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COMMON INCREASING DISPERSIONS OF CERTAIN LINEAR SECOND ORDER DIFFERENTIAL EQUATIONS

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1. Introduction

Borůvka ([4], [5]) and Staněk ([6] – [8]) investigated a structure of intersection of groups of dispersions in two linear second order differential equations, where at least one of them was oscillatory. The problem of the present paper is to study this intersection in assuming that at least one of those equations is disconjugate.

2. Basic concepts and lemmas

Throughout we shall concern ourselves with differential equations of the type

$$y'' = q(t)\,y, \qquad q \in C^0(\mathbf{R}), \tag{q}$$

only.

A function $\alpha \in C^0(\mathbf{R})$ is called the (first) phase of (q) if there exist independent solutions u, v of (q) such that

$$\operatorname{tg} \alpha(t) = u(t)/v(t) \qquad \text{for } t \in \mathbf{R} - \{t;\, v(t) = 0\}.$$

If α is a phase of (q), then $\alpha \in C^3(\mathbf{R})$, $\alpha'(t) \neq 0$ for $t \in \mathbf{R}$ and the coefficient q of (q) is uniquely determined by the phase α as: $q(t) = -\{\alpha, t\} - \alpha'^2(t)$, $t \in \mathbf{R}$, where $\{\alpha, t\} := \alpha'''(t)/(2\alpha'(t)) - (3/4)\,(\alpha''(t)/\alpha'(t))^2$ denotes the Schwarz derivative of the function α at the point t.

We indicate by $\mathfrak{E}$ the set of phases relative to the equation $y'' = -y$. This set constitutes a group with respect to the composition of functions, whereby $\varepsilon \in \mathfrak{E}$ exactly if

$$\operatorname{tg} \varepsilon(t) = (a_{11} + a_{12} \operatorname{tg} t)/(a_{21} + a_{22} \operatorname{tg} t), \tag{1}$$

for all $t \in \mathbf{R}$ where the expressions on both sides of (1) being meaningful and $\det a_{ij} \neq 0$.

Equation (q) is disconjugate if every (nontrivial) solution of this equation has one zero at most. Every disconjugate equation (q) is either pure disconjugate or specially disconjugate.

Definition 1 ([2]). *Equation* (q) *is pure disconjugate* (*specially disconjugate*) *exactly if there exists a phase* α *of* (q) *such that*

$$\lim_{t \to -\infty} \alpha(t) = 0, \qquad \lim_{t \to \infty} \alpha(t) = \pi/2,$$

$$(\lim_{t \to -\infty} \alpha(t) = -\pi/2, \qquad \lim_{t \to \infty} \alpha(t) = \pi/2).$$

A function $X \in C^3(\mathbf{R})$, $X'(t) \neq 0$ for $t \in \mathbf{R}$ is called the complete dispersion (of the 1st kind) of (q) if $X(\mathbf{R}) = \mathbf{R}$ and X is a solution (on $\mathbf{R}$) of a third order nonlinear differential equation

$$-\{X, t\} + X'^2 \,.\, q(X) = q(t).$$

The function $X(t) := t + \pi$, $t \in \mathbf{R}$, is a complete dispersion of (q) exactly if the function q is π-periodic. The complete dispersion of (q) is for brevity called the dispersion of (q).

Let $\mathscr{L}_q(\mathscr{L}_q^+)$ denote the set of all (increasing) dispersions of (q) constituting a group under the rule of composition of functions. Then:

(i) $X \in \mathscr{L}_q$ exactly if the function $u(X(t))/\sqrt{|X'(t)|}$ for every solution u of (q) is again a solution of (q) on $\mathbf{R}$.

(ii) Let α be a phase of (q) $I := \alpha(\mathbf{R})$. If $X \in \mathscr{L}_q$, then there exists an $\varepsilon \in \mathfrak{E}$: $X(t) = \alpha^{-1}\varepsilon\alpha(t)$, $t \in \mathbf{R}$, and vice versa, if a composite function $\alpha^{-1}\varepsilon\alpha$ is defined on $\mathbf{R}$ for an $\varepsilon \in \mathfrak{E}, \varepsilon(I) = I$, then $X \in \mathscr{L}_q$ for $X := \alpha^{-1}\varepsilon\alpha$.

All the above definitions and properties are given in [2], [3].

Let $\mathscr{S} \subset \mathscr{L}_q^+$ be a subgroup of the group $\mathscr{L}_q^+$. Conformably with [4], [5] let us say that $\mathscr{S}$ is a planar group if there exists a unique $X \in \mathscr{S}$ to every point $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}$, such that $X(t_0) = x_0$.

Let (q) be an oscillatory equation. It then follows from [1], [4], [5], [7] that $\mathscr{S} \subset \mathscr{L}_q^+$ is a planar group exactly if there exists a phase α of (q) such that $\mathscr{S} = \{\alpha^{-1}(\alpha(t) + a);\, a \in \mathbf{R}\}$.

