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DERIVATIONS ON THE ALGEBRA OF DIFFERENTIAL FORMS OF HIGHER ORDER ON A MANIFOLD

JAROSLAV CARBOL and JIŘÍ VANŽURA

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Dedicated to Prof. Miroslav Laitoch on his 60th birthday

In this paper we consider a C^∞-differentiable paracompact manifold M. All the objects on M will also be of class C^∞. Our main goal is to describe the structure of derivations on the graded algebra of forms of order $\leqq r$ on M. We generalize the results of A. Frölicher and A. Nijenhuis (see [1]) concerning derivations on the de Rham algebra.

We start with the description of the above mentioned graded algebra. Let $T(M)$ denote the tangent bundle of M, and let $J^rT(M)$ be its r-th jet prolongation ($r \geqq 0$). (It is worth noticing that all our results not involving the exterior differential remain valid if we replace $T(M)$ by arbitrary vector bundle over M.) We consider $\Lambda(J^rT(M))^* = \bigoplus_{p=-\infty}^{\infty} \Lambda^p(J^rT(M))^*$, i.e. the bundle of exterior algebras associated with the dual $(J^rT(M))^*$ of the vector bundle $J^rT(M)$. We denote by $\Phi^r = \bigoplus_{p=-\infty}^{\infty} \Phi^r_p$ the algebra of sections (over M) of $\Lambda(J^rT(M))^*$ with the usual multiplication. Φ^r is obviously a commutative (in the graded sense) and associative graded algebra over the reals. We shall call it algebra of forms of order $\leqq r$ on M.

There is another way of describing Φ^r. Let $\mathfrak{X}(M)$ denote as usual the real Lie algebra of vector fields on M. By a p-form of order $\leqq r$ on M we shall call any real p-form φ on $\mathfrak{X}(M)$ with values in smooth functions on M satisfying the following property: If $x_0 \in M$ is a point, and if $X_1, \dots, X_p \in \mathfrak{X}(M)$ are such that $j^r_{x_0}(X_i) = 0$ for some $1 \leq i \leq p$ (where $j^r_{x_0}$ denotes the r-th jet at the point x_0), then $(\varphi(X_1, \dots, X_p))(x_0) = 0$. It is easy to see that the vector space of p-forms of order $\leqq r$ on M can be identified with Φ^r_p. We shall use this identification in the sequel without any further comment.

If $0 \leqq s \leqq r$ then the natural projection $J^rT(M) \to J^sT(M)$ induces a homomorphism $\Phi^s \to \Phi^r$ of graded algebras. It is obviously injective, which enables us to consider Φ^s as a graded subalgebra of Φ^r. As already mentioned, we intend to study derivations on the algebra Φ^r. We recall

Definition 1. A derivation of degree k on Φ^s with values in Φ^r is any real linear mapping $D : \Phi^s \to \Phi^r$ satisfying

(i) $D(\Phi^s_p) \subset \Phi^r_{p+k}$

(ii) $D(\varphi_p \wedge \varphi_q) = D\varphi_p \wedge \varphi_q + (-1)^{kp}\varphi_p \wedge D\varphi_q$

for any $\varphi_p \in \Phi^s_p$, $\varphi_q \in \Phi^s_q$.

In the case $r = s$ we call D simply derivation of degree k on Φ^r.

Along the same lines as in [1] one can prove the following three lemmas:

Lemma 1. Let D be a derivation on Φ^s with values in Φ^r, and let $\varphi, \psi \in \Phi^s_p$. If $\varphi/U = \psi/U$ with U being an open subset of M, then there is also $(D\varphi)/U = (D\psi)/U$.

Lemma 2. Any derivation D on Φ^s with values in Φ^r is uniquely determined by its values on Φ^s_0 and Φ^s_1.

Lemma 3. Any linear mapping $D : \Phi^s_0 \oplus \Phi^s_1 \to \Phi^r$ satisfying

(i) $D\Phi^s_p \subset \Phi^r_{p+k}$ for $p = 0, 1$

(ii) $D(\varphi_p \wedge \varphi_q) = D\varphi_p \wedge \varphi_q + (-1)^{kp}\varphi_p \wedge D\varphi_q$ for $p + q \leqq 1$, $\varphi_p \in \Phi^s_p$, $\varphi_q \in \Phi^s_q$

can be extended to a derivation of degree k on Φ^s with values in Φ^r.

