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ACTA UNIVERSITATIS PALACKIANAE OLOMUCENSIS FACULTAS RERUM NATURALIUM
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NON-MODULAR AND NON-DISTRIBUTIVE PRIMITIVE ORDERED SUBSETS OF LATTICES

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Abstract

Lattices not containing an isomorphic copy of a member from a given set of finite ordered sets as a primitive subset form a variety of lattices. In the paper some collections of primitive ordered sets characterising the variety of distributive lattices and two small non-distributive varieties of lattices are shown.

Key words: Primitive ordered subset of a lattice, lattice variety, distributive (modular) ordered set, distributive (modular) lattice.

MS Classification: 06B20, 06A99

In the paper [4] modular and distributive ordered sets are introduced and some of their properties are shown. Forbidden subsets of these types of ordered sets are described in [1]; these configurations are isomorphic to the sets $R_1, R_2, R_3, R_4, R_5, R_6, S_1, S_2, S_3, S_4$ and S_5. (See Figures 1 and 4.) R. Wille in [6] defined the notion of a primitive subset of a lattice and showed that the lattices not containing an isomorphic copy of a member from a given set of finite ordered sets as a primitive set form a variety of lattices.

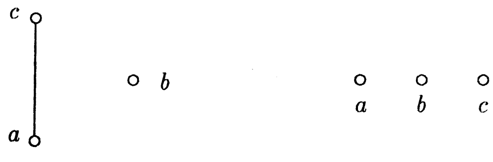
In this paper there is shown which of the ordered sets $R_1, \ldots, R_4, S_1, \ldots, S_4$, characterize the class of all distributive lattices $\mathcal{D}$ and there are studied the varieties characterized by the sets R_1 and S_1.

If L is a lattice, $g, h \in L$, then $\theta(g,h)$ will denote the principal congruence relation on L generated by the pair $\langle g, h\rangle$. The trivial congruence relation on L will be denoted by ω.

Let L be a lattice, P a finite non-void subset of L. Then P is called a primitive subset of L if

$$\bigwedge(\theta(x, x \vee y);\ x, y \in P,\ x \neq x \vee y) \neq \omega.$$

Lemma 1 *If L is a distributive lattice, then it contains no primitive subset isomorphic to the ordered set R_1 or to the set S_1 (See Fig. 1.)*



R_1 Fig. 1 S_1

Proof Let L be a distributive lattice.

a) Let us suppose that L contains a subset $P_1 = \{x, y, z\}$ isomorphic to R_1 (Fig. 2.).

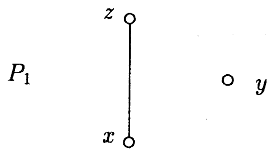


Fig. 2

Consider the congruence relations

$$\theta(x, z),\ \theta(x, x \vee y),\ \theta(y, x \vee y),\ \theta(y, y \vee z),\ \theta(z, y \vee z)$$

and denote by ψ_1 their intersection.

Let $p, q \in L,\ p \equiv q(\psi_1)$. Then by [2, Theorem II.3.3] we have

$$p \vee z = q \vee z \quad \text{and} \quad p \wedge z = q \wedge z,$$

therefore $p = q$. This implies $\psi_1 = \omega$, and so P_1 is not a primitive subset of L.

b) Let L contain a subset $P_2 = \{x, y, z\}$ isomorphic to S_1. (Fig. 3.)

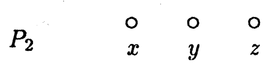


Fig. 3

Denote by ψ_2 the intersection of the congruence relations

$$\theta(x, x \vee y),\ \theta(y, x \vee y),\ \theta(x, x \vee z),\ \theta(z, x \vee z),\ \theta(y, y \vee z),\ \theta(z, y \vee z).$$

Let $p, q \in L$, $p \equiv q(\psi_2)$. Then e.g.

$$p \wedge x = q \wedge x,\ p \wedge z = q \wedge z,\ p \vee (x \vee z) = q \vee (x \vee z),$$

hence we have

$$p \wedge (x \vee z) = q \wedge (x \vee z),$$

and therefore $p = q$. This means $\psi_2 = \omega$, thus P_2 is not a primitive subset of L. □