Lemma 1. *Letting* $\mathfrak{E}_1 := \{\varepsilon|_{(0,\pi/2)};\ \varepsilon \in \mathfrak{E},\ \varepsilon(0) = 0,\ \varepsilon(\pi/2) = \pi/2\}$, *then* $\bar{\varepsilon} \in \mathfrak{E}_1$ *exactly if*

$$\bar{\varepsilon}(t) = \operatorname{arctg}(A \operatorname{tg} t), \qquad t \in (0, \pi/2), \tag{2}$$

with $A > 0$.

Proof. ($\Rightarrow$) Let $\bar{\varepsilon} \in \mathfrak{E}_1$. Then $\bar{\varepsilon} = \varepsilon|_{(0,\pi/2)}$, where $\varepsilon \in \mathfrak{E}$, $\varepsilon(0) = 0$, $\varepsilon(\pi/2) = \pi/2$. Let for ε be (1) true with $\det a_{ij} \neq 0$. Conditions $\varepsilon(0) = 0$ and $\varepsilon(\pi/2) = \pi/2$ imply

$a_{11} = 0$ and $a_{22} = 0$, respectively. Putting $A := a_{12}/a_{21}$ then it follows from $\operatorname{sign} \varepsilon' = 1$ that $A > 0$, hence $\operatorname{tg} \bar{\varepsilon}(t) = A \operatorname{tg} t$, $t \in (0, \pi/2)$, whence (2) follows.

($\Leftarrow$) Let $A > 0$ and the function $\bar{\varepsilon}$ be defined by (2). Then $\operatorname{tg} \bar{\varepsilon}(t) = A \operatorname{tg} t$ for $t \in (0, \pi/2)$. Let $\varepsilon(t) = \bar{\varepsilon}(t)$ for $t \in (0, \pi/2)$ and $\operatorname{tg} \varepsilon(t) = A \operatorname{tg} t$ throughout where the function $\operatorname{tg} t$ is defined and $\varepsilon \in C^{\circ}(\mathbf{R})$. Then $\varepsilon \in \mathfrak{E}$, $\operatorname{sign} \varepsilon' = 1$, $\varepsilon(0) = 0$, $\varepsilon(\pi/2) = \pi/2$, hence $\bar{\varepsilon} \in \mathfrak{E}_1$.

Lemma 2. *Let* (q) *be pure disconjugate equation where its phase* α *satisfies*

$$\lim_{t \to -\infty} \alpha(t) = 0, \qquad \lim_{t \to \infty} \alpha(t) = \pi/2. \tag{3}$$

Then

$$\mathscr{L}_q^+ = \{\alpha^{-1}(\operatorname{arctg}(A \operatorname{tg} \alpha(t))); A > 0\}.$$

Proof. Let (3) be valid for a phase α of the pure disconjugate equation (q) and $\mathfrak{E}_1$ be defined like in Lemma 2. Then $\mathscr{L}_q^+ = \alpha^{-1}\mathfrak{E}_1\alpha := \{\alpha^{-1}\varepsilon\alpha;\ \varepsilon \in \mathfrak{E}_1\}$ and the assertion follows from Lemma 1.

Corollary 1. *Let* (q) *be a pure disconjugate equation. Then* $\mathscr{L}_q^+$ *is a planar group.*

Proof. Let α be a phase of (q) satisfying (3) and $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}$. With reference to Lemma 1 it suffices to show the existence of exactly one number $A > 0$: $\alpha^{-1}(\operatorname{arctg}(A \operatorname{tg} \alpha(t_0))) = x_0$. It becomes evident that only positive solution of the last equation is the number $A := (\operatorname{tg} \alpha(x_0))/(\operatorname{tg} \alpha(t_0))$.

Example 1. Let $k > 0$. The equation

$$y'' = k^2 y \tag{4}$$

has independent solutions e^{kt}, e^{-kt}, hence the function $\alpha(t) := \operatorname{arctg}(e^{2kt})$, $t \in \mathbf{R}$ is its phase. On account of the fact that (3) is valid for α, we see that equation (4) is pure disconjugate, further, $\alpha^{-1}(t) = (1/2k) \ln (\operatorname{tg} t)$ for $t \in (0, \pi/2)$, and so for $A > 0$ we have $\alpha^{-1}(\operatorname{arctg}(A \operatorname{tg} \alpha(t))) = (1/2k) \ln (Ae^{2kt}) = (\ln A)/2k + t$, whence, with respect to Lemma 2, $\mathscr{L}_{k^2}^+ = \{t + c;\ c \in \mathbf{R}\}$.

