr{P}$-extension of $\mathscr{H}^{2k+1}$. Suppose that this extension is not free. Then there is at least one point $B \in \mathscr{P}^{2k+2} \setminus \mathscr{P}^{2k+1}$ such that there exist three distinct lines $h, k, l \in \mathscr{L}^{2k+1}$ passing through B in $\mathscr{H}^{2k+2}$. Consider a net $\mathscr{M} = \cup \widetilde{\mathscr{H}}^n$ with a maximal free extension chain of $\mathscr{H}^{2k+1}$. The homomorphism (id, id) of $\mathscr{H}^{2k+1}$ into $\mathscr{M}$ is extendable to a homomorphism (ψ, Ψ) of $\mathscr{H}^{2k+2}$ into $\mathscr{M}$, where $B^{\psi} \tilde{I} h^{\Psi}$, $B^{\psi} \tilde{I} k^{\Psi}$, $B^{\psi} \tilde{I} l^{\Psi}$, hence $B^{\psi} \in \mathscr{P}^{2k+1}$. Since $h^{\Psi} = h$, $k^{\Psi} = k$, $l^{\Psi} = l$, then necessarily $B^{\Psi} = B$ and $B \in \mathscr{P}^{2k+1}$ in contradiction to our assumption.

Theorem 1.22. Let $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_{\iota})_{\iota \in \mathscr{I}}, \mathrm{I})$ be a halfnet. Then there exists a net $\mathscr{N}$ freely generated by the halfnet $\mathscr{H}$. If $\mathscr{N}$, $\mathscr{N}'$ are two nets freely generated by $\mathscr{H}$, then they are isomorphic.

Proof. The existence of a freely generated net by a helfnet $\mathscr{H}$ follows from 1.19. Let $\mathscr{N}$, $\mathscr{N}'$ be two nets freely generated by $\mathscr{H}$. Then the identity homomorphism of $\mathscr{H} \subset \mathscr{N}$ into $\mathscr{N}'$ is extendable to a homomorphism $(\varphi, \Phi) : \mathscr{N} \to \mathscr{N}'$. Since $N^{(\varphi, \Phi)}$ is a net generated by H, then necessarily $\mathscr{N}^{(\varphi, \Phi)} = \mathscr{N}'$. Similarly, the identity homomorphism of $\mathscr{H} \subset \mathscr{N}'$ into $\mathscr{N}$ is extendable to a homomorphism $(\psi, \Psi) : \mathscr{N}' \to \mathscr{N}$ and again $\mathscr{N}'^{(\psi, \Psi)} = \mathscr{N}$. Then $\mathscr{N}^{(\varphi, \Phi)(\psi, \Psi)} = \mathscr{N}$ and

$$X^{\varphi\psi} = (X^{\varphi})^{\psi} = X^{\psi} = X, \qquad \forall X \in \mathscr{P}$$

and

$$p^{\Phi\Psi} = (p^{\Phi})^{\Psi} = p^{\Psi} = p, \qquad \forall p \in \mathscr{L}$$

and thus $\varphi\psi = \text{id}$ and $\Phi\Psi = \text{id}$. Similarly $\psi\varphi = \text{id}$ and $\Psi\Phi = \text{id}$. This implies that (φ, Φ) and (ψ, Ψ) are reciprocal isomorphisms of $\mathscr{N}$ into $\mathscr{N}'$. Thus $\mathscr{N}, \mathscr{N}'$ are isomorphic nets.

Theorem 1.23. (This theorem is a generalization of theorem 1.4. from [3] p. 511, for nets of any arbitrary degree.) Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_{\iota})_{\iota \in \mathscr{I}}, \mathrm{I})$ be a net. Then there exists a free net $\mathscr{M}$ and a homomorphism (φ, Φ) of $\mathscr{M}$ into $\mathscr{N}$ such that $\mathscr{M}^{(\varphi, \Phi)} = \mathscr{N}$.

Proof. Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_{\iota})_{\iota \in \mathscr{I}}, \mathrm{I})$ be an arbitrary net. Let $h \in \mathscr{L}_{\alpha}$ $(\alpha \in \mathscr{I})$. Let us make

$$\begin{aligned} \mathscr{L}' &:= \{h\} \\ \mathscr{P}' &:= \{X \in \mathscr{P} \mid X \mathrm{I} h\} \\ \mathscr{L}'_{\alpha} &:= \{h\} \\ \mathscr{L}'_{\iota} &:= \emptyset \quad \forall \iota \in \mathscr{I} \setminus \{\alpha\} \\ \mathrm{I}' &:= \mathrm{I} \cap (\mathscr{P}' \times \mathscr{L}') \end{aligned}$$

Then clearly $\mathscr{H}' = (\mathscr{L}', \mathscr{P}', (\mathscr{L}'_{\iota})_{\iota \in \mathscr{I}}, \mathrm{I}')$ is a subhalfnet of the net $\mathscr{N}$ generating $\mathscr{N}$, for there doesnot exist any closed subhalfnet in $\mathscr{N}$, containing $\mathscr{H}'$ being different from $\mathscr{N}$. Namely, every point of $\mathscr{N}$ being not in $\mathscr{H}'$ is joinable with the points of $\mathscr{H}'$ by means of lines of different pencils.

Furthermore let $\beta \in \mathscr{I} \setminus \{\alpha\}$. Let us make $\mathscr{H}^0 = (\mathscr{P}^0, \mathscr{L}^0, (\mathscr{L}_\iota^0)_{\iota \in \mathscr{I}}, I^0)$ as follows:

$$\begin{aligned}
\mathscr{P}^0 &:= \emptyset \\
\mathscr{L}^0 &:= \{h\} \cup \{p \mid p \in \mathscr{L}_\beta, h \sqcap p \in \mathscr{P}'\} \\
\mathscr{L}_\alpha^0 &:= \{h\} \\
\mathscr{L}_\beta^0 &:= \mathscr{L}_\beta \\
\mathscr{L}_\iota^0 &:= \emptyset \quad \forall \iota \in \mathscr{I} \setminus \{\alpha, \beta\} \\
I^0 &:= \emptyset
\end{aligned}$$

Consider the maximal chain of the free extension of the halfnet $\mathscr{H}^0$. Then $\mathscr{M} = \cup \mathscr{H}^n$ is a free net. The identity homomorphism of $\mathscr{H}^0 \subset \mathscr{M}$ onto $\mathscr{H}^0 \subset \mathscr{N}$ is extendable to the homomorphism $(\varphi, \Phi) : \mathscr{M} \to \mathscr{N}$ and since $\mathscr{N}$ is generated by $\mathscr{H}^0$, then necessarily $\mathscr{N} = \mathscr{M}^{(\varphi, \Phi)}$.

Definition 1.24. Let $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, I)$ be a halfnet. Let $\xi, \eta \in \mathscr{I}$; $\xi \neq \eta$, $P \in \mathscr{P}$. The halfnet $\mathscr{H}$ is called a $P_{\xi,\eta}$-*centered* if the following conditions are satisfied:

(i) $\forall R \in P \quad \forall \iota \in \mathscr{I} \quad \exists p \in \mathscr{L}_\iota RIp$,

(ii) $\forall p \in \mathscr{L} \setminus \mathscr{L}_\xi \; \{X \in \mathscr{P} \mid XIp, XI\xi(P)\} \neq \emptyset$
$\forall p \in \mathscr{L} \setminus \mathscr{L}_\eta \; \{X \in \mathscr{P} \mid XIp, XI\eta(P)\} \neq \emptyset$.

Note that a net is $P_{\xi,\eta}$-centered for its any point P and any pair of different indices ξ, η.