Lemma 2 *If L is a lattice, $P \subseteq Q \subseteq L$, and if P is not a primitive subset of L, then Q is not a primitive subset of L, too.* □

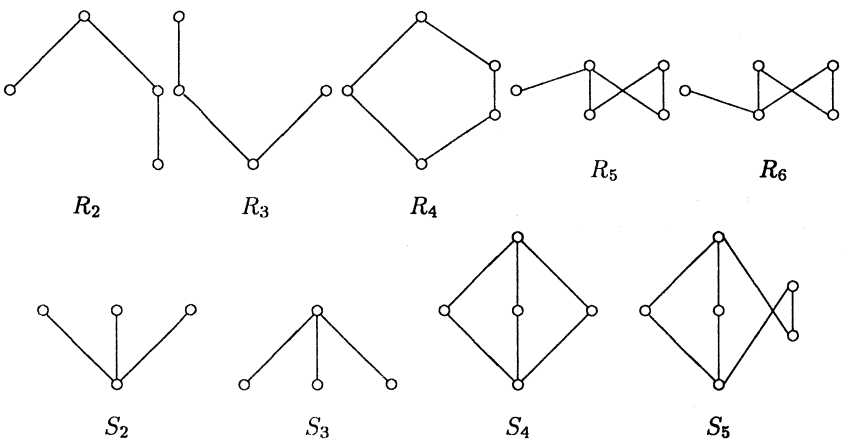


Fig. 4

Corollary 3 *If L is a distributive lattice, then it contains no primitive subset isomorphic to some of the ordered sets $R_1, \ldots, R_6, S_1, \ldots, S_5$.* □

Let $\mathcal{P}$ be a set of finite ordered sets. Denote $Equ(\mathcal{P})$ the class of all lattices that do not contain an isomorphic copy of a member of P as a primitive subset. By [6] (see also [2, Theorem V.2.6]), $Equ(\mathcal{P})$ is a variety of lattices.

Let us denote

$$P_{i,j} = \{R_i, S_j\},\ i = 1, \ldots, 6;\ j = 1, \ldots, 5.$$

The variety of all distributive lattices will be denoted by $\mathcal{D}$.

Theorem 4 $Equ(\mathcal{P}_{i,j}) = D$, $i = 1, 2, 3, 4$; $j = 1, 2, 3, 4$.

Proof By Corollary 3, $\mathcal{D} \subseteq Equ(\mathcal{P}_{i,j})$, $i = 1,2,3,4$; $j = 1,2,3,4$.
Let L be a non-distributive lattice.

a) First, let L contain a sublattice $L_1 = \{x, y, z, u, v\}$ isomorphic to the pentagon R_4. (See Fig. 5.)

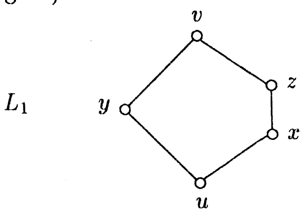


Fig. 5

If $\psi_1 = \bigwedge(\theta(r, r \vee s);\ r, s \in L_1,\ r \neq r \vee s)$, then we have $x \equiv z(\psi_1)$. This means that L_1 is a primitive subset of L. Hence, by Lemma 2,

$$Q_1 = \{x, y, z\}, \quad Q_2 = \{x, y, z, v\} \quad \text{and} \quad Q_3 = \{x, y, z, u\}$$

are primitive subsets of L, too.

b) Now, let L contain a sublattice $L_2 = \{x, y, z, u, v\}$ isomorphic to the diamond S_4 (See Fig. 6.)