Lemma 3. *Letting* $\mathfrak{E}_2 = \{\varepsilon|_{(-\pi/2, \pi/2)};\ \varepsilon \in \mathfrak{E},\ \varepsilon(-\pi/2) = -\pi/2,\ \varepsilon(\pi/2) = \pi/2\}$ *yields* $\varepsilon \in \mathfrak{E}_2$ *exactly if*

$$\bar{\varepsilon}(t) = \operatorname{arctg}(A \operatorname{tg} t + B), \qquad t \in (-\pi/2, \pi/2), \tag{5}$$

where $A > 0$, B *are numbers.*

Proof. ($\Rightarrow$) Let $\bar{\varepsilon} \in \mathfrak{E}_2$, whereupon $\bar{\varepsilon}(t) = \varepsilon(t)$ for $t \in (-\pi/2, \pi/2)$, with $\varepsilon \in \mathfrak{E}$, $\varepsilon(-\pi/2) = -\pi/2$, $\varepsilon(\pi/2) = \pi/2$. Since $\operatorname{tg} \varepsilon(t) = (a_{11} + a_{12} \operatorname{tg} t)/(a_{21} + a_{22} \operatorname{tg} t)$, where $\det a_{ij} \neq 0$, the conditions $\varepsilon(-\pi/2) = -\pi/2$, $\varepsilon(\pi/2) = \pi/2$ imply $a_{22} = 0$. $a_{12}/a_{21} > 0$. Putting $A := a_{12}/a_{21}$, $B := a_{11}/a_{21}$, then $\operatorname{tg} \varepsilon(t) = A \operatorname{tg} t + B$, which shows that (5) is valid.

($\Leftarrow$) Let $\bar{\varepsilon}$ be defined by (5), where $A > 0$, B are numbers. Then $\operatorname{sign} \bar{\varepsilon}' = 1$, $\operatorname{tg} \bar{\varepsilon}(t) = A \operatorname{tg} t + B$. Thus there exists an $\varepsilon \in \mathfrak{E}$: $\varepsilon(-\pi/2) = -\pi/2$, $\varepsilon(\pi/2) = \pi/2$ and $\varepsilon(t) = \bar{\varepsilon}(t)$ for $t \in (-\pi/2, \pi/2)$, whence as a necessary consequence, $\bar{\varepsilon} \in \mathfrak{E}_2$.

Lemma 4. *Let* (q) *be a specially disconjugate equation with* α *being its phase which satisfies*

$$\lim_{t\to-\infty} \alpha(t) = -\pi/2, \qquad \lim_{t\to\infty} \alpha(t) = \pi/2. \tag{6}$$

Then

$$\mathscr{L}_q^+ = \{\alpha^{-1}(\operatorname{arctg}(A \operatorname{tg} \alpha(t) + B));\ A(>0),\ B \in \mathbf{R}\}. \tag{7}$$

Proof. Let α be a phase of the specially disconjugate equation (q) satisfying (6) and $\mathfrak{E}_2$ be defined like in Lemma 3. Then $\mathscr{L}_q^+ = \alpha^{-1}\mathfrak{E}_2\alpha := \{\alpha^{-1}\bar{\varepsilon}\alpha;\ \bar{\varepsilon} \in \mathfrak{E}_2\}$. The assertion immediately follows from Lemma 3.

Lemma 5. *Let* (q) *be a specially disconjugate equation with* α *being its phase which satisfies* (6). *Then* $\mathscr{S}$ *is a planar subgroup of the group* $\mathscr{L}_q^+$ *exactly if*

$$\mathscr{S} = \{\alpha^{-1}(\operatorname{arctg}(\operatorname{tg} \alpha(t) + b));\ b \in \mathbf{R}\}. \tag{8}$$

Proof. ($\Rightarrow$) Let $\mathscr{S}$ be a planar subgroup of the group $\mathscr{L}_q^+$ and $A_1, A_2, B_1, B_2 \in \mathbf{R}, 0 < A_1 < A_2$. Putting $X_i(t) := \alpha^{-1}(\operatorname{arctg}(A_i \operatorname{tg} \alpha(t) + B_i)), t \in \mathbf{R}, i = 1, 2$, then, by Lemma 4 $X_i \in \mathscr{L}_q^+$, $X_1 \neq X_2$ and there exists one and only one $t_0 \in \in (-\pi/2, \pi/2)$: $X_1(t_0) = X_2(t_0)$. Consequently, X_1, X_2 do not simultaneously belong into $\mathscr{S}$. Since $\mathrm{id}_{\mathbf{R}} \in \mathscr{S}$, we see that (8) holds.

($\Leftarrow$) If $\mathscr{S}$ is defined by (8), then with respect to Lemma 4 $\mathscr{S} \subset \mathscr{L}_q^+$ and it can be easily verified that $\mathscr{S}$ is a planar group.

Corollary 2. *Let* (q) *be a specially disconjugate equation having a* π*-periodic coefficient* q *and* $\mathscr{S}$ *be a planar subgroup of the group* $\mathscr{L}_q^+$ *and* $X_0(t) := t + \pi$, $t \in \mathbf{R}$. *Then* $X_0 \in \mathscr{S}$.