Theorem 1.25. (The theorem is a generalization of theorem 1.8. in [3] p. 524.) Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, I)$ be a net, $\mathscr{H}' = (\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota \in \mathscr{I}}, I')$ be its subhalfnet with $P \in \mathscr{P}'$, $\xi, \eta \in \mathscr{I}$; $\xi \neq \eta$. Then there exists an explicitely defined $P_{\xi,\eta}$-centered halfnet $\mathscr{H}^*$ such that

(a) $\mathscr{H}^* \subset \mathscr{N}$,

(b) $\mathscr{H}^*$ is generated by $\mathscr{H}$,

(c) every subhalfnet $\mathscr{K}$ satisfying $\mathscr{H} \subset \mathscr{K} \subset \mathscr{H}^*$; $\mathscr{K} \neq \mathscr{H}^*$ is not $P_{\xi,\eta}$-centered.

Proof: We see that the $P_{\xi,\eta}$-centered halfnets containing $\mathscr{H}$ exist in $\mathscr{N}$; for instance, $\mathscr{N}$ itself. Let $\mathscr{H}^*$ be the intersection of all $P_{\xi,\eta}$-centered subhalfnets of the net $\mathscr{N}$ containing $\mathscr{H}$. Now it is easy to verify that $\mathscr{H}^*$ is a $P_{\xi,\eta}$-centered subhalfnet of $\mathscr{N}$ satisfying (c). Suppose that there exists a halfnet $\mathscr{K} \subset \mathscr{N}$ such that $\mathscr{H} \subset \mathscr{K} \subset \mathscr{H}^*$, with $\mathscr{K}$ closed in $\mathscr{H}^*$. Then clearly, $\mathscr{K}$ is $P_{\xi,\eta}$-centered, as well, and therefore $\mathscr{K} = \mathscr{H}^*$, so that $\mathscr{H}^*$ is generated by $\mathscr{H}$.

From here on we shall use the notation $(\mathscr{H}, P, \xi, \eta)$ for the halfnet $\mathscr{H}^*$ from theorem 1.25.

Corollary 1.26. Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, I)$ be a net freely generated by its subhalfnet $\mathscr{K} = (\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota \in \mathscr{I}}, I')$ and let $P \in \mathscr{P}'$, $\xi, \eta \in \mathscr{I}$; $\xi \neq \eta$. Then $\mathscr{N}$ is freely generated by $(\mathscr{K}, \mathrm{P}, \xi, \eta)$. This corollary is an immediate consequence of theorem 1.25.

Note. Obviously $((\mathscr{K}, \mathrm{P}, \xi, \eta), \mathrm{P}, \xi, \eta) = (\mathscr{K}, \mathrm{P}, \xi, \eta)$.

Definition 1.27. Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I})$ be a net freely generated by its $\mathrm{P}_{\xi,\eta}$-centered subhalfnet $(\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota \in \mathscr{I}}, \mathrm{I} \cap (\mathscr{P}' \times \mathscr{L}'))$. Let the following implication hold:

$$\mathrm{R} \in \mathscr{P}' \Rightarrow \mathrm{RI}\xi(\mathrm{P}) \ \vee \ \mathrm{RI}\eta(\mathrm{P}) \tag{2}$$

Then the net $\mathscr{N}$ is called the $\mathrm{P}_{\xi,\eta}$-*centered free net.*

Theorem 1.28. A net $\mathscr{N}$ is a $\mathrm{P}_{\xi,\eta}$-centered free net if and only if $\mathscr{N}$ is a free net.

Proof. I. Let $\mathscr{N} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota \in \mathscr{I}}, \mathrm{I})$ be a $\mathrm{P}_{\xi,\eta}$-centered free net generated by a halfnet $(\mathscr{K}, \mathrm{P}, \xi, \eta) = (\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota \in \mathscr{I}}, \mathrm{I} \cap (\mathscr{P}' \times \mathscr{L}')$. Let us put

$$\begin{aligned}
\mathscr{P}^* &:= \emptyset \\
\mathscr{L}^*_\iota &:= \emptyset \quad \forall \iota \in \mathscr{I} \setminus \{\xi, \eta\} \\
\mathscr{L}^*_\xi &:= \mathscr{L}'_\xi \\
\mathscr{L}^*_\eta &:= \{\eta(\mathrm{P})\} \quad \eta(\mathrm{P}) \in \mathscr{L}'_\eta \\
\mathscr{L}^* &:= \mathscr{L}^*_\xi \cup \mathscr{L}^*_\eta \\
\mathrm{I}^* &:= \emptyset
\end{aligned}$$

Clearly $(\mathscr{P}^*, \mathscr{L}^*, (\mathscr{L}^*_\iota)_{\iota \in \mathscr{I}}, \mathrm{I}^*)$ is a halfnet freely generating $\mathscr{N}$. Since $\mathscr{P}^* = \emptyset$ the net $\mathscr{N}$ is a free net.

II. Let $\mathscr{N}$ be a free net generated by its subhalfnet $\mathscr{H} = (\mathscr{P}'', \mathscr{L}'', (\mathscr{L}''_\iota)_{\iota \in \mathscr{I}}, \mathrm{I} \cap \cap (\mathscr{P}'' \times \mathscr{L}''))$, with $\mathscr{P}'' = \emptyset$. In the maximal chain of free extension of $\mathscr{H}$ we have $\mathscr{H} = \mathscr{H}^1$, but $\mathscr{H}^1 \neq \mathscr{H}^2$. Let P be an arbitrary point of $\mathscr{H}^2$. Then there exist exactly two lines h, k of distinct pencils such that $\mathrm{PI}h$, $\mathrm{PI}k$. Let us constitute a halfnet $\mathscr{H}^* \subset \mathscr{N}$ for which $\mathscr{P}^* = \{\mathrm{P}\}$, $\mathscr{L}^* = \mathscr{L}'' \setminus \{h, k\}$ holds. Obviously $\mathscr{H}^2$ is freely generated by $\mathscr{H}^*$ and therefore $\mathscr{N}$ is freely generated by $\mathscr{H}^*$ as well. Then, however, $\mathscr{N}$ is freely generated even by its $\mathrm{P}_{\xi,\eta}$-centered subhalfnet $(\mathscr{H}^*, \mathrm{P}, \xi, \eta)$ for any $\xi, \eta \in \mathscr{I}$; $\xi \neq \eta$. Now we show that the halfnet $(\mathscr{H}^*, \mathrm{P}, \xi, \eta)$ satisfies implication (2). Suppose that there exists a point Q in $\mathscr{H}^*$ such that Q is neither on $\xi(\mathrm{P})$ nor on $\eta(\mathrm{P})$. By definition 1.24.(i) there exists a line $p \in \mathscr{L}_\iota$ for every $\iota \in \mathscr{I}$ so that $\mathrm{QI}p$, however at most two of them are in $\mathscr{H}$. Therefore there should exist a $\mathrm{P}_{\xi,\eta}$-centered halfnet $\mathscr{K}$ in which Q is on two lines only and $\mathscr{H} \subset \mathscr{K} \subset (\mathscr{H}^*, \mathrm{P}, \xi, \eta)$, which, according to theorem 1.25., is not possible.