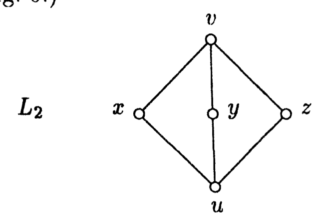


Fig. 6

Then for each elements $e, f, g, h \in L_2$, $e \neq f$, we have $g \equiv h(\theta(e, f))$, thus L_2 is a primitive subset of L. And therefore, by Lemma 2,

$$P_1 = \{x, y, z\}, \quad P_2 = \{x, y, z, u\} \quad \text{and} \quad P_3 = \{x, y, z, v\}$$

are primitive subsets of L, too. □

Denote $\mathcal{P}_{0,1} = \{S_1\}$, $\mathcal{P}_{1,0} = \{R_1\}$. Let $\mathcal{N}_5$ be the smallest non-modular lattice variety, i.e. the variety generated by the lattice $N_5 = R_4$, and $\mathcal{M}_3$ the variety generated by the lattice $M_3 = S_4$.

Theorem 5 $Equ(\mathcal{P}_{0,1}) = \mathcal{N}_5$.

Proof By Theorem 4, we have $\mathcal{D} \subseteq Equ(\mathcal{P}_{0,1})$. Lattice N_5 contains no primitive subset isomorphic to S_1, thus $\mathcal{N}_5 \subseteq Equ(\mathcal{P}_{0,1})$. Moreover, $\mathcal{M}_3 \notin Equ(\mathcal{P}_{0,1})$.

Hence, let us consider the lattices L_1–L_{15}. (See Fig. 7.)

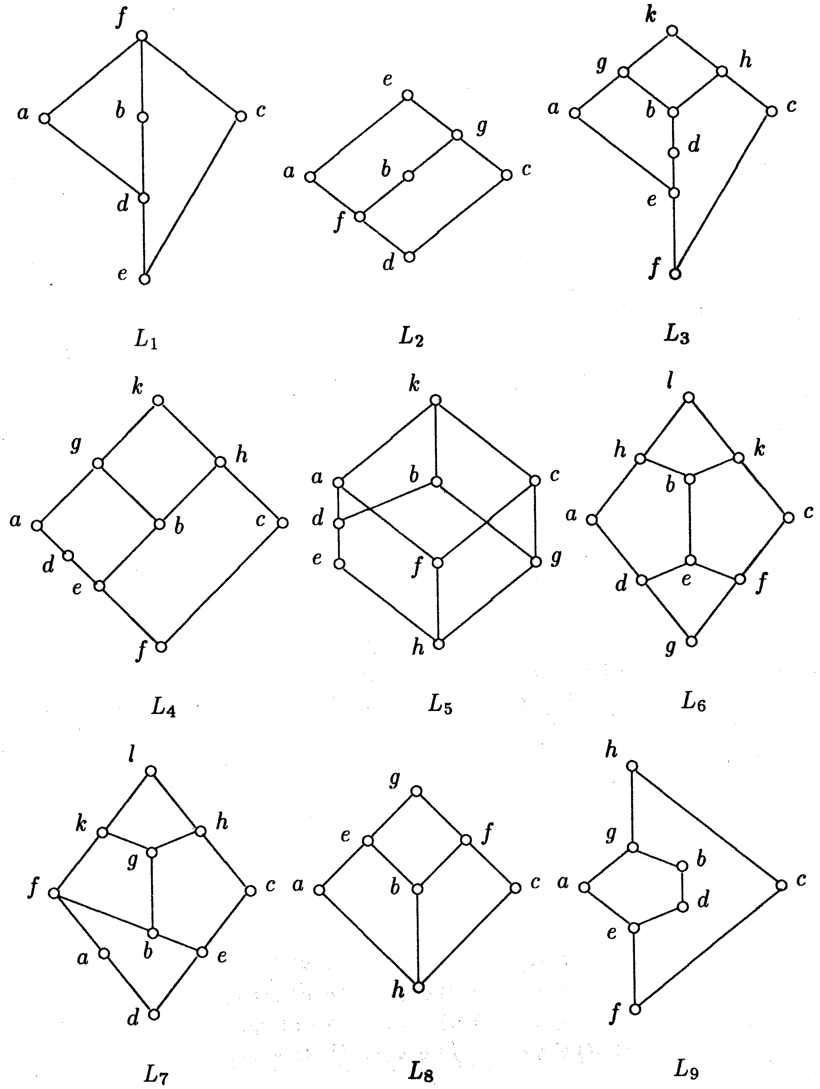


Fig. 7 (part 1)