Proof. The function X_0 is the dispersion of (q) and therefore $X_0 \in \mathscr{L}_q^+$. Let $X_0 \notin \mathscr{S}$ and α be a phase of (q) satisfying (6). From Lemmas 4 and 5 follows then the existence a positive number $A \neq 1$ and a number B: $\alpha^{-1}(\operatorname{arctg}(A \operatorname{tg} \alpha(t) + B)) = = t + \pi$ for $t \in \mathbf{R}$. Putting $p(t) := \alpha^{-1}(\operatorname{arctg} t)$, $t \in \mathbf{R}$, yields from the last equality that $p(At + B) = p(t) + \pi$, $t \in \mathbf{R}$. If we put $t := -B/(A - 1)$ in the last equality, we obtain $p(-B/(A - 1)) = p(-B/(A - 1)) + \pi$, which leads to a contradiction.

Remark 1. It follows from Corollary 1 and Lemma 5 that every disconjugate equation (q) has exactly one planar group of increasing dispesions.

Example 2. Equation

$$y'' = 0 \tag{9}$$

has independent solutions 1 and t, and the function $\alpha(t) := \operatorname{arctg} t$, $t \in \mathbf{R}$, is clearly its phase. The phase α satisfyies (6), hence equation (9) is specially disconjugate. It follows from Lemma 4 that $\mathscr{L}_0^+ = \{At + B;\ A(>0),\ B \in \mathbf{R}\}$ and from Lemma 5 and Remark 1 we find that $\mathscr{S} := \{t + C;\ C \in \mathbf{R}\}$ is the only one planar subgroup of the group $\mathscr{L}_0^+$.

3. Main results

Theorem 1. *Let* (p), (q) *be pure disconjugate equations. Further, let a phase* α *of* (q) *and a phase* β *of* (p) *satisfy*

$$\lim_{t\to-\infty}\alpha(t) = \lim_{t\to-\infty}\beta(t) = 0, \qquad \lim_{t\to\infty}\alpha(t) = \lim_{t\to\infty}\beta(t) = \pi/2. \tag{10}$$

Then $\mathscr{L}_p^+ = \mathscr{L}_q^+$ *exactly if*

$$\beta(t) = \operatorname{arctg}(C(\operatorname{tg}\alpha(t))^k), \qquad t \in \mathbf{R}, \tag{11}$$

with k, C *being positive number.*

Proof. ($\Rightarrow$) Let (10) hold for phases α and β of the pure disconjugate equations (q) and (p), respectively. Let $\mathscr{L}_p^+ = \mathscr{L}_q^+$. Then, with respect to Lemma 2, there exists to every positive number B a positive number $A = A(B)$:

$$\alpha^{-1}(\operatorname{arctg}(A\operatorname{tg}\alpha(t))) = \beta^{-1}(\operatorname{arctg}(B\operatorname{tg}\beta(t))), \qquad t \in \mathbb{R}. \tag{12}$$

Putting $s(t) := \operatorname{tg}(\alpha(\beta^{-1}(\operatorname{arctg} t)))$, $t \in (0, \infty)$, yields $s \in C^3(\mathbf{R})$, $s(t) > 0$, $s'(t) > 0$ for $t \in (0, \infty)$. On making use of the function s enables us to write (12) as

$$As(t) = s(Bt), \qquad t \in (0, \infty), \tag{13}$$

whence it follows that the function $A = A(B)$ has a continuous derivative on $(0, \infty)$. Differentiating (13) first with respect to the variable t and then with respect to the parameter B we get

$$As'(t) = Bs'(Bt),$$
$$A's(t) = ts'(Bt),$$

whereupon

$$(BA'(B))/A(B) = (ts'(t))/s(t), \qquad t, B \in (0, \infty).$$

Then there necessarily exists a $k > 0$:

$$s'(t)/s(t) = k/t, \qquad t \in (0, \infty),$$

hence $s(t) = Ct^k$, where $C > 0$. According $\operatorname{tg}(\alpha(\beta^{-1}(\operatorname{arctg} t))) = Ct^k$ whence (11) immediately follows.

($\Leftarrow$) Let (10) and (11) hold for phases α and β relative to equations (q) and (p), respectively, with $k > 0$, $C > 0$. Then $\beta^{-1}(t) = \alpha^{-1}(\operatorname{arctg}(\sqrt[k]{(1/C)\operatorname{tg} t}))$ for $t \in (0, \pi/2)$ and for $B > 0$ we obtain

$$\beta^{-1}(\operatorname{arctg}(B\operatorname{tg}\beta(t))) = \alpha^{-1}(\operatorname{arctg}(B^{1/k}\operatorname{tg}\alpha(t))) = \alpha^{-1}(\operatorname{arctg}(A\operatorname{tg}\alpha(t))),$$

where $A := B^{1/k}$. From this and from Lemma 2 it follows that $\mathscr{L}_p^+ = \mathscr{L}_q^+$.