2. Partial $\mathfrak{I}$-loops

Definition 2.1. Let H be a set, $o \in \mathrm{H}$ and let $\mathrm{Dom}(+) \subset \mathrm{H} \times \mathrm{H}$ and $+ : \mathrm{Dom}(+) \to \to \mathrm{H}$ be a partial binary operation in H. $\mathbf{H} := (\mathrm{H}, o, +)$ is called a *half-loop* whenever

(i) $\forall\, a \subset \mathrm{H}\quad (a, o), (o, a) \in \mathrm{Dom}(+)\quad a + o = o + a = a$,

(ii) $\forall\, a, d, f \in \mathrm{H}\quad (a, d), (a, f) \in \mathrm{Dom}(+)\quad a + d = a + f \Rightarrow d = f$,
$\forall\, a, b, c \in \mathrm{H}\quad (b, a), (c, a) \in \mathrm{Dom}(+)\quad b + a = c + a \Rightarrow b = c$.

A half-loop $(\mathrm{H}, o, +)$ is called a *semi-loop* whenever $\mathrm{Dom}(+) = \mathrm{H} \times \mathrm{H}$. A semi-loop $(\mathrm{H}, o, +)$ is called a *loop* whenever

(iii) $\forall\, a, b \in \mathrm{H}\quad \exists!\, (x, y) \in \mathrm{H} \times \mathrm{H}\quad a + x = b, y + x = b$.

Definition 2.2. Let S, $\mathfrak{J}$ be nonempty sets, $(\sigma_\iota)_{\iota \in \mathfrak{J}}$ a system of permutations of S, $(+_\iota)_{\iota \in \mathfrak{J}}$ a system of partial binary operations on S, $o \in \mathrm{S}$ and let the following conditions be satisfied:

1. $\exists\, \vartheta \in \mathfrak{J}, \sigma_\vartheta = \mathrm{id}_\mathrm{S}$,
2. $\forall\, \iota \in \mathfrak{J}, o^{\sigma_\iota} = o$,
3. $\forall\, \iota \in \mathfrak{J}\ (\mathrm{S}, o, +_\iota)$ is a half-loop,
4. for every $\alpha, \beta \in \mathfrak{J};\ \alpha \neq \beta$ and every $a, b \in \mathrm{S}$ there exists at most one pair $(x, y) \in \mathrm{S} \times \mathrm{S}$ such that $x^{\sigma_\alpha} +_\alpha y = a \wedge x^{\sigma_\beta} +_\beta y = b$,
5. $\forall\, a, b \in \mathrm{S}\quad \forall\, \iota \in \mathfrak{J},\quad (a^{\sigma_\iota}, b) \in \mathrm{Dom}(+_\iota) \Leftrightarrow (a, b) \in \mathrm{Dom}(+_\vartheta)$.

Then $\mathbf{S} := (\mathrm{S}, o, (\sigma_\iota)_{\iota \in \mathfrak{J}}, (+_\iota)_{\iota \in \mathfrak{J}})$ is called a *partial $\mathfrak{J}$-loop*.

Definition 2.3. A partial $\mathfrak{J}$-loop $\mathbf{S}$ is termed a *$\mathfrak{J}$-loop* whenever

3'. $\forall\, \iota \in \mathfrak{J}\ (\mathrm{S}, o, +_\iota)$ is a loop,
4'. $\forall\, \alpha, \beta \in \mathfrak{J};\ \alpha \neq \beta\quad \forall\, a, b \in \mathrm{S}\quad \exists!\, (x, y) \in \mathrm{S} \times \mathrm{S}$

$$x^{\sigma_\alpha} +_\alpha y = a \qquad x^{\sigma_\beta} +_\beta y = b$$

Obviously 5 is valid for every $\mathfrak{J}$-loop.

Construction 2.4. Given a partial $\mathfrak{J}$-loop $\mathbf{S} = (\mathrm{S}, o, (\sigma_i)_{\iota \in \mathfrak{J}}, (+_i)_{\iota \in \mathfrak{J}})$. Let ω_1, $\omega_2 \notin \mathfrak{J};\ \omega_1 \neq \omega_2$, and $\sigma_\vartheta = \mathrm{id}_\mathrm{S}$. Let us put

$$\begin{aligned}
\mathscr{I}\ &:= \mathfrak{J} \cup \{\omega_1, \omega_2\}\\
\mathscr{P}\ &:= \mathrm{Dom}\,(+_\vartheta)\\
\mathscr{L}\ &:= \mathrm{I} \times \mathrm{S}\\
\mathscr{L}_\alpha &:= \{\alpha\} \times \mathrm{S}, \alpha \in \mathscr{I}\\
\mathrm{I}\ &:= \bigcup_{a \in \mathrm{S}} \{((x, y), (\omega_1, a)) \mid (x, y) \in \mathrm{P}, x = a\} \cup\\
&\quad \bigcup_{b \in \mathrm{S}} \{((x, y), (\omega_2, b)) \mid (x, y) \in \mathrm{P}, y = b\} \cup\\
&\quad \bigcup_{\substack{c \in \mathrm{S}\\ \iota \in \mathfrak{J}}} \{((x, y), (\iota, c)) \mid (x, y) \in \mathrm{P}, x^{\sigma_\iota} +_i y = c\}
\end{aligned}$$

With respect to properties 3. and 4. of the partial $\mathfrak{J}$-loop $\mathbf{S}$ we see that

$(\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ is a halfnet. We denote the halfnet determined by the partial $\mathfrak{I}$-loop S, by writing $\mathscr{N}(\mathrm{S}, \omega_1, \omega_2)$.

Proposition 2.5. Let $(\mathrm{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}}) = \mathbf{S}$ be a partial $\mathfrak{I}$-loop. Then the halfnet $\mathscr{N}(\mathrm{S}, \omega_1, \omega_2)$ is $(o, o)_{\omega_1\omega_2}$-centered.

Proof: Obviously $(o, o) \in \mathscr{P}$, because $(o, o) \in \mathrm{Dom}(+_\vartheta)$; $o +_\vartheta o = o$. If $x \in \mathrm{S}$, then $o +_\vartheta x = x \in \mathrm{S}$ and therefore $(o, x) \in \mathscr{P}$, $x +_\vartheta o = x \in \mathrm{S}$ and $(x, o) \in \mathscr{P}$ as well. Let $\mathrm{R} = (r, q) \in \mathscr{P}$. Then $(r, q) \in \mathrm{Dom}(+_\vartheta)$ and therefore $(\mathrm{r}^{\sigma_\iota}, q) \in \mathrm{Dom}(+_\iota)$ $\forall \iota \in \mathfrak{I}$ by 5. of definition 2.3. Thus $\mathrm{RI}(\omega_1, r)$, $\mathrm{RI}(\omega_2, q)$, $\mathrm{RI}(\iota, z_\iota)\ \forall \iota \in \mathfrak{I}$, where $z_\iota = r^{\sigma_\iota} +_\iota q$, and the condition (i) of the definition regarding the $\mathrm{P}_{\xi,\eta}$-centered halfnet is satisfied. Furthermore it holds:

$$(\omega_2, x) \sqcap (\omega_1, o) = (o, x) \quad \forall x \in \mathrm{S}$$
$$(\iota, \ x) \sqcap (\omega_1, o) = (o, x) \quad \forall \iota \in \mathfrak{I}, x \in \mathrm{S}$$
$$(\omega_1, x) \sqcap (\omega_2, o) = (x, o) \quad \forall x \in \mathrm{S}$$
$$(\iota, \ x) \sqcap (\omega_2, o) = (x^{\sigma_\iota}, o) \quad \forall \iota \in \mathfrak{I}, x \in \mathrm{S}$$

So the condition (ii) of definition 1.24. is satisfied for the point (o, o) and for the pair of indices ω_1, ω_2.

Now it can be readily verified:

Corollary 2.6. Let $\mathbf{L} = (\mathrm{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ be a $\mathfrak{I}$-loop, $\xi, \eta \notin \mathfrak{I}$. Then $\mathscr{N}(\mathbf{L}, \xi, \eta)$ is a net.