Corollary to Lemma 3. There are no non-trivial derivations on Φ^s with values in Φ^r of degree $k \leqq -2$.

Now we start with the study of special derivations. We introduce

Definition 2. A derivation D on Φ^s with values in Φ^r is called derivation of type i_* if it satisfies $D\Phi^s_0 = 0$.

We must now slightly generalize the definition of the $\overline{\wedge}$-product from [1]. Let V and W be real vector spaces, and let $\pi : W \to V$ be a homomorphism. Let $\varphi \in \Lambda^p V^*$, and $L \in V \otimes \Lambda^{k+1} W^*$. We define $\varphi \overline{\wedge} L \in \Lambda^{p+k} W^*$ by the formula

$$(\varphi \overline{\wedge} L)(w_1, \ldots, w_{p+k}) =$$
$$= \frac{1}{(p-1)!(k+1)!} \Sigma \operatorname{sg} \alpha \,.\, \varphi(L(w_{\alpha_1}, \ldots, w_{\alpha_{k+1}}), \pi w_{\alpha_{k+2}}, \ldots, \pi w_{p+k})$$

Here $w_1, \ldots, w_{p+k}$ are elements of W, the sum is taken over all permutations α of $p + k$ elements, and $\operatorname{sg} \alpha$ denotes the sign of α. (If $p = 0$ we define $\varphi \overline{\wedge} L = 0$.) Taking $\psi \in \Lambda^q V^*$ it is a matter of computations to show that there is

$$(\varphi \wedge \psi) \overline{\wedge} L = (\varphi \overline{\wedge} L) \wedge \pi^*\psi + (-1)^{kp}\pi^*\varphi \wedge (\psi \overline{\wedge} L),$$

where $\pi^* : \Lambda V^* \to \Lambda W^*$ is induced by π. It is completely obvious that the $\overline{\wedge}$-product can be applied to vector bundles. We take $J^sT(M)$ and $J^rT(M)$ in the role of V and W respectively. In the role of π we take the natural projection $J^rT(M) \to$ $\to J^sT(M)$. Let φ be a section of $\Lambda^p(J^sT(M))^*$, i.e. $\varphi \in \Phi^s$, and let L be a section of $J^sT(M) \otimes \Lambda^{k+1}(J^rT(M))^*$. Then we can define

$$i_L\varphi = \varphi \,\overline{\wedge}\, L.$$

It follows from the above formula that i_L is a derivation on Φ^s with values in Φ^r of degree k, and it is obvious that i_L is of type i_*.

Proposition 1. Any derivation D on Φ^s with values in Φ^r of degree $k \geqq -1$ which is of type i_* can be uniquely expressed in the form $D = i_L$, where L is a section of the bundle $J^sT(M) \otimes \Lambda^{k+1}(J^rT(M))^*$.

Proof: Let $U \subseteq M$ be an open subset, and let $u_1, \ldots, u_a$ be a local basis of $J^sT(M)$ on U. Let $\omega^1, \ldots, \omega^a$ be the corresponding dual basis of $(J^sT(M))^*$ on U. We define L on U by the formula

$$L = \sum_{i=1}^{a} u_i \otimes D\omega^i.$$

One can check that this definition does not depend on the choice of the basis. The rest of the proof we leave to the reader.

Similarly as on the de Rham algebra we can define the exterior derivation d on Φ^r. It is a derivation with values in Φ^r of degree 1. For its definition we shall use our second description of Φ^r. Let $\varphi \in \Phi^r_p$, and let $X_1, \ldots, X_{p+1} \in \mathfrak{X}(M)$. We define $d\varphi$ by the usual formula

$$\mathrm{d}\varphi(X_1, \ldots, X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} X_i\varphi(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) +$$
$$+ \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{p+1}).$$

The reader can verify that $d\varphi$ really belongs to Φ^r and that $d^2 = 0$. Moreover it is obvious that the exterior derivation d on Φ^r restricted to the subalgebra Φ^s coincides with the exterior derivation on Φ^s. By virtue of this fact we shall not distinguish between these two derivations.