Example 3. Let $k > 0$ and (p) be a pure disconjugate equation. Then the function $\alpha(t) := \operatorname{arctg}(e^{2kt})$, $t \in \mathbf{R}$, is a phase of equation (4). Let $k_1 > 0$, $C > 0$ and $\beta(t) := \operatorname{arctg}(C(\operatorname{tg}\alpha(t))^{k_1})$, $t \in \mathbf{R}$. Then $\beta(t) = \operatorname{arctg}(Ce^{st})$ with $s := 2kk_1$ and so

$\operatorname{tg}\beta(t) = Ce^{st}$. Then $-\{\operatorname{tg}\beta(t), t\} = s^2/2$. Thus, with respect to Theorem 1, $\mathscr{L}^+_{k^2} = \mathscr{L}^+_p$ is satisfies for a pure disconjugate equation (p) exactly if $p(t) = \text{const}\ (> 0)$.

Theorem 2. *Let* (q) *be a pure disconjugate equation and* α *be its phase satisfying* (3). *Let* (p) *be an oscillatory equation. Then* $\mathscr{L}^+_q \subset \mathscr{L}^+_p$ *exactly if the function*

$$\beta(t) = k \ln(\operatorname{tg}\alpha(t)), \qquad t \in \mathbf{R}, \tag{14}$$

with $k > 0$, *is an increasing phase of* (p).

Proof. Let $\mathscr{L}^+_q \subset \mathscr{L}^+_p$. $\mathscr{L}^+_q$ is a planar group by Corollary 1 and there exists a phase β_1 of (p), $\operatorname{sign}\beta_1' = 1$:

$$\mathscr{L}^+_q = \{\beta_1^{-1}(\beta_1(t) + a);\ a \in \mathbf{R}\}.$$

It follows from Lemma 2 that there exist to $A > 0$ a unique number $a = a(A) > 0$:

$$\alpha^{-1}(\operatorname{arctg}(A \operatorname{tg}\alpha(t))) = \beta_1^{-1}(\beta_1(t) + a), \qquad t \in \mathbf{R}. \tag{15}$$

Setting $s(t) := \beta_1(\alpha^{-1}(\operatorname{arctg} t))$, $t \in (0, \infty)$, yields $s \in C^3(0, \infty)$, $s'(t) > 0$ for $t \in (0, \infty)$. On making use of s enables us to write (15) as

$$s(At) = s(t) + a,\ t \in (0, \infty). \tag{16}$$

From this it follows that the function $a = a(A)$ has a continuous derivative on $(0, \infty)$. Differentiating (16) first with respect to the variable t and then with respect to the parameter A gives

$$As'(At) = s'(t),$$
$$ts'(At) = a'(A),$$

whence

$$ts'(t) = Aa'(A), \qquad t, A \in (0, \infty).$$

So there exists a $k > 0$: $ts'(t) = k$ and we have $s(t) = k \ln t + C$, where $C \in \mathbf{R}$ Then $\beta_1(\alpha^{-1}(\operatorname{arctg} t)) = k \ln t + C$ and $\beta_1(t) = k \ln(\operatorname{tg}\alpha(t)) + C$. Setting $\beta(t) := \beta_1(t) - C$, $t \in \mathbf{R}$, then β is an increasing phase of (p) and (14) holds.

Conversely, let β be defined by (14) with $k > 0$ being a phase of (p). Since $\beta(\mathbf{R}) = \mathbf{R}$, we can see that (q) is an oscillatory equation. Next we have

$$\beta^{-1}(t) = \alpha^{-1}(\operatorname{arctg} e^{t/k}), \qquad t \in \mathbf{R},$$

and for $a \in \mathbf{R}$

$$\beta^{-1}(\beta(t) + a) = \alpha^{-1}(\operatorname{arctg} e^{(1/k)(k \ln(\operatorname{tg}\alpha(t) + a))}) =$$
$$= \alpha^{-1}(\operatorname{arctg}(A \operatorname{tg}\alpha(t))),$$

where $A := e^{a/k}$. This implies $\mathscr{L}^+_q = \{\beta^{-1}(\beta(t) + c);\ c \in \mathbf{R}\} \subset \mathscr{L}^+_p$.

