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MODAL OPERATORS ON ORDERED SETS

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Let A be an *ordered set*, $X \subseteq A$. Then we put $L(X) = \{a \in A; a \leqq x \text{ for all } x \in X\}$, $U(X) = \{b \in A;\ x \leqq b \text{ for all } x \in X\}$. If $x_1, \dots, x_n \in A$, then we shall write $L(x_1, \dots, x_n)$ and $U(x_1, \dots, x_n)$ instead of $L(\{x_1, \dots, x_n\})$ and $U(\{x_1, \dots, x_n\})$, respectively. An ordered set A is called *lower directed* if $L(x, y) \neq \varnothing$ for each x, $y \in A$. Let us remind that a *closure operator* on an ordered set A is any mapping $\varphi : A \to A$ such that for each $x, y \in A$ it is

1. $x \leqq \varphi(x)$, 2. $\varphi\varphi(x) = \varphi(x)$, 3. $x \leqq y \Rightarrow \varphi(x) \leqq \varphi(y)$.

Macnab, in [1], introduces the notion of a *modal operator* on a $\wedge$-semilattice L as a mapping $\varphi : L \to L$ such that for each $x, y \in L$ it is

1. $x \leqq \varphi(x)$, 2. $\varphi\varphi(x) = \varphi(x)$, 3. $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$.

In this paper, that notion is generalized for an arbitrary ordered set and there are studied its properties.

Definition. If A is an ordered set, then a mapping $\varphi : A \to A$ is called a ***modal operator*** on A if for each $x, y \in A$ it is

(1) $x \leqq \varphi(x)$;

(2) $\varphi\varphi(x) = \varphi(x)$;

(3) $U(\varphi(L(x, y))) = U(L(\varphi(x), \varphi(y)))$.

Note. a) The identity is a modal operator for any ordered set. b) If a lower directed set A contains the greatest element 1, then the mapping φ such that $\varphi(x) = 1$ for each $x \in A$ is a modal operator on A.

Theorem 1. *Any modal operator is a closure operator.*

Proof. Let $x, y \in A$, $x \leqq y$. Then

$$\varphi(y) \in U(L(\varphi(x), \varphi(y))) = U(\varphi(L(x, y))) = U(\varphi(L(x))) =$$
$$= U(\varphi(L(x, x))) = U(L(\varphi(x), \varphi(x))) = U(\varphi(x)),$$

hence $\varphi(x) \leqq \varphi(y)$.

Note. Let us show that there exist closure operators which are not modal operators. We can consider the ordered set A specified by the diagram in Figure 1.

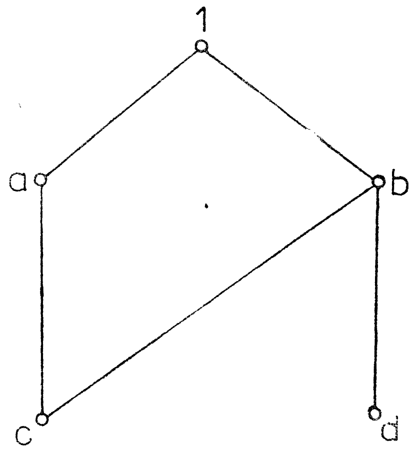

Fig. 1

We denote by φ a mapping such that

$$\varphi(1) = \varphi(a) = 1, \qquad \varphi(b) = \varphi(d) = b, \qquad \varphi(c) = c.$$

It is clear that φ is a closure operator on A. But it is

$$U(\varphi(L(c, d))) = U(\varphi(\emptyset)) = U(\emptyset) = A,$$
$$U(L(\varphi(c), \varphi(d))) = U(L(c, b)) = U(c) = \{c, a, b, 1\},$$

that means φ is not a modal operator.

But it holds the following assertion.

Theorem 2. *A closure operator on an ordered set A is a modal operator on A if and only if*

$$(*)\ \forall\, x, y \in A; \qquad U(\varphi(L(x, y))) \subseteq U(L(\varphi(x), \varphi(y))).$$

Proof. Let φ be a closure operator on A, x, y, $z \in A$. If $\varphi(z) \in \varphi(L(x, y))$, then $\varphi(z) \in L(\varphi(x), \varphi(y))$, and thus $\varphi(L(x, y)) \subseteq L(\varphi(x), \varphi(y))$. Hence $U(L(\varphi(x), \varphi(y))) \subseteq$ $\subseteq U(\varphi(L(x, y)))$, therefore φ is a modal operator.

Note. If φ satisfies conditions (1) and (2) from the definition of a modal operator, then the condition (*) can be satisfied without φ being a closure operator.