Theorem 2.7. Let $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ be an $0_{\xi,\eta}$-centered halfnet. Then there exists a partial $\mathfrak{I}$-loop S such that $\mathscr{N}(\mathrm{S}, \xi, \eta)$ and $(\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ are isomorphic.

Proof. Let us define: $\mathrm{S} := \{\mathrm{X} \in \mathscr{P} \mid \mathrm{XI}\xi(0)\}$, $\mathfrak{I} := \mathscr{I} \setminus \{\xi, \eta\}$. We choose $\vartheta \in \mathfrak{I}$. Let $\sigma_\iota : \mathrm{S} \to \mathrm{S}$ $(\forall \iota \in \mathfrak{I})$ be a mapping defined as follows:

$$x^{\sigma_\iota} = \iota(\vartheta(x) \sqcap \eta(0)) \sqcap \xi(0), \qquad \forall x \in \mathrm{S} \qquad \text{(fig. 1)}$$

Define the operation $+_\iota$ to every $\iota \in \mathfrak{I}$ as follows: $(a^{\sigma_\iota}, b) \in \mathrm{Dom}(+_\iota) :\Leftrightarrow$ there exists a point $\xi(\iota(a) \sqcap \eta(0)) \sqcap \eta(b)$. Let for $(a^{\sigma_\iota}, b) \in \mathrm{Dom}(+_\iota)$ hold:

$$a^{\sigma_\iota} +_\iota b := \iota(\xi(\iota(a) \sqcap \eta(0)) \sqcap \eta(b)) \sqcap \xi(0), \qquad \text{(fig. 2)}$$

and next $x^{\sigma_\vartheta} = \vartheta(\vartheta(x) \sqcap \eta(0)) \sqcap \xi(0) = x \ \forall x \in \mathrm{S}$. Hence $\sigma_\vartheta = \mathrm{id}_\mathrm{S}$.

$$0^{\sigma_\iota} = \iota(\vartheta(0) \sqcap \eta(0)) \sqcap \xi(0) = \iota(0) \sqcap \xi(0) = 0, \forall \iota \in \mathfrak{I}.$$

The unigueness of the partial operations $+_\iota$ in S follows from the properties of $0_{\xi,\eta}$-centered halfnet $(\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$. Clearly, the point 0 is the zero element in all partial operations $+_\iota$. Both cancellation laws for S are easily verified from the above construction. Hence $(\mathrm{S}, 0, +_\iota)$ are half-loops for every $\iota \in \mathfrak{I}$. Since any two lines of the distinct pencils intersect at most in one point, there holds condition 4.

of definition 2.2. If $a^{\sigma_\iota} +_\iota b$ is defined for $a, b \in \mathrm{S}$, $\iota \in \mathfrak{I}$, then $(a^{\sigma_\iota}, b) \in \mathrm{Dom}(+_\iota)$. Hence there exists a point $\xi(\iota(a^{\sigma_\iota}) \sqcap \eta(0)) \sqcap \eta(b) = \xi(\vartheta(a) \sqcap \eta(0)) \sqcap \eta(b)$. This is however a necessary and sufficient condition for the purpose of $(a^{\sigma_\iota}, b) \in \mathrm{Dom}(+_\iota)$ $\forall \iota \in \mathfrak{I}$ which proves the validity of (5) of definition 2.2. We have thus proved that $(\mathrm{S}, 0, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ is a partial $\mathfrak{I}$-loop.

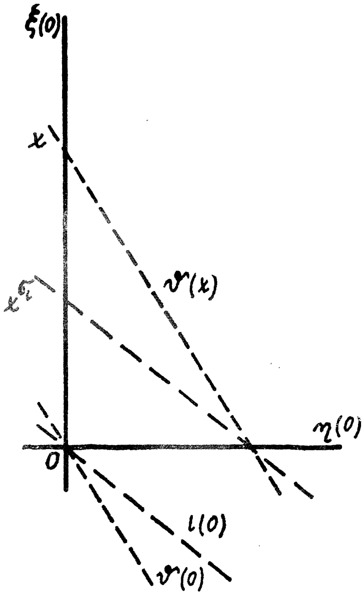

Fig. 1.

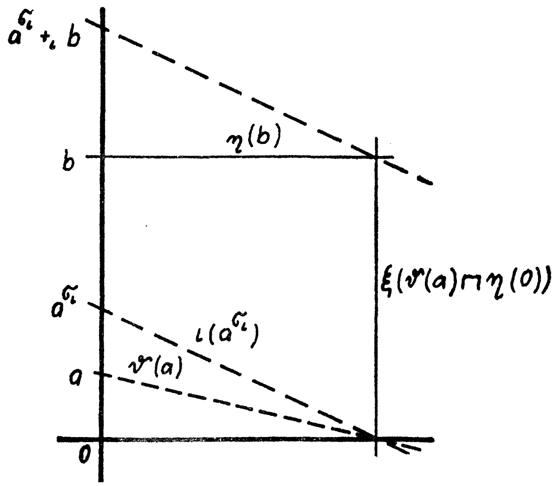

Fig. 2.

Let us define a pair of mappings $\sigma : \mathrm{Dom}(+_\vartheta) \to \mathscr{P}$,

$$\Sigma : \mathscr{I} \times \mathrm{S} \to \mathscr{L} \text{ as follows:}$$

$$\sigma(a, b) = \xi(\vartheta(a) \sqcap \eta(0)) \sqcap \eta(b),$$

$$\Sigma(\iota, c) = \iota(c) \text{ whenever } \iota \in \mathscr{I} \setminus \{\xi\},$$

$$\Sigma(\xi, c) = \xi(\vartheta(c) \sqcap \eta(0)).$$

It is obvious that the mappings σ, Σ are single-valued and that (σ, Σ) is an isomorphism of the halfnet $\mathscr{N}(\mathrm{S}, \xi, \eta)$ onto the halfnet $(\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$. The dependence of a partial $\mathfrak{I}$-loop on the $0_{\xi,\eta}$-centered halfnet $\mathscr{H} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ will be written as $\mathbf{L}(\mathscr{H}, 0, \xi, \eta)$ and $\mathbf{L}((\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I}), 0, \xi, \eta)$, respectively. The proof is complete.

Note. 1. If $\mathrm{S} = (\mathrm{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ is a partial $\mathfrak{I}$-loop, $\omega_1, \omega_2, \omega_1', \omega_2' \notin \mathfrak{I}$; $\omega_1 \neq \omega_2$, $\omega_1' \neq \omega_2'$, then the halfnets $\mathscr{N}(\mathrm{S}, \omega_1, \omega_2)$, $\mathscr{N}(\mathrm{S}, \omega_1', \omega_2')$ are clearly isomorphic. Therefore we will write $\mathscr{N}(\mathrm{S})$ in place of $\mathscr{N}(\mathrm{S}, \omega_1, \omega_2)$ whenever this does not lead to any confusion.

2. It is easily verified that the halfnets $\mathcal{N}(\mathbf{L}((\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota\in\mathcal{I}}, \mathrm{I}), 0, \xi, \eta), \xi, \eta)$ and $((\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota\in\mathcal{I}}, \mathrm{I}), 0, \xi, \eta)$ are isomorphic.