Example 4. Let (p) be an oscillatory equation and $\mathscr{L}^+_{k^2} \subset \mathscr{L}^+_p$ with $k > 0$. It follows from Example 1 that the function $\alpha(t) := \operatorname{arctg} e^{2kt}$, $t \in \mathbf{R}$, is a phase of equation (4) satisfying (3). And then there exists by Theorem 2 a $k_1 > 0$ such that the function $\beta_1(t) := k_1 \ln(\operatorname{tg}\alpha(t))$, $t \in \mathbf{R}$, is a phase of (p). Since $\beta_1(t) =$

$= k_1 \ln(\text{tg}(\text{arctg}\, e^{2kt})) = 2kk_1t = \lambda t$, where $\lambda := 2kk_1 (> 0)$, we get $p(t) = = -\{\beta_1, t\} - \beta_1'^2(t) = -\lambda^2$. Suppose that $k_2 > 0$ and $\beta_2(t) := k_2 \ln(\text{tg}\, \alpha(t))$ $(= 2kk_2t)$, $t \in \mathbf{R}$. Then $-\{\beta_2, t\} - \beta_2'^2(t) = -4k^2k_2^2$, whence, with respect to Theorem 2, $\mathscr{L}_{k^2}^+ \subset \mathscr{L}_p^+$ for an oscillatory equation (p) exactly if $p(t) = \text{const} (< 0)$.

Remark 2. It follows from Examples 3 and 4 that equations $y'' = \lambda y$, $\lambda \neq 0$, have a planar group of dispersions $\mathscr{S} = \{t + c;\ c \in \mathbf{R}\}$ in common.

Theorem 3. *Let* (p), (q) *be specially disconjugate equations with* $\mathscr{S}_p$ *and* $\mathscr{S}_q$ *being planar subgroups of the groups* $\mathscr{L}_p^+$ *and* $\mathscr{L}_q^+$, *respectively.*
Then $\mathscr{S}_p = \mathscr{S}_q$ *exactly if* $p = q$.

Proof. Suppose that $\mathscr{S}_p = \mathscr{S}_q$. Let α and β be respectively phases of (q) and (p) satisfying

$$\lim_{t\to-\infty} \alpha(t) = \lim_{t\to-\infty} \beta(t) = -\pi/2, \qquad \lim_{t\to\infty} \alpha(t) = \lim_{t\to\infty} \beta(t) = \pi/2.$$

It then follows from Lemma 5 that there exists to every number A a unique number $B = B(A)$ such that

$$\alpha^{-1}(\text{arctg}(\text{tg}\,\alpha(t) + A)) = \beta^{-1}(\text{arctg}(\text{tg}\,\beta(t) + B)), \qquad t \in \mathbf{R}. \tag{16}$$

Setting $s(t) := \text{tg}(\beta(\alpha^{-1}(\text{arctg}\, t)))$, $t \in \mathbf{R}$, gives $s \in C^3(\mathbf{R})$, $s'(t) > 0$ for $t \in \mathbf{R}$ and (16) may be written as

$$s(t + A) = s(t) + B, \qquad t \in \mathbf{R}. \tag{17}$$

From this we get $B \in C^3(\mathbf{R})$ and differentiating (17) first with respect to the variable t and then with respect to the parameter A gives

$$s'(t + A) = s'(t),$$
$$s'(t + A) + B'(A).$$

Thus $B'(A) = s'(t)$ and $s'(t) = k$ for $t \in \mathbf{R}$ with $k > 0$. This yields $s(t) = kt + c$, where $c \in \mathbf{R}$. From the definition of the function s we obtain $\beta(t) = \text{arctg}(k\,\text{tg}\,\alpha(t) + + c)$ and a calculation shows that β is a phase of (q). Therefore $p = q$.

Contrawise the proof becomes evident.

Theorem 4. *Let* (p) *be a pure disconjugate equation and* (q) *be a specially disconjugate equation. Let* $\mathscr{S}_q$ *be a planar subgroup of the group* $\mathscr{L}_q^+$. *Let* α *and* β *be phases of* (q) *and* (p) *satisfying* (6) *and* $\lim_{t\to-\infty} \beta(t) = 0$, $\lim_{t\to\infty} \beta(t) = \pi/2$, *respectively. Then* $\mathscr{L}_p^+ = \mathscr{S}_q$ *exactly if*

$$\beta(t) = \text{arctg}(c\, e^{k\,\text{tg}\,\alpha(t)}), \qquad t \in \mathbf{R}, \tag{18}$$

with c, k being positive constants.

Proof. ($\Rightarrow$) Suppose that $\mathscr{L}_p^+ = \mathscr{S}_q$. It then follows from Lemmas 2 and 5 that there exists to every number A a unique positive number $B = B(A)$:

$$\alpha^{-1}(\text{arctg}(\text{tg}\,\alpha(t) + A)) = \beta^{-1}(\text{arctg}(B\,\text{tg}\,\beta(t))), \qquad t \in \mathbf{R}. \tag{19}$$

Setting $s(t) := \mathrm{tg}\,(\beta(\alpha^{-1}(\mathrm{arctg}\, t)))$, $t \in \mathbf{R}$, then $s \in C^3(\mathbf{R})$, $s(t) > 0$, $s'(t) > 0$ for $t \in \mathbf{R}$ and (19) may be written as

$$s(t + A) = Bs(t), \qquad t \in \mathbf{R}, \tag{20}$$

from which we find that $B \in C^3(\mathbf{R})$ and differentiating (20) first with respect to the variable t and then with respect to the parameter A gives

$$s'(t + A) = B(A)\, s'(t),$$
$$s'(t + A) = B'(A)\, s(t).$$

Then $B'(A)/B(A) = s'(t)/s(t)$ and thus $s'(t)/s(t) = k$, $t \in \mathbf{R}$, with k being a positive number. From this we get $s(t) = ce^{kt}$, where c is a positive number and we see that (18) is valid.