Let us consider the ordered set $A = \{0, a, 1\}$, where $0 < a < 1$, and the mapping $\varphi : A \to A$ such that

$$\varphi(0) = \varphi(1) = 1, \qquad \varphi(a) = a.$$

Then it is $0 \leqq a$, but $\varphi(0) \not\leqq \varphi(a)$. But at the same time

$$\begin{aligned} &U(\varphi(L(0, a))) = \{1\}, && U(L(\varphi(0), \varphi(a))) = \{a, 1\}; \\ &U(\varphi(L(0, 1))) = \{1\}, && U(L(\varphi(0), \varphi(1))) = \{1\}; \\ &U(\varphi(L(a, 1))) = \{1\}, && U(L(\varphi(a), \varphi(1))) = \{a, 1\}; \\ &U(\varphi(L(0))) = \{1\}, && U(L(\varphi(0))) = \{1\}; \\ &U(\varphi(L(a))) = \{1\}, && U(L(\varphi(a))) = \{a, 1\}; \\ &U(\varphi(L(1))) = \{1\}, && U(L(\varphi(1))) = \{1\}. \end{aligned}$$

Theorem 3. *If $(A, \leqq, \wedge)$ is a semilattice, then $\varphi : A \to A$ is a modal operator on $(A, \leqq)$ if and only if it is a modal operator on $(A, \wedge)$.*

Proof. a) Let φ be a modal operator on $(A, \wedge)$ and $z \in A$. Then $L(x, y) = L(x \wedge y)$ and hence $z \in U(\varphi(L(x, y)))$ if and only if $z \geqq \varphi(v)$ for each $v \leqq x \wedge y$. This is, by Theorem 1, equivalent with $z \geqq \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ and that holds if and only if $z \in U(\varphi(x) \wedge \varphi(y)) = U(\varphi(x), \varphi(y))$.

b) Let us suppose that φ is a modal operator on $(A, \leqq)$. As $x \wedge y \leqq x, y$, it is, by Theorem 1, $\varphi(x \wedge y) \leqq \varphi(x) \wedge \varphi(y)$. Let $z \in A$ be such element that $\varphi(x \wedge y) \leqq z$. Then

$$\begin{aligned} z \in U(L(\varphi(x \wedge y))) &= U(L(\varphi(x \wedge y, x \wedge y))) = U(\varphi(L(x \wedge y, x \wedge y))) = \\ &= U(\varphi(L(x, y))) = U(L(\varphi(x), \varphi(y))) = U(\varphi(x) \wedge \varphi(y))), \end{aligned}$$

hence $z \geqq \varphi(x) \wedge \varphi(y)$. Therefore $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$.

Note. In [2] it is introduced the notion of a *translation* of an ordered set as a mapping $\varphi : A \to A$ satisfying

$$\forall\, x, y \in A; \qquad \varphi(U(x, y)) = U(\varphi(x), y).$$

(In [2] is used the dual notion.) It is proved that translations are closure operators, too, and that any two translations commute. Let us show that in contrast to translations modal operators do not commute in general.

Let us consider the ordered set $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. We define $\varphi(0) = \varphi(a) = a$, $\varphi(b) = \varphi(1) = 1$, $\psi(0) = \psi(a) = \psi(b) = b$, $\psi(1) = 1$. It can be easily shown that φ and ψ are modal operators on A. But it holds

$$\varphi\psi(a) = \varphi(b) = 1, \qquad \psi\varphi(a) = \psi(a) = b,$$

i.e. $\varphi\psi = \psi\varphi$.

If φ is any mapping of A into A, then we put $I_\varphi = \{a \in A;\ \varphi(a) = a\}$.

Note. It is evident that if φ is a closure operator on an ordered set A, then it holds:

(a) $\forall\, a, b \in I_\varphi$; $a \wedge b$ exists $\Rightarrow a \wedge b \in I_\varphi$.

In [2] it is proved that for any translation φ on A moreover

(b) $\forall\, x \in A,\, b \in I_\varphi$; $b \leqq x \Rightarrow x \in I_\varphi$.

But, in general, for modal operators (b) is not satisfied. See e.g. the modal operator φ from the last note.

Theorem 4. *If for closure operators φ and ψ on A it is $I_\varphi = I_\psi = I$, then $\varphi = \psi$.*

Proof. Let $x \in A$. As, by condition 2 from the definition of a closure operator, it is $\varphi(x), \psi(x) \in I$, it holds also $\varphi\psi(x) = \psi(x)$ and $\psi\varphi(x) = \varphi(x)$. But then evidently $\psi(x) \leqq \varphi\psi(x) = \varphi(x)$ and $\varphi(x) \leqq \varphi\psi(x) = \psi(x)$, hence $\varphi(x) = \psi(x)$.

Theorem 5. *For any closure operators φ and ψ on A it is $\varphi \leqq \psi$ if and only if $\varphi\psi = \psi$.*

Proof. If $\varphi \leqq \psi$, $x \in A$, then $\varphi\psi(x) \leqq \psi\psi(x) = \psi(x)$ and $\psi(x) \leqq \varphi\psi(x)$, therefore $\varphi\psi = \psi$.