Definition 2.8. Let $\mathbf{S} = (\mathrm{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{J}}, (+_\iota)_{\iota\in\mathfrak{J}})$ and $\mathbf{S}' = (\mathrm{S}', o', (\sigma'_\iota)_{\iota\in\mathfrak{J}}, (+'_\iota)_{\iota\in\mathfrak{J}})$ be partial $\mathfrak{J}$-loops. By a *homomorphism of the partial* $\mathfrak{J}$*-loop* $\mathbf{S}$ *into the partial* $\mathfrak{J}$*-loop* $\mathbf{S}'$ we mean a mapping $\pi : \mathrm{S} \to \mathrm{S}'$ with the following conditions:

(1) $o^\pi = o'$,

(2) $(a^{\sigma_\iota}, b) \in \mathrm{Dom}(+_\iota) \Rightarrow ((a^\pi)^{\sigma_\iota'}, b^\pi) \in \mathrm{Dom}(+'_\iota)$ and $(a^{\sigma_\iota} +_\iota b)^\pi = (a^\pi)^{\sigma_\iota'} +'_\iota b^\pi$.

Definition 2.9. We say that the partial $\mathfrak{J}$-loop $\mathbf{S} = (\mathrm{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{J}}, (+_\iota)_{\iota\in\mathfrak{J}})$ is *embedded* into the partial $\mathfrak{J}$-loop $\mathbf{S}' = (\mathrm{S}', o', (\sigma'_\iota)_{\iota\in\mathfrak{J}}, (+'_\iota)_{\iota\in\mathfrak{J}})$ if $\mathrm{S} \subset \mathrm{S}'$, $o = o'$, $\sigma_\iota = \sigma'_\iota/\mathrm{S}$, $+_\iota = +'_\iota/\mathrm{S}\times\mathrm{S}$ and id_S is a homomorphism of $\mathbf{S}$ into $\mathbf{S}'$.

Note. It is easy to verify:

a) If $\mathbf{S}$ is a proper partial $\mathfrak{J}$-loop embedded in $\mathbf{S}'$, then $\mathcal{N}(\mathbf{S})$ is a proper subhalfnet of the halfnet $\mathcal{N}(\mathbf{S}')$.

b) If $(\mathcal{H}, 0, \xi, \eta)$ is a proper subhalfnet of the halfnet $(\mathcal{H}', 0, \xi, \eta)$, then $\mathbf{L}(\mathcal{H}, 0, \xi, \eta)$ is a proper embedded partial $\mathfrak{J}$-loop in the partial $\mathfrak{J}$-loop $\mathbf{L}(\mathcal{H}', 0, \xi, \eta)$.

With respect to the above note it follows from theorem 2.7:

Corollary 2.10. Let $\mathcal{N} = (\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota\in\mathcal{I}}, \mathrm{I})$ be a net. Then $\mathbf{L}(\mathcal{N}, \xi, \eta)$ is a $\mathfrak{J}$-loop for any $\xi, \eta \in \mathcal{I}$; $\xi \neq \eta$.

Theorem 2.11. Let π be a homomorphism of the partial $\mathfrak{J}$-loop $\mathbf{S} = (\mathrm{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{J}}, (+_\iota)_{\iota\in\mathfrak{J}})$ into the partial $\mathfrak{J}$-loop $\mathbf{S}' = (\mathrm{S}', o', (\sigma'_\iota)_{\iota\in\mathfrak{J}}, (+'_\iota)_{\iota\in\mathfrak{J}})$, $\omega_1, \omega_2 \notin \mathfrak{J}$; $\omega_1 \neq \omega_2$. Let two mappings $\varphi : \mathrm{Dom}(+\mathrm{id}_\mathrm{S}) \to \mathrm{Dom}(+'\mathrm{id}_{\mathrm{S}'})$, $\Phi : (\mathfrak{J} \cup \{\omega_1, \omega_2\})\times\mathrm{S} \to (\mathfrak{J} \cup \{\omega_1, \omega_2\})\times\mathrm{S}'$ be given as follows:

$$(a, b)^\varphi := (a^\pi, b^\pi), \qquad (\iota, c)^\Phi := (\iota, c^\pi)$$

Then the pair of mappings (φ, Φ) is a homomorphism of the halfnet $\mathcal{N}(\mathbf{S})$ into the halfnet $\mathcal{N}(\mathbf{S}')$.

Proof. Let (a, b) be a point in $\mathcal{N}(\mathbf{S})$, i.e. (a^{σ_ι}, b) $\mathrm{Dom}(+_\iota)$ $\iota \in \mathfrak{J}$. Then there exists a $z \in \mathrm{S}$ such that $z = a^{\sigma_\iota} +_\iota b$ for a $\iota \in \mathfrak{J}$. Clearly, $a^\pi, b^\pi \in \mathrm{S}'$ and hence (by (2) of definition 2.8.) $(a^\pi)^{\sigma_\iota'} +'_\iota b^\pi = (a^{\sigma_\iota} +_\iota b)^\pi = z^\pi \in \mathrm{S}'$. Thus $((a^\pi)^{\sigma_\iota'}, b^\pi) \in \mathrm{Dom}(+'_\iota)$, and from here (a^π, b^π) is a point in $\mathcal{N}(\mathbf{S}')$ and the mapping φ is a single-valued mapping of the points from $\mathcal{N}(\mathbf{S})$ into the set of points of $\mathcal{N}(\mathbf{S}')$. Let $0 := (o, o)$. Then $0^\varphi = (o, o)^\varphi = (o^\pi, o^\pi) = (o', o') =: 0'$. We have then for the points (a, y), (a, z) on the line (ξ, a) : $(a, y)^\varphi = (a^\pi, y^\pi)$, $(a, z)^\varphi = (a^\pi, z^\pi)$ which leads to $(a, y)^\varphi\, I'(\xi, a^\pi) = (\xi, a)^\Phi$ and also to $(a, z)^\varphi\, I'(\xi, a^\pi) = (\xi, a)^\Phi$. Similarly: (x, b), (y, b) $\mathrm{I}(\eta, b)$ and $(x, b)^\varphi = (x^\pi, b^\pi)$, $(y, b)^\varphi = (y^\pi, b^\pi)$, hence $(x, b)^\varphi, (y, b)^\varphi\, \mathrm{I}'(\eta, b)^\Phi$. Let us now consider the points (x, y), $(x^{\sigma_\iota} +_\iota y, 0)$ on the same ι-line in $\mathcal{N}(\mathbf{S})$. Then the points $(x, y^\varphi = (x^\pi, y^\pi)$, $(x^{\sigma_\iota} +_i y, 0) = ((x^{\sigma_\iota} +_\iota y)^\pi, 0')) = ((x^\pi)^{\sigma_\iota'} +'_\iota y^\pi, 0')$ are on the same ι-line in $\mathcal{N}(\mathbf{S}')$.

Theorem 2.12. Let $\mathbf{S} = (S, o, (\sigma_\iota)_{\iota \in \mathfrak{J}}, (+_\iota)_{\iota \in \mathfrak{J}})$, $\mathbf{S}' = (S', o', (\sigma'_\iota)_{\iota \in \mathfrak{J}}, (+'_\iota)_{\iota \in \mathfrak{J}})$ be two partial $\mathfrak{J}$-loops and let $\xi, \eta \notin \mathfrak{J}$; $\xi \neq \eta$ and $\sigma_\vartheta = \mathrm{id}_S$, $\sigma'_\vartheta = \mathrm{id}_{S'}$, $\vartheta \in \mathfrak{J}$. Let besides (φ, Φ) be a homomorphism of the halfnet $\mathcal{N}(\mathbf{S})$ into the halfnet $\mathcal{N}(\mathbf{S})$ such that $(o, o)^\varphi = (o', o')$, $(\xi, o)^\Phi = (\xi, o')$, $(\eta, o)^\Phi = (\eta, o')$ and $\{(\vartheta, c)^\Phi \mid c \in S\} \subset \subset \{(\vartheta, c') \mid c' \in S'\}$. Then there exists a homomorphism π of the partial $\mathfrak{J}$-loop $\mathbf{S}$ into the partial $\mathfrak{J}$-loop $\mathbf{S}'$ such that $(x, y)^\varphi = (x^\pi, y^\pi)$ and $(\iota, c)^\Phi = (\iota, c^\pi)$ for all $x, y, c \in S$ and all $\iota \in \mathfrak{J} \cup \{\xi, \eta\}$.