($\Leftarrow$) Suppose that (18) is valid for phases α, β, with c, k being positive numbers. Then $\beta^{-1}(t) = \alpha^{-1}\left(\mathrm{arctg}\left(\frac{1}{k}\ln\left(\frac{1}{c}\,\mathrm{tg}\, t\right)\right)\right)$ for $t \in (0, \pi/2)$ and for $B > 0$

$$\beta^{-1}(\mathrm{arctg}\,(B\,\mathrm{tg}\,\beta(t))) = \alpha^{-1}\left(\mathrm{arctg}\left(\frac{1}{k}(\ln B + k\,\mathrm{tg}\,\alpha(t))\right)\right) =$$
$$= \alpha^{-1}(\mathrm{arctg}\,(\mathrm{tg}\,\alpha(t) + A),$$

where $A := (\ln B)/k$. From this and from Lemma 5 we get $\mathscr{L}_q^+ = \mathscr{S}_p$.

Theorem 5. *Let* (q) *be a special disconjugate equation and* (p) *be an oscillatory equation. Let* α *be a phase of* (q) *satisfying* (6) *and* $\mathscr{S}_q$ *be a planar subgroup of the group* $\mathscr{L}_q^+$. *Then* $\mathscr{S}_q \subset \mathscr{L}_p^+$ *exactly if*

$$\beta(t) := k\,\mathrm{tg}\,\alpha(t), \qquad t \in \mathbf{R}, \tag{21}$$

(*where k is a positive number*) *is an increasing phase of* (p).

Proof. ($\Rightarrow$) Suppose that $\mathscr{S}_q \subset \mathscr{L}_p^+$. Then there exists an increasing phase β_1 of (p): $\mathscr{S}_q = \{\beta_1^{-1}(\beta_1(t) + a);\ a \in \mathbf{R}\}$. It then follows from Lemma 5 that there exists to every number A a unique number $B = B(A)$

$$\alpha^{-1}(\mathrm{arctg}\,(\mathrm{tg}\,\alpha(t) + A)) = \beta_1^{-1}(\beta_1(t) + B), \qquad t \in \mathbf{R}. \tag{22}$$

Setting $s(t) := \beta_1(\alpha^{-1}(\mathrm{arctg}\, t))$, $t \in \mathbf{R}$, then $s \in C^3(\mathbf{R})$, $s'(t) > 0$ for $t \in \mathbf{R}$ and (22) may be written as

$$s(t + A) = s(t) + B(A), \qquad t, A \in \mathbf{R}, \tag{23}$$

whence as a necessary consequence $B \in C^3(\mathbf{R})$. Differentiating (23) first with respect to the variable t and then with respect to the parameter A yields

$$s'(t + A) = s'(t),$$
$$s'(t + A) = B'(A),$$

which establishes the existence of $k > 0$: $s'(t) = k$. Thus for an $a \in \mathbf{R}$ we have $s(t) = kt + a$, $t \in \mathbf{R}$. Then $\beta_1(\alpha^{-1}(\mathrm{arctg}\, t)) = kt + a$, whence $\beta_1(t) = k\,\mathrm{tg}\,\alpha(t) +$

$+ a$. Setting $\beta(t) := \beta_1(t) - a$, $t \in \mathbf{R}$, then β is an increasing phase of (p) having the form (21).

($\Leftarrow$) Let the function β defined by (21) with k being a positive number, be an increasing phase of (p). Since $\beta(\mathbf{R}) = \mathbf{R}$, we see that (p) is an oscillatory equation. Next we have $\beta^{-1}(t) = \alpha^{-1}(\operatorname{arctg}(t/k))$ and for $B \in \mathbf{R}$

$$\beta^{-1}(\beta(t) + B) = \alpha^{-1}(\operatorname{arctg}(\operatorname{tg}\alpha(t) + A)),$$

where $A := B/k$, whence as a necessary consequence of Lemma 5 $\mathcal{S}_q \subset \mathcal{L}_p^+$.

Example 5. The function $\alpha(t) := \operatorname{arctg} t$, $t \in \mathbf{R}$, is a phase of the specially disconjugate equation (9) satisfying (6). $\mathcal{S} := \{t + a;\ a \in \mathbf{R}\}$ is a planar subgroup of the group $\mathcal{L}_0^+$. Let $k > 0$ and $\beta(t) := k \operatorname{tg}\alpha(t)\ (= kt)$, $t \in \mathbf{R}$. It then follows from Theorem 5 that $\mathcal{S} \subset \mathcal{L}_p^+$, where (p) in an oscillatory equation exactly if $p(t) := -k^2$, $t \in \mathbf{R}$, where k is a positive number.

