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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 34 (1995), No. 1, 69--74

Persistent URL: `http://dml.cz/dmlcz/120334`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **34** (1995) 69–74

Grassmann Formula for Certain Type of Modules

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(Received June 30, 1994)

Abstract

The article is devoted to the validity of Grassmann formula and Steinitz theorem for **R**-space in the sense of [1]. It is shown that the formula and the theorem above are true if the powers of the maximal ideal $\mathcal{J}$ of **R** fulfil the descending chain condition.

Key words: R-space, bases of R-space, dual space.

MS Classification: 13C99

According to [1] we define:

Definition 1 Let **A** be a local ring. Let **M** be a finitely generated **A**-module. Then **M** is an **A**-*space of finite dimension* if there exists $\underline{E}_1, \dots, \underline{E}_n$ in **M** with

(a) $\mathbf{M} = \mathbf{A}\underline{E}_1 \oplus \cdots \oplus \mathbf{A}\underline{E}_n$

(b) the map $\mathbf{A} \to \mathbf{A}\underline{E}_i$ defined by $x \mapsto x\underline{E}_i$ is an **A**-isomorphism for $1 \leq i \leq n$.

Remark 2 A module **M** over a local ring **A** is an **A**-space if and only if it is a free finitely dimensional module.

Definition 3 A local ring **A** having the maximal ideal $\mathcal{J}$ with

$$\mathcal{J}^m = \{0\} \ \wedge \ \mathcal{J}^{m-1} \neq \{0\}$$

is called *a local ring of order m*.

*Supported by grant No 201/95/1631 of the Grant Agency of Czech Republic

Agreement 4 In the following text we denote by $\mathbf{A}$ the local ring of order m (introduced by 3) with the maximal ideal $\mathcal{J}$. By $\mathbf{M}$ we denote the finite-dimensional $\mathbf{A}$-space. By $\mathbf{M}^*$ we denote the $\mathbf{A}$-module of linear forms $\mathbf{M}$ into $\mathbf{A}$. Evidently, $\mathbf{M}^*$ is a $\mathbf{A}$-space (*dual* $\mathbf{A}$*-space* to $\mathbf{M}$).
Any submodule of $\mathbf{M}$ being also an $\mathbf{A}$-space will be called an $\mathbf{A}$-*subspace*.

Proposition 5 *Let* $\mathcal{G} = \{\underline{E}_1, \dots \underline{E}_r\}$ *be some system of generators of a* $\mathbf{A}$-*space* $\mathbf{M}$. *If* $\underline{U}_1, \dots, \underline{U}_k$ *are linearly independent elements from* $\mathbf{M}$ *then:*

(1) $k \leq r$

(2) by a suitable renumbering of elements $\underline{E}_1, \dots \underline{E}_r$,

$$\mathcal{H} = \{\underline{U}_1, \dots, \underline{U}_k, \underline{E}_{k+1}, \dots, \underline{E}_r\}$$

will be a set of generators of $\mathbf{M}$.

(3) If the system $\mathcal{G}$ *is linearly independent then* $\mathcal{H}$ *is also linearly independent.*

Proof If $m = 1$ [i.e. $\mathcal{J} = (0)$, $\mathcal{J}^0 = (1)$] then $\mathbf{A}$ is a field and in this case the proposition is well known. Further, let us assume $m > 1$.

First, let $k = 1$, (1) is fulfilled evidently.
(2): let $\underline{U}_1$ be linearly independent,

$$\underline{U}_1 = \sum_{i=1}^{r} x_i \underline{E}_i \qquad (*)$$

We show, that there exists at least one unit among $x_1, \dots x_r$. In fact, in the opposite case multiplying $(*)$ by $h \in \mathcal{J}^{m-1}$, $h \neq 0$ we get: $h\underline{U}_1 = \underline{o}$ i.e. $\underline{U}_1$ is linearly dependent—a contradiction. Let for example x_1 be a unit, then it follows from $(*)$:

$$\underline{E}_1 = x_1^{-1}\underline{U}_1 + \sum_{j=2}^{r}(-x_j x_1^{-1})\underline{E}_j.$$

Consequently $[\underline{U}_1, \underline{E}_2, \dots, \underline{E}_r] = \mathbf{M}$.

Now, we finish our proof by induction for k, supposing, that (1), (2) are fulfilled for $k - 1$.

As $\underline{U}_1, \dots \underline{U}_k$ are linearly independent, then $\underline{U}_{,1} \dots, \underline{U}_{k-1}$ are linearly independent as well. By the induction supposition we have by a suitable renumbering of $\underline{E}_i$: $[\underline{U}_1, \dots \underline{U}_{k-1}, \underline{E}_k, \dots \underline{E}_r] = \mathbf{M}$.