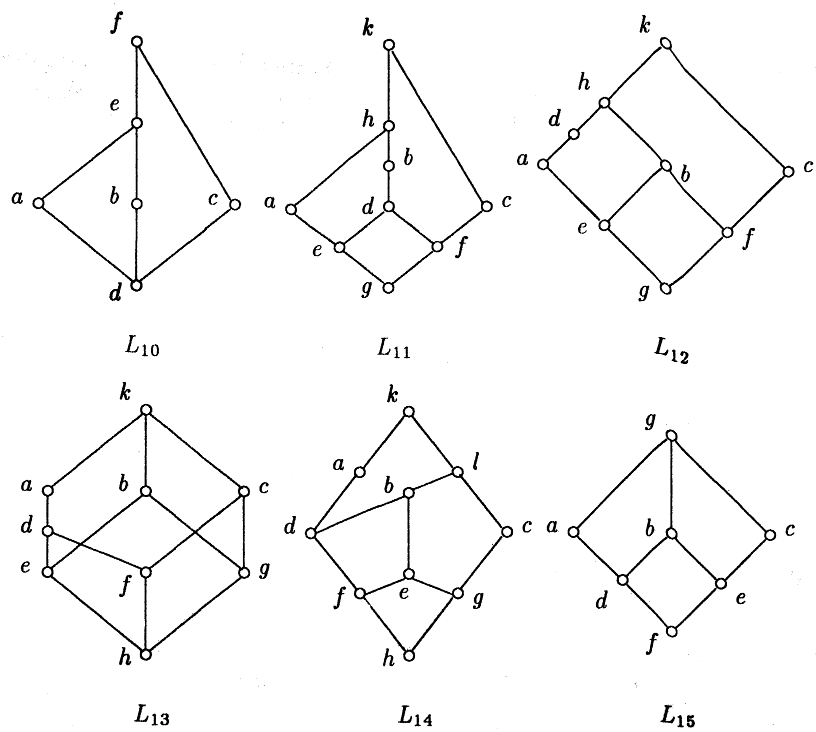


Fig. 7 (part 2)

By [5], each of these lattices generates a variety covering $\mathcal{N}_5$ in the lattice of varieties of lattices. And by [3], every variety of lattices that properly contains $\mathcal{N}_5$ includes one of the lattices $M_3, L_1, \ldots, L_{15}$.

For L_i $(i = 1, \ldots, 15)$ put

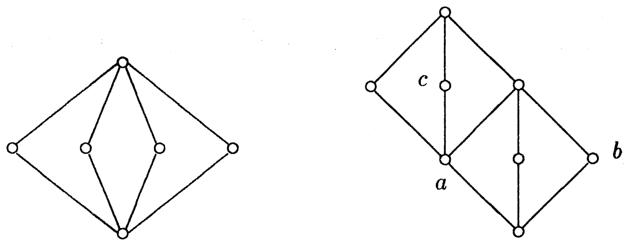
$$\psi_i = \bigwedge(\theta(x, x \vee y);\ x, y \in \{a, b, c\},\ x \neq x \vee y).$$

Then we have e.g.

$(e, c) \in \psi_1$	$(e, b) \in \psi_6$	$(b, d) \in \psi_{11}$
$(f, b) \in \psi_2$	$(b, g) \in \psi_7$	$(a, d) \in \psi_{12}$
$(e, d) \in \psi_3$	$(b, d) \in \psi_8$	$(a, d) \in \psi_{13}$
$(a, d) \in \psi_4$	$(b, d) \in \psi_9$	$(b, e) \in \psi_{14}$
$(e, d) \in \psi_5$	$(c, f) \in \psi_{10}$	$(b, g) \in \psi_{15}$.

But this means that $\psi_i \neq \omega$ for each $i = 1, \ldots, 15$, hence $L_i \notin Equ(\mathcal{P}_{0,1})$, $i = 1, \ldots, 15$. Therefore $\mathcal{N}_5 = Equ(\mathcal{P}_{0,1})$. □

Let $\mathcal{M}_4$ denote the variety of lattices generated by the lattice M_4, and $\mathcal{M}_{3,3}$ the variety of lattices generated by the lattice $M_{3,3}$. (See Fig. 8.)