Example 6. Equation

$$y'' = \frac{2t^2 - 1}{(1 + t^2)^2}\, y \tag{24}$$

has a phase $\alpha(t) := \operatorname{arctg}(t + t^3/3)$, $t \in \mathbf{R}$, satisfying (6), hence it is specially disconjugate. Since $\alpha^{-1}(t) = \sqrt[3]{\frac{3}{2}\operatorname{tg} t + \sqrt{\frac{9}{4}\operatorname{tg}^2 t + 1}} + \sqrt[3]{\frac{3}{2}\operatorname{tg} t - \sqrt{\frac{9}{4}\operatorname{tg}^2 t + 1}}$, $t \in (-\pi/2, \pi/2)$, we see that by Lemma 5 $\mathcal{S} :=$

$$:= \left\{\sqrt[3]{\frac{3}{2}\left(t + \frac{t^3}{3} + B\right) + \sqrt{\frac{9}{4}\left(t + \frac{t^3}{3} + B\right)^2 + 1}} + \sqrt[3]{\frac{3}{2}\left(t + \frac{t^3}{3} + B\right) - \sqrt{\frac{9}{4}\left(t + \frac{t^3}{3} + B\right)^2 + 1}};\ B \in \mathbf{R}\right\}$$

is a planar subgroup of the group of dispersions of equation (24). Following Theorem 4 $\mathcal{S} = \mathcal{L}_p^+$, where (p) is a pure disconjugate equation exactly if $\beta(t) := \operatorname{arctg}(ce^{k(t + t^3/3)})$, $t \in \mathbf{R}$, with $c > 0$, $k > 0$, is a phase of (p). Following Theorem 5 $\mathcal{S} \subset \mathcal{L}_r^+$, where (r) is an oscillatory equation exactly if $\gamma(t) := k(t + t^3/3)$, $t \in \mathbf{R}$, is an increasing phase of (r).

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SPOLEČNÉ ROSTOUCÍ DISPERSE JISTÝCH LINEÁRNÍCH DIFERENCIÁLNÍCH ROVNIC DRUHÉHO ŘÁDU

Souhrn

Funkce $X \in C^3(\mathbf{R})$, $X'(t) > 0$ pro $t \in \mathbf{R}$, se nazývá (úplná) rostoucí disperse rovnice (q): $q'' = q(t)\,y$, $q \in C^\circ(\mathbf{R})$, jestliže $X(\mathbf{R}) = \mathbf{R}$ a X je řešením rovnice

$$-\{X, t\} + X'^2\, q(X) = q(t),$$

kde $\{X, t\} = X'''(t)((2X'(t)) - (3/4)\,(X''(t)(X'(t))^2$. Množina rostoucích dispersí rovnice (q) tvoří vzhledem k operaci skládání funkcí grupu $\mathscr{L}_q^+$. Nechť (q) je diskonjugovaná rovnice. Pak $\mathscr{L}_q^+$ je nejvýše dvouparametrická spojitá grupa. Užitím teorie fází pro lineární diferenciální rovnice druhého řádu jsou v práci uvedeny všechny rovnice typu (p): $y'' = p(t)\,y$, $p \in C^\circ(\mathbf{R})$, které jsou buď diskonjugované, anebo oscilatorické a pro které platí $\mathscr{L}_q^+ \subset \mathscr{L}_p^+$.

СОВМЕСТНЫЕ ВОЗРАСТАЮЩИЕ ДИСПЕРСИИ НЕКОТОРЫХ ЛИНЕЙНЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ ВТОРОГО ПОРЯДКА

Резюме

Функция $X \in C^3(\mathbf{R})$, $X'(t) > 0$ для $t \in \mathbf{R}$, называется возрастающей (полной) дисперсией уравнения (q) : $y'' = q(t)y$, $q \in C^\circ(\mathbf{R})$, если $X(\mathbf{R}) = \mathbf{R}$ и решением уравнения

$$-\{X, t\} + X'^2 \,.\, q(X) = q(t),$$

где $\{X, t\} = \alpha'''(t)/(2\alpha'(t)) - (3/4)\,(\alpha''(t)/\alpha'(t))^2$. Множество возрастающих дисперсий уравнения (q) является относительно операции сложения функций группой $\mathscr{L}_q^+$. Пусть (q) уравнение без сопняженных точек. Тогда $\mathscr{L}_q^+$ наиболее двухпараметрическая непрерывная группа. С помощью теории фаз линейных дифференциальных уравнений 2-ого порядка приводится в работе все уравнения типа (p) : $y'' = p(t)y$, $p \in C^\circ(\mathbf{R})$, которые или без сонряженных точек или колеблющиеся и для которых имеет место $\mathscr{L}_q^+ \subset \mathscr{L}_p^+$.