Proof. Let us define the mapping $\pi : S \to S'$ as follows: $(x, o)^\varphi = (x^\pi, o')$.

Now $(o, x), (o, o)$ are on (ξ, o) in $\mathcal{N}(S)$ and thus $(o, x)^\varphi, (o, o)^\varphi$ are on $(\xi, o)^\Phi = (\xi, o')$ in $\mathcal{N}(S')$. Consequently $(o, x)^\varphi = (o', z)$ for some $z \in S'$. Furthermore $(x, o), (o, x)$ are on (ϑ, c) for some $c \in S$. Hence $(x, o)^\varphi = (x^\pi, o')$ and $(o, x)^\varphi = (o', z)$ are on the same ϑ-line in $\mathcal{N}(S')$. Therefore $(x^\pi)^{\sigma_\vartheta'} +'_\vartheta o' = o' +'_\vartheta z$ or $z = x^\pi$ and from here $(o, x)^\varphi = (o', x^\pi)$. Finally, $(x, y)^\varphi = (x^\pi, y^\pi)$, because $(x, y)^\varphi = (w, z)$ and $(x, o)^\varphi = (x^\pi, o')$ are on the same ξ-line in $\mathcal{N}(\mathbf{S}')$. Thus $w = x^\pi$, and the points $(x, y)^\varphi = (w, z)$ and $(o, y)^\varphi = (o', y_\sigma)$ are on the same η-line in $\mathcal{N}(\mathbf{S}')$. Consequently $z = y^\pi$.

Let us now consider $(x, y), (x^{\sigma_\iota} +_\iota y, o)\ \mathrm{I}(\iota, c)\ \ c \in S$. Then the points $(x, y)^\varphi = (x^\pi, y^\pi)$ and $(x^{\sigma_\iota} +_\iota y, o)^\varphi = ((x^{\sigma_\iota} +_\iota y)^\pi, o')$ are on the same ι-line in $\mathcal{N}(\mathbf{S}')$. Hence $(x^{\sigma_\iota} +_\iota y)^\pi = (x^\pi)^{\sigma_\iota'} +'_\iota y^\pi$, i.e. π is a homomorphism of the partial $\mathfrak{J}$-loop $\mathbf{S}$ into the partial $\mathfrak{J}$-loop $\mathbf{S}'$.

Definition 2.13. Let $\mathbf{S} = (S, o, (\sigma_\iota)_{\iota \in \mathfrak{J}}, (+_\iota)_{\iota \in \mathfrak{J}})$ be a partial $\mathfrak{J}$-loop embedded in the partial $\mathfrak{J}$-loop $\mathbf{S}' = (S', o', (\sigma'_\iota)_{\iota \in \mathfrak{J}}, (+'_\iota)_{\iota \in \mathfrak{J}})$. We say that the partial $\mathfrak{J}$-loop $\mathbf{S}$ is *closed* in the partial $\mathfrak{J}$-loop $\mathbf{S}'$ if the following statements hold:

(i) If two of the elements $x, y, z \in S'$ belong to S and $x^{\sigma_\iota'} +'_\iota y = z$ holds in $\mathbf{S}'$ for some $\iota \in \mathfrak{J}$. Then the third element is in S as well.

(ii) If $a, b \in S$ and there are $v, w \in S'$ such that $v^{\sigma_\alpha'} +'_\alpha w = a \wedge v^{\sigma_\alpha'} +'_\alpha w = b$, $\alpha, \beta \in \mathfrak{J}$; $\alpha \neq \beta$, then $v, w \in S$ as well.

Obviously, any $\mathfrak{J}$-loop $\mathbf{S}$ embedded in the partial $\mathfrak{J}$-loop $\mathbf{S}'$ is closed in $\mathbf{S}'$

Definition 2.14. Let $\mathbf{S} = (S, o, (\sigma_\iota)_{\iota \in \mathfrak{J}}, (+_\iota)_{\iota \in \mathfrak{J}})$ be a partial $\mathfrak{J}$-loop embedded in the partial $\mathfrak{J}$-loop $\mathbf{S}' = (S', o, (\sigma'_\iota)_{\iota \in \mathfrak{J}}, (+'_\iota)_{\iota \in \mathfrak{J}})$.

A. We say that the partial $\mathfrak{J}$-loop $\mathbf{S}'$ is *generated* by the partial $\mathfrak{J}$-loop $\mathbf{S}$ if $\mathbf{S}'$ is the only partial $\mathfrak{J}$-loop containing $\mathbf{S}$ and is closed in $\mathbf{S}'$.

B. We say that the partial $\mathfrak{J}$-loop $\mathbf{S}'$ is *free over* the partial $\mathfrak{J}$-loop $\mathbf{S}$ if every homomorphism of $\mathbf{S}$ into some partial $\mathfrak{J}$-loop $\mathbf{K}$ is extendable to a homomorphism of $\mathbf{S}'$ into $\mathbf{K}$.

C. We say that the partial $\mathfrak{J}$-loop $\mathbf{S}'$ is *freely generated* by $\mathbf{S}$, if $\mathbf{S}'$ is free over its generating embedded partial $\mathfrak{J}$-loop $\mathbf{S}$.

Proposition 2.15. The partial $\mathfrak{I}$-loop $\mathbf{S} = (S, o, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ is closed in the partial $\mathfrak{I}$-loop $\mathbf{S}' = (S', o', (\sigma'_\iota)_{\iota\in\mathfrak{I}}, (+'_\iota)_{\iota\in\mathfrak{I}})$ if and only if the halfnet $\mathcal{N}(\mathbf{S}, \xi, \eta) = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ is closed in the halfnet $\mathcal{N}(\mathbf{S}', \xi, \eta) = (\mathscr{P}', \mathscr{L}', (\mathscr{L}'_\iota)_{\iota\in\mathscr{I}}, \mathrm{I}')$.

Proof. I. Let the partial $\mathfrak{I}$-loop $\mathbf{S}$ be closed in the partial $\mathfrak{I}$-loop $\mathbf{S}'$.

1. Let $(a, b) \in \mathscr{P}$. Then $a, b \in S$ and let $(\alpha, c) \in \mathscr{L}'$, whereby $(a, b)\,\mathrm{I}'(\alpha, c)$. Then $c \in S'$ and $a^{\sigma_\alpha'} +'_\alpha b = c$ in $\mathbf{S}'$. Since $a, b \in S$ and $\mathbf{S}$ is closed in $\mathbf{S}'$, then necessarily $c \in S$ and $(\alpha, c) \in \mathscr{L}$.