Conversely, if $\varphi\psi = \psi$, then $\varphi(x) \leqq \varphi\psi(x) = \psi(x)$.

Note. If φ and ψ are translations on A, then for each $x, y \in A$ it is $\varphi\psi(U(x, y)) = \varphi(U(\psi(x), y)) = U(\varphi\psi(x), y)$, hence the composition of two translations is a translation as well. Let us show that the composition of two modal operators need not be, in general, a modal operator (not even a closure operator).

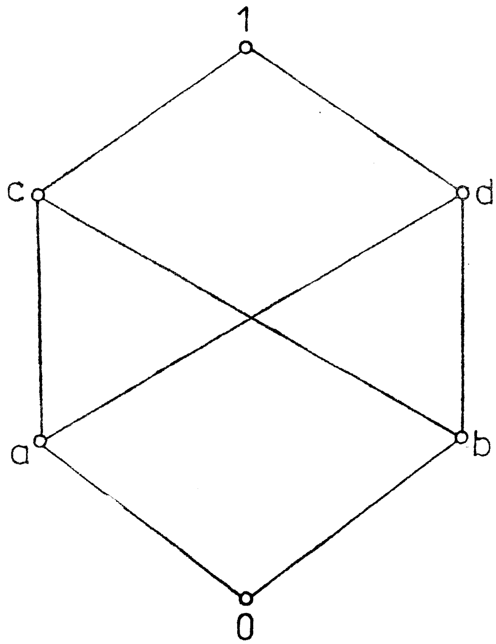

Fig. 2

Let us consider the ordered set B specified by the diagram of Figure 2.

Put

$$\varphi(0) = \varphi(a) = a, \qquad \varphi(b) = \varphi(c) = c, \qquad \varphi(d) = \varphi(1) = 1;$$
$$\psi(0) = \psi(b) = b, \qquad \psi(a) = \psi(d) = d, \qquad \psi(c) = \psi(1) = 1.$$

It holds that φ and ψ are modal operators on B but

$$\varphi\psi\varphi\psi(0) = \varphi\psi\varphi(b) = \varphi\psi(c) = \varphi(1) = 1,$$
$$\varphi\psi(0) = \varphi(b) = c,$$

thus $\varphi\psi\varphi\psi \neq \varphi\psi$.

Theorem 6. *If φ and ψ are closure operators on A, then the following conditions are equivalent:*

1. $\varphi\psi = \psi\varphi$.
2. $\varphi\psi$ *and* $\psi\varphi$ *are closure operators.*
3. $\varphi\psi\varphi\psi = \varphi\psi$ *and* $\psi\varphi\psi\varphi = \psi\varphi$.

Proof. $1 \Rightarrow 2$: Let $\varphi\psi = \psi\varphi$. Then for each $x, y \in A$ it is

1. $x \leqq \varphi(x) \leqq \varphi\psi(x)$;
2. $\varphi\psi\varphi\psi(x) = \varphi\varphi\psi\psi(x) = \varphi\varphi\psi(x) = \varphi\psi(x)$;
3. $x \leqq y \Rightarrow \psi(x) \leqq \psi(y) \Rightarrow \varphi\psi(x) \leqq \varphi\psi(y)$.

Therefore $\varphi\psi = \psi\varphi$ is a closure operator

$2 \Rightarrow 3$: Trivial.

$3 \Rightarrow 1$: If $\varphi\psi\varphi\psi = \varphi\psi$ and $\psi\varphi\psi\varphi = \psi\varphi$, then for each $x \in A$ it is

$$\varphi\psi(x) = \varphi\psi\varphi\psi(x) \geqq \psi\varphi\psi(x) \geqq \psi\varphi(x),$$
$$\psi\varphi(x) = \psi\varphi\psi\varphi(x) \geqq \varphi\psi\varphi(x) \geqq \varphi\psi(x),$$

thus $\varphi\psi = \psi\varphi$.

Corollary. a) [2, Théorème 4] *Any two translations of an ordered set commute.*

b) *If modal operators φ and ψ on a $\wedge$-semilattice commute, then $\varphi\psi$ and $\psi\varphi$ are modal operators as well.*

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MODÁLNÍ OPERÁTORY NA USPOŘÁDANÝCH MNOŽINÁCH

Souhrn

V článku je zaveden a studován pojem modálního operátoru, který je speciálním případem uzávěrového operátoru na uspořádané množině.

МОДАЛЬНЫЕ ОПЕРАТОРЫ НА УПОРЯДОЧЕННЫХ МНОЖЕСТВАХ

Резюме

В статье введено и изучается понятие модального оператора, который является специальным случаем оператора замыкания на упорядоченных множествах.