M_4 $M_{3,3}$

Fig. 8

Theorem 6 $Equ(\mathcal{P}_{1,0}) \supseteq \mathcal{M}_4$, $Equ(\mathcal{P}_{1,0}) \supseteq \mathcal{M}_{3,3}$.

Proof Obviously, M_4 contains no subset isomorphic to R_1. The lattice M_3 is congruence trivial, hence $\{a, b, c\}$ is a primitive subset of $\mathcal{M}_{3,3}$. □

Let M denote the variety of all modular lattices.

Corollary 7 $\mathcal{M}_3 \subset Equ(\mathcal{P}_{1,0}) \subset \mathcal{M}$. □

Remark 1 Now, let us consider $Equ(\mathcal{P}_{0,5})$, the variety of lattices not containing a primitive subset isomorphic to the ordered set S_5. It is clear that if a lattice contains a primitive subset isomorphic to S_5, then it also contains a subset isomorphic to the lattice L_{16}. (See Fig. 9.)
Therefore every lattice having at most eight elements belongs to $Equ(\mathcal{P}_{0,5})$. It means that $Equ(\mathcal{P}_{0,5})$ contains e.g. all three varieties covering the variety $\mathcal{M}_3$.

L_{16}

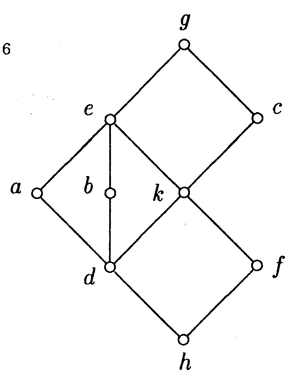


Fig. 9

Moreover, a unique subset of L_{16} isomorphic to S_5 is $A = \{a, b, d, e, c, f\}$. But in L_{16},

$$\theta(a, c) = \{\{a, b, k, d, e\}, \{h, f\}, \{c, g\}\},$$
$$\theta(e, e \vee c) = \{\{e, g\}, \{k, c\}, \{a\}, \{b\}, \{d\}, \{h\}, \{f\}\},$$

hence

$$\theta(a,e) \wedge \theta(e, e \vee c) = \omega,$$

and this means that A is not a primitive subset of L_{16}.
Therefore $L_{16} \in Equ(\mathcal{P}_{0,5})$ and thus every lattice which fails to belong to $Equ(\mathcal{P}_{0,5})$ has at least ten elements.

Remark 2 Consider $Equ(\mathcal{P}_{5,0})$, the variety of lattices not containing a primitive subset isomorphic to R_5. We have

$$Equ(\mathcal{P}_{0,1}) \subseteq Equ(\mathcal{P}_{5,0}),$$

$$Equ(\mathcal{P}_{1,0}) \subseteq Equ(\mathcal{P}_{5,0}).$$

In addition, every lattice which does not belong to $Equ(\mathcal{P}_{5,0})$ contains a subset isomorphic to the lattice L_{17}. (See Fig. 10.) Hence every lattice L such that $L \notin Equ(\mathcal{P}_{5,0})$ has at least eight elements.

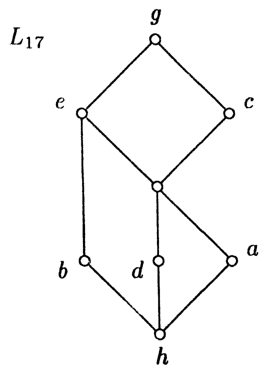


Fig. 10

Moreover, in L_{17},

$$(c,f) \in \bigwedge(\theta(x, x \vee y);\ x,y \in \{b,d,a,e,c\},\ x \neq x \vee y)$$

hence $\{b,d,a,e,c\}$ is a primitive subset of L_{17}, and so $L_{17} \notin Equ(\mathcal{P}_{5,0})$.

The ordered set R_6 is the dual case of R_5, hence for R_6 we obtain analogical results as for R_5.

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