Now

$$\underline{U}_k \in \mathbf{M} \Rightarrow \underline{U}_k = \sum_{i=1}^{k-1} x_i \underline{U}_i + \sum_{j=k}^{r} x_j \underline{E}_j \qquad (**)$$

Let us derive that there exists at least one unit among $x_k, \dots, x_r$. Otherwise after multiplying $(**)$ by $h \in \mathcal{J}^{m-1}$, $h \neq 0$ we would obtain:

$$(hx_1)\underline{U}_1 + \dots + (hx_{k-1})\underline{U}_{k-1} - h\underline{U}_k = \underline{o}$$

which contradicts to a linear independence of $\underline{U}_1, \dots \underline{U}_k$.

Let for example x_k be a unit. Then from (**) we have:

$$\underline{E}_k = (-x_k x_1)\underline{U}_1 + \cdots + (-x_k x_{k-1})\underline{U}_{k-1} + x_k \underline{U}_k + \\ + (-x_k x_{k+1})\underline{E}_{k+1} + \cdots + (-x_k x_r)\underline{E}_r.$$

It follows from this that: $[\underline{U}_1, \dots \underline{U}_k, \underline{E}_{k+1}, \dots, \underline{E}_r] = \mathbf{M}$, i.e. (2) is true.

By the induction supposition we have that $k - 1 \leq r$. From (**) it follows that $k - 1 = r$ implies the linear dependence of $\underline{U}_1, \dots \underline{U}_k$, which is not possible, i.e. (1) is true.

It follows from the previous part of this proof that a linear independence of the system $\mathcal{G}$ implies the linear independence of the system $\mathcal{H}$.

Corollary 6

(a) *If the* **A**-*space* **M** *has one basis consisting of* n *elements then any its basis consists of the same number* n *elements. The number* n *is called the dimension of* **M**. *(It is true in every free module over a commutative ring.*[1]

(b) *From every system of generators of* **M** *we may select a basis of* **M**. *(It is valid over every local ring (according to Nakayama's lemma)).*[2]

Moreover in our case:

(c) *Any linearly independent system can be completed to a basis of* **M**.

(d) *Every maximal linearly independent system in* **M** *forms a basis of* **M**.

Considering (I.4.) Corollary in [1] and 6. (c) above we obtain the following theorem:

Theorem 7 *K is a direct summand of* **M** *if and only if K is an* **A**-*subspace of* **M**.

Definition 8 A linear form ϕ **M** into **A** is called *the epiform* if ϕ is a surjective homomorphism **M** into **A**.

Lemma 9 *Let ϕ be a linear form* **M** *into* **A**. *Then the following are equivalent*

(1) *ϕ is an epiform*

(2) $\operatorname{Im}\phi \not\subset \mathfrak{I}$

(3) $1 \in \operatorname{Im}\phi$

(4) *ϕ is a linearly independent element of* $\mathbf{M}^*$.

Proof The validity of this lemma is evident.

Definition 10 *A free $n-1$-dimensional submodule of* **M** *is called a hyperplane of the* **M**.

[1]See [2]
[2]See [1]

Proposition 11 *Let $N \subseteq \mathbf{M}$. N is a hyperplane of $\mathbf{M}$ if and only if there exists the epiform ϕ such that $N = \operatorname{Ker}\phi$.*

Proof 1) Let N be a hyperplane of $\mathbf{M}$. Let $\{\underline{V}_1, \dots, \underline{V}_{n-1}\}$ be a basis of N. Then (by 6.) there is $\underline{V}_n \in \mathbf{M}$ such that $\{\underline{V}_1, \dots, \underline{V}_{n-1}, \underline{V}_n\}$ is a basis of $\mathbf{M}$.

Take $\underline{X} \in \mathbf{M}$, $\underline{X} = \sum_{i=1}^{n} x_i \underline{V}_i$. Then the linear form $\phi : \mathbf{M} \to \mathbf{A}$ defined by $\phi(\underline{X}) = x_n$ is the epiform with $\operatorname{Ker}\phi = N$.

2) Let ϕ be an epiform and let $\{\underline{E}_1, \dots, \underline{E}_n\}$ be a basis of the $\mathbf{A}$-space $\mathbf{M}$. Putting $\phi(\underline{E}_i) = a_i$, $1 \le i \le n$, we get for $\underline{X} \in \mathbf{M}$,

$$\underline{X} = \sum_{i=1}^{n} x_i \underline{E}_i, \qquad \phi(\underline{X}) = \sum_{i=1}^{n} x_i a_i.$$

As ϕ is an epiform then for some j, $1 \le j \le n$, a_j is a unit. Without loss of generality, assume a_n is a unit. For any j, $1 \le j \le n$ let us put

$$\underline{V}_j = a_n \underline{E}_j - a_j \underline{E}_n.$$

Evidently each of them turns the form ϕ to zero.