Definition 3. Let D be a derivation on Φ^s with values in Φ^r of degree k. D is called derivation of type d_* if it satisfies

$$Dd = (-1)^k dD.$$

Let us take a derivation i_L on Φ^s with values in Φ^r of degree $k - 1$ and type i_*. It is easy to see that $d_L = [i_L, d] = i_L d - (-1)^{k-1} d i_L$ is a derivation on Φ^s with values in Φ^r of degree k and that this derivation is of type d_*. If $s = 0$ we get moreover

Proposition 2. Any derivation D on Φ^0 with values in Φ^r of degree k and type d_* can be uniquely expressed in the form $D = [i_L, d]$, where i_L is a derivation of type i_* as above.

Before we start with the proof of this proposition we shall describe the following construction. Let D be any derivation on Φ^s with values in Φ^r of degree k. We take any point $x \in M$ and fix arbitrary vectors $w_1, \ldots, w_k$ from the fibre of $J^rT(M)$ over x. Let f be any function defined around x. We look at the mapping $f \to Df(w_1, \ldots, w_k) \in \mathbf{R}$. Since D is a derivation we can see that this mapping is a tangent vector of M at x. Thus we get a section $\varrho(D)$ of $T(M) \otimes \Lambda^k(J^rT(M))^*$ defined by

$$(\varrho(D)(w_1, \ldots, w_k))f = Df(w_1, \ldots, w_k).$$

Proof of Proposition 2: We notice first that any derivation D_1 on Φ^0 with values in Φ^r of type d_* is uniquely determined by its values on Φ_0^0. Taking any $\varphi \in \Phi_1^0$ and any chart with coordinate functions $(x^1, \ldots, x^m)$ we can write $\varphi = \sum_{i=1}^{m} f_i dx^i$. Since D_1 is of type d_* we get

$$D_1\varphi = \sum_{i=1}^{m} D_1(f_i dx^i) = \sum_{i=1}^{m} (D_1 f_i \wedge dx^i + f_i D_1 dx^i) = \\ = \sum_{i=1}^{m} (D_1 f_i \wedge dx^i + (-1)^k f_i d D_1 x^i).$$

We have thus shown that the values of D_1 on Φ_0^0 determine their values on Φ_1^0, and therefore by virtue of Lemma 2 the derivation D_1 is completely determined.

For any $\varphi \in \Phi_0^0$ we have

$$d_{\varrho(D)}\varphi = i_{\varrho(D)}d\varphi - (-1)^{k-1} d i_{\varrho(D)}\varphi = d\varphi \mathbin{\overline{\wedge}} \varrho(D) = d\varphi(\varrho(D)) = (\varrho(D))\,\varphi = D\varphi.$$

Both D and $d_{\varrho(D)}$ are derivations of type d_*. They coincide on Φ_0^0 and thus by the first part of the proof $D = d_{\varrho(D)}$.

We introduce now the following notations:

$\mathscr{R}_s^r$ = the set of all derivations on Φ^s with values in Φ^r

$\mathscr{I}_s^r = \{D \in \mathscr{R}_s^r;\ D \text{ is of type } i_*\}$

$\mathscr{D}_0^r = \{D \in \mathscr{R}_0^r;\ D \text{ is of type } d_*\}$.

All the above three sets have a natural structure of vector space. $\mathscr{I}_s^r$ is a subspace of $\mathscr{R}_s^r$. The corresponding inclusion we denote by ι. The mapping $D \in \mathscr{R}_s^r \to d_{\varrho(D)} \in \mathscr{D}_0^r$ defines a homomorphism $\sigma : \mathscr{R}_s^r \to \mathscr{D}_0^r$. We have seen in the proof of the previous proposition that for any function $\varphi \in \Phi_0^0$ we have $\sigma(D)\,\varphi = D\varphi$. We shall need this formula below.