2. a) Let (α, a), $(\beta, b) \in \mathscr{L}$ $\alpha, \beta \in \mathfrak{I}$; $\alpha \neq \beta$. Then $a, b \in S$. Let $(v, w) := (\alpha, a) \sqcap (\beta, b) \in \mathscr{P}'$. Then $v, w \in S'$ and $v^{\sigma_\alpha'} +'_\alpha w = a$, $v^{\sigma_\beta'} +'_\beta w = b$. Since $a, b \in S$ and $\mathbf{S}$ is closed in $\mathbf{S}'$, then necessarily $v, w \in S$ and $(v, w) \in \mathscr{P}$.

b) Let (ξ, a), $(\beta, c) \in \mathscr{L}$, $\beta \in \mathfrak{I}$, $\xi \notin \mathfrak{I}$. Then $a, c \in S$. Let $(\xi, a) \sqcap (\beta, c) \in \mathscr{P}'$. Then there exists $b \in S'$ such that $a^{\sigma_\beta'} +'_\beta b = c$ and $(\xi, a) \sqcap (\beta, c) = (a, b)$. Since $\mathbf{S}$ is closed in $\mathbf{S}'$, then necessarily $b \in S$, and $(a, b) \in \mathscr{P}$.

c) Let (α, c), $(\eta, b) \in \mathscr{L}$, $\alpha \in \mathfrak{I}$, $\eta \in \mathscr{I} \setminus \mathfrak{I}$. Then $b, c \in S$. Let $(\alpha, c) \sqcap (\eta, b) \in \mathscr{P}'$. Then there exists $a \in S'$ such that $a^{\sigma_\alpha'} +'_\alpha b = c$ in S' and $(\alpha, c) \sqcap (\eta, b) = (a, b)$. Since $\mathbf{S}$ is closed in $\mathbf{S}'$, then necessarily $a \in S$, and $(a, b) \in \mathscr{P}$.

II. Let the halfnet $\mathcal{N}(\mathbf{S}, \xi, \eta)$ be closed in the halfnet $\mathcal{N}(\mathbf{S}', \xi, \eta)$.

1. Let $a, b, c \in S'$ and $a^{\sigma_\iota'} +'_\iota b = c$.

a) Let $a, b \in S$, $c \in S'$. Then $(a, b) \in \mathscr{P}$, $(\iota, c) \in \mathscr{L}'$. Since $a^{\tau_\iota'} +'_\iota b = c$, then necessarily $(a, b)\,\mathrm{I}'(\iota, c)$ in $\mathcal{N}(\mathbf{S}')$ and because $\mathcal{N}(\mathbf{S})$ is closed in $\mathcal{N}(\mathbf{S}')$, we obtain $(\iota, c) \in \mathscr{L}$. Thus $c \in S$.

b) Let $a, c \in S$, $b \in S'$. Then (ξ, a), $(\iota, c) \in \mathscr{L}$, $(a, b) \in \mathscr{P}'$ and $(a, b)\,\mathrm{I}'(\iota, c)$ because $a^{\sigma_\iota'} +'_\iota b = c$. Hence $(a, b) = (\xi, a) \sqcap (\iota, c)$. Since $\mathcal{N}(\mathbf{S})$ is closed in $\mathcal{N}(\mathbf{S}')$, then necessarily $(a, b) \in \mathscr{P}$, and $b \in S$.

c) Let $b, c \in S$, $a \in S'$. Then (η, b), $(\iota, c) \in \mathscr{L}$, $(a, b) \in \mathscr{P}'$ and $(a, b)\,\mathrm{I}'(\iota, c)$. Then $(a, b) = (\eta, b) \sqcap (\iota, c)$. Since $\mathcal{N}(\mathbf{S})$ is closed in $\mathcal{N}(\mathbf{S}')$, we have $(a, b) \in \mathscr{P}$, and $a \in S$.

2. Let $a, b \in S$, $v, w \in S'$ and $v^{\sigma_\alpha'} +'_\alpha w = a$, $v^{\sigma_\beta'} +'_\beta w = b$. Then $(v, w) = (\alpha, a) \sqcap (\beta, b)$. Since $a, b \in S$, then necessarily (α, a), $(\beta, b) \in \mathscr{L}$, and since $\mathcal{N}(\mathbf{S})$ is closed in $\mathcal{N}(\mathbf{S}')$, we have $(v, \mathrm{w}) \in \mathscr{P}$, and $v, \mathrm{w} \in S$.

Proposition 2.16. The partial $\mathfrak{I}$-loop $\mathbf{S}' = (S', o', (\sigma'_\iota)_{\iota\in\mathfrak{I}}, (+'_\iota)_{\iota\in\mathfrak{I}})$ is generated by its embedded partial $\mathfrak{I}$-loop $\mathbf{S} = (S, o', (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ exactly if $\mathcal{N}(\mathbf{S}')$ is generated by $\mathcal{N}(\mathbf{S})$.

Proof. I. Let the partial $\mathfrak{I}$-loop $\mathbf{S}'$ be generated by the partial $\mathfrak{I}$-loop $\mathbf{S}$. Suppose that there exists a halfnet $\mathcal{M} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I}) \neq \mathcal{N}(\mathbf{S}')$ such that $\mathcal{N}(\mathbf{S}) \subset \mathcal{M} \subset \mathcal{N}(\mathbf{S}')$ and that $\mathcal{M}$ is closed in $\mathcal{N}(\mathbf{S}')$. Then there must exist a partial $\mathfrak{I}$-loop $\mathbf{L}(\mathcal{M}, 0, \xi, \eta) \neq \mathbf{S}'$ closed in $\mathbf{S}'$ which is impossible.

II. Let $\mathcal{N}(\mathbf{S}')$ be a halfnet generated by the halfnet $\mathcal{N}(\mathbf{S})$. Suppose next that there exists a partial $\mathfrak{I}$-loop $\mathbf{K} \neq \mathbf{S}'$ such that $\mathbf{S} \subset \mathbf{K} \subset \mathbf{S}'$ and being closed in $\mathbf{S}'$. Then the halfnet $\mathcal{N}(\mathbf{K})$ must be closed in $\mathcal{N}(\mathbf{S}')$ which is imposible.

Proposition 2.17. The partial $\mathfrak{J}$-loop $\mathbf{S}' = (S', o', (\sigma'_\iota)_{\iota\in\mathfrak{J}}, (+'_\iota)_{\iota\in\mathfrak{J}})$ is freely generated by its embedded partial $\mathfrak{J}$-loop $\mathbf{S} = (S, o', (\sigma_\iota)_{\iota\in\mathfrak{J}}, (+_\iota)_{\iota\in\mathfrak{J}})$ precisely if the halfnet $\mathcal{N}(\mathbf{S}')$ is freely generated by the halfnet $\mathcal{N}(\mathbf{S})$.

Proof. I. Let $\mathcal{N}(\mathbf{S}')$ be freely generated by $\mathcal{N}(\mathbf{S})$. Let π be a homomorphism of the partial $\mathfrak{J}$-loop $\mathbf{S}$ into a $\mathfrak{J}$-loop $\mathbf{K}$. Then $\mathcal{N}(\mathbf{K})$ is a net and, by theorem 2.11., there exists a homomorphism (φ, Φ) of $\mathcal{N}(\mathbf{S})$ into $\mathcal{N}(\mathbf{K})$ extendable to a homomorphism (ψ, Ψ) of $\mathcal{N}(\mathbf{S}')$ into $\mathcal{N}(\mathbf{K})$ (cf. theorem 1.20). By theorem 2.12. there is a homomorphism $\varrho : \mathbf{S}' \to \mathbf{K}$ where $\varrho/\mathbf{S} = \pi$. Therefore the partial $\mathfrak{J}$-loop $\mathbf{S}'$ is freely generated by the partial $\mathfrak{J}$-loop $\mathbf{S}$.

II. Let $\mathbf{S}'$ be freely generated by $\mathbf{S}$. Let (φ, Φ) be a homomorphism of the halfnet $\mathcal{N}(\mathbf{S})$ into a net $\mathcal{M} = (\mathcal{P}, \mathcal{L}, (\mathcal{L}_\iota)_{\iota\in\mathcal{J}}, \mathrm{I})$. Since $\mathcal{N}(\mathbf{S})$ is an $0_{\xi,\eta}$-centered halfnet for one of its point 0 and a pair of indices $\xi, \eta \in \mathfrak{J}$; $\xi \neq \eta$, then $\mathcal{M}$ is necessarily $0^\varphi_{\xi,\eta}$-centered and $\mathbf{L}(\mathcal{M}, 0^\varphi, \xi, \eta)$ is a $\mathfrak{J}$-loop. By theorem 2.12. there exists a homomorphism $\pi : \mathbf{S} \to \mathbf{L}(\mathcal{M}, 0^\varphi, \xi, \eta)$ extendable to a homomorphism $\varrho : \mathbf{S}' \to \mathbf{L}(\mathcal{M}, 0^\varphi, \xi, \eta)$. Then, by theorem 2.11., there exists a homomorphism (ψ, Ψ) of the halfnet $(\mathcal{N}(\mathbf{S}')$ into the net $\mathcal{M}$ where $(\psi, \Psi)/\mathcal{N}(\mathbf{S}) = (\varphi, \Phi)$. Hence $\mathcal{N}(\mathbf{S}')$ is freely generated by $\mathcal{N}(\mathbf{S})$ (see theorem 1.21.).

Definition 2.18. Let $\mathbf{L} = (S, o, (\sigma_\iota)_{\iota\in\mathfrak{J}}, (+_\iota)_{\iota\in\mathfrak{J}})$ be a $\mathfrak{J}$-loop freely generated by its embedded partial $\mathfrak{J}$-loop $\mathbf{S}' = (S', o, (\sigma'_\iota)_{\iota\in\mathfrak{J}}, (+'_\iota)_{\iota\in\mathfrak{J}})$. Let next the following implication hold:

$$a, b \in S \Rightarrow ((a, b) \in \mathrm{Dom}(+'_\mathfrak{J}) \Leftrightarrow a = 0 \vee b = 0). \tag{3}$$

Then $\mathbf{L}$ is called a *free $\mathfrak{J}$-loop*.

Theorem 2.19. The $\mathfrak{J}$-loop $\mathbf{L} = (\mathbf{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{J}}, (+_\iota)_{\iota\in\mathfrak{J}})$ is a free $\mathfrak{J}$-loop if and only if $\mathcal{N}(\mathbf{L})$ is a free net.

Proof. The $\mathfrak{J}$-loop $\mathbf{L}$ is freely generated by its embedded partial $\mathfrak{J}$-loop $\mathbf{S}$ if and only if $\mathcal{N}(\mathbf{L})$ is freely generated by its subhalfnet $\mathcal{N}(\mathbf{S})$ (according to proposition 2.17). Besides, implication (3) follows from implication (2) of definition 1.27, and vice versa. These facts and theorem 1.28. prove the validity of theorem 2.19.

Theorem 2.20. Let $\mathbf{S} = (S, o, (\sigma_\iota)_{\iota\in\mathfrak{J}}, (+_\iota)_{\iota\in\mathfrak{J}})$ be a partial $\mathfrak{J}$-loop. Then there exists exactly one (except for an isomorphism) $\mathfrak{J}$-loop $\mathbf{L}$ which is freely generated by $\mathbf{S}$.

Theorem 2.21. Let $\mathbf{L} = (S, o, (\sigma_\iota)_{\iota\in\mathfrak{J}}, (+_\iota)_{\iota\in\mathfrak{J}})$ be a $\mathfrak{J}$-loop. Then there exist a free $\mathfrak{J}$-loop $\mathbf{L}' = (S', o', (\sigma'_\iota)_{\iota\in\mathfrak{J}}, (+'_\iota)_{\iota\in\mathfrak{J}})$ and a homomorphism π of $\mathbf{L}$ into $\mathbf{L}'$ such that $\{x^\pi \mid x \in S'\} = S$.

With regard to proposition 2.17. and theorem 2.19. the last two theorems follow from theorems 1.22. and 1.23. about nets.

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Shrnutí

POLOTKÁNĚ A PARCIÁLNÍ $\mathfrak{I}$-LUPY

JAROSLAVA JACHANOVÁ

V článku se zobecňují některé výsledky z [3], týkající se 3-tkání a lup pro k-tkáně a jim odpovídající lupové algebry. První část je věnována problematice polotkání a jejich rozšiřování. Konstruuje se zde řetězec maximálního rozšíření a zavádí se pojem volné tkáně a $P_{\xi,\eta}$-centrální volné tkáně.

Ve druhé části je definována parciální $\mathfrak{I}$-lupa, která slouží ke koordinatisaci $P_{\xi,\eta}$-centrálních polotkání. Tato struktura je jistým zobecněním $\mathfrak{I}$-lup z [5]. Dále se zde studuje homomorfismus parciálních $\mathfrak{I}$-lup a volných $\mathfrak{I}$-lup v souvislosti s homomorfismem polotkání a volných tkání.

Buď $\mathscr{H} = (\mathscr{P}, \mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ polotkáň. Pak existuje tkáň $\mathscr{N}$, která je volně generována polotkání $\mathscr{H}$. Jestliže $\mathscr{N}, \mathscr{N}'$ jsou dvě tkáně volně generované polotkání $\mathscr{H}$, pak jsou isomorfní.

Buď $\mathscr{N} = (\mathscr{P}, \mathscr{L}, (\mathscr{L}_\iota)_{\iota\in\mathscr{I}}, \mathrm{I})$ tkáň. Pak existuje volná tkáň $\mathscr{M}$ a homomorfismus (φ, Φ) tkáně $\mathscr{M}$ do $\mathscr{N}$ takový, že $\mathscr{M}^{(\varphi,\Phi)} = \mathscr{N}$.

$\mathscr{N}$ je $P_{\xi,\eta}$-centrální volná tkáň právě tehdy, když je to volná tkáň.

$\mathbf{L} = (\mathrm{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ je volná $\mathfrak{I}$-lupa právě tehdy, když $\mathscr{N}(\mathbf{L}, \xi, \eta)$ je volná tkáň.

Buď $\mathbf{S} = (\mathrm{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ parciální $\mathfrak{I}$-lupa. Potom existuje, až na isomorfismus, právě jedna $\mathfrak{I}$-lupa $\mathbf{L}$, která je volně generována $\mathbf{S}$.

Buď $\mathbf{L} = (\mathrm{S}, o, (\sigma_\iota)_{\iota\in\mathfrak{I}}, (+_\iota)_{\iota\in\mathfrak{I}})$ $\mathfrak{I}$-lupa. Pak existuje volná $\mathfrak{I}$-lupa $\mathbf{L}' = (\mathrm{S}', o', (\sigma'_\iota)_{\iota\in\mathfrak{I}}, (+'_\iota)_{\iota\in\mathfrak{I}})$ a homomorfismus π $\mathbf{L}$ do $\mathbf{L}'$ tak, že $\{x^\pi \mid x \in \mathrm{S}'\} = \mathrm{S}$.

Резюме

ПОЛОСЕТИ И ЧАСТИЧНЫЕ $\mathfrak{I}$-ЛУПЫ

ЯРОСЛАВА ЯХАНОВА

В этой работе обобщаются для к-сетей (и для их соответствующих алгебраических выражений) некоторые результаты знакомые для 3-сетей.

В первой главе сосредоточнится внимание на полосети и их разширения. Показывается, что каждая сеть гомоморфным образом некоторой свободной сети.

Во второй главе вводится понятие частичной $\mathfrak{I}$-лупы — алгебраической структуры при помощи которой координатизируется полосеть степени к. В дальнейшем здесь изучаются гомоморфизмы частичных $\mathfrak{I}$-луп и их свойства.