Proposition 3. The sequence (with $0 \leqq s \leqq r$)

$$0 \to \mathscr{I}_s^r \xrightarrow{\iota} \mathscr{R}_s^r \xrightarrow{\sigma} \mathscr{D}_0^r \to 0$$

is exact.

Proof: ι is an inclusion and therefore is injective. If $D \in \mathscr{I}_s^r$ then $\varrho(D) = 0$, which shows that $\operatorname{im} \iota \subseteq \ker \sigma$. Now let $D \in \mathscr{R}_s^r$ be such that $\sigma(D) = 0$. For any function $\varphi \in \Phi_0^s = \Phi_0^0$ we have $D\varphi = \sigma(D)\,\varphi = 0$ so that $D \in \mathscr{I}_s^r$, and thus $\ker \sigma \subseteq \operatorname{im} \iota$. Finally let $D \in \mathscr{D}_0^r$. We denote by π' the natural projection $J^sT(M) \to \to T(M)$. Let us choose any injective vector bundle homomorphism $\pi_1 : T(M) \to \to J^sT(M)$ such that $\pi' \circ \pi_1 = id$.Then $\varrho(D)$ is a section of $T(M) \otimes \Lambda^k(J^rT(M))^*$ and $\pi_1\varrho(D)$ is a section of $J^sT(M) \otimes \Lambda^k(J^rT(M))^*$. Thus $d_{\pi_1\varrho(D)} \in \mathscr{R}_s^r$. For any $\varphi \in \Phi_0^0$ we get

$$(\sigma(d_{\pi_1\varrho(D)}))\,\varphi = d_{\pi_1\varrho(D)}\varphi = d\varphi(\pi_1\varrho(D)) = d\varphi(\varrho(D)) = D\varphi.$$

This shows that $\sigma(D_1)$ and D coincide on Φ_0^0. Since they are both of type d_* we have $\sigma(D_1) = D$. This finishes the proof.

Proposition 4. The exact sequence (for any $r \geqq 0$)

$$0 \to \mathscr{I}_0^r \xrightarrow{\iota} \mathscr{R}_0^r \xrightarrow{\sigma} \mathscr{D}_0^r \to 0$$

has a natural splitting.

Proof: The natural splitting $\sigma' : \mathscr{D}_0^r \to \mathscr{R}_0^r$ is the inclusion $\mathscr{D}_0^r \subseteq \mathscr{R}_0^r$. For any $D \in \mathscr{D}_0^r$ and any $\varphi \in \Phi_0^0$ we have namely $(\sigma\sigma'(D))\,\varphi = (\sigma(D))\,\varphi = D\varphi$. Again $\sigma\sigma'(D)$ and D are of type d_* and thus $\sigma\sigma'(D) = D$.

Corollary: Every derivation $D \in \Phi_0^r$ has a unique decomposition $D = D_1 + D_2$ with D_1 being of type i_* and D_2 of type d_*.

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Souhrn

DERIVACE NA ALGEBŘE DIFERENCIÁLNÍCH FOREM VYŠŠÍHO ŘÁDU NA VARIETĚ

JAROSLAV CARBOL a JIŘÍ VANŽURA

V práci se zkoumá struktura prostoru derivací algebry diferenciálních forem vyššího řádu na diferencovatelné varietě. Získané výsledky zobecňují známé výsledky Frölichera a Nijenhuise o derivacích de Rhamovy algebry forem nultého řádu.

Резюме

ДИФФЕРЕНЦИРОВАНИЯ НА АЛГЕБРЕ ДИФФЕРЕНЦИАЛЬНЫХ ФОРМ ВЫСШЕГО ПОРЯДКА НА МНОГООБРАЗИИ

ЯРОСЛАВ ЦАРБОЛ и ЙИРЖИ ВАНЖУРА

В статье изучается структура пространства дифференцирований алгебры дифференциальных форм высшего порядка на дифференцируемом многообразии. Полученные результаты обобщают известные результаты Фрёлихера и Нейенхейса о дифференцированиях алгебры де Рама дифференциальных форм нулевого порядка.