Let us suppose that $\sum_{j=1}^{n-1} b_j \underline{V}_j = \underline{o}$. Then

$$\sum_{j=1}^{n-1} (b_j a_n) \underline{E}_j - \left(\sum_{j=1}^{n-1} b_j a_j \right) \underline{E}_n = \sum_{j=1}^{n-1} b_j \left(a_n \underline{E}_j - a_j \underline{E}_n \right) = \sum_{j=1}^{n-1} b_j \underline{V}_j = \underline{o} \Rightarrow$$

$$\Rightarrow b_j a_n = 0 \Rightarrow b_j = 0 \quad (\text{for any } j = 1, \dots, n-1)$$

Let us consider the hyperplane $N = [\underline{V}_1, \dots, \underline{V}_{n-1}]$.

a) Clearly, $N \subseteq \operatorname{Ker}\phi$.

b) Let $\underline{X} = \sum_{i=1}^{n} x_i \underline{E}_i$ belong to $\operatorname{Ker}\phi$. Then

$$\sum_{i=1}^{n-1} a_i x_i + a_n x_n = 0.$$

And we may express x_n in the form

$$x_n = \sum_{i=1}^{n-1} (-a_i a_n^{-1}) x_i.$$

It follows from this

$$\underline{X} = \sum_{j=1}^{n-1} x_j \underline{E}_j + x_n \underline{E}_n = \sum_{j=1}^{n-1} x_j \underline{E}_j + \sum_{i=1}^{n-1} (-a_i a_n^{-1}) x_i \underline{E}_n =$$

$$= a_n^{-1} \left(\sum_{j=1}^{n-1} x_j \left(a_n \underline{E}_j - a_j \underline{E}_n \right) \right) = a_n^{-1} \left(\sum_{j=1}^{n-1} x_j \underline{V}_j \right)$$

i.e. $\underline{X} \in N$.

Proposition 12 *Let ϕ, ψ be epiforms. Then* $\operatorname{Ker}\phi = \operatorname{Ker}\psi$ *if and only if there exists a unit $e \in \mathbf{A}$ such that $\phi = e\psi$.*

Proof If there exists a unit e for that $\phi = e\psi$ then the both forms have obviously the same kernel. Now, let N be the common kernel of the form ϕ and ψ. Let $\{\underline{V}_1, \dots, \underline{V}_{n-1}\}$ be a basis of N. Then there exists a $\underline{V}_n \in \mathbf{M}$ such that $[\underline{V}_1, \dots, \underline{V}_{n-1}, \underline{V}_n] = \mathbf{M}$. Then

$$\forall \underline{X} \in \underline{M}: \quad \underline{X} = \sum_{i=1}^{n} x_i \underline{V}_i \Rightarrow \phi(\underline{X}) = x_n \phi(\underline{V}_n) \text{ and } \psi(\underline{X}) = x_n \psi(\underline{V}_n).$$

As ϕ, ψ are epiforms $\phi(\underline{V}_n)$, $\psi(\underline{V}_n)$ are units and therefore we can find e for which $\phi(\underline{V}_n) = e.\psi(\underline{V}_n)$. Hence $\phi = e\psi$.

Definition 13 *Let $K \subseteq \mathbf{M}$ be a submodule. Then by $\mathbb{A}(K)$ we denote the set*

$$\{\phi \in \mathbf{M}^*;\ \phi(\underline{X}) = 0,\ \forall \underline{X} \in K\}$$

Let $K \subseteq \mathbf{M}^*$ be a submodule. Then by $\mathbb{A}(K)$ we denote the set

$$\{\ \underline{X} \in \mathbf{M};\ \phi(\underline{X}) = 0,\ \forall \phi \in K\}$$

If we take
(a) the validity of corollaries of Proposition 5,
(b) the system of coordinate linear forms $\xi_1, \dots, \xi_n$ defined by

$$\xi_j\left(\sum_{i=1}^{n} x_i \underline{U}_i\right) = x_j, \qquad 1 \le j \le n,$$

with respect to the basis $\mathcal{U} = \langle \underline{U}_1, \dots, \underline{U}_n \rangle$ forms the basis of $\mathbf{M}^*$ dual to $\mathcal{U}$,
(c) every basis of $\mathbf{M}^*$ is dual to exactly one basis of $\mathbf{M}$
we may prove the following two lemmas in a similar way to the case of vector space.

Lemma 14 *If $K \subseteq \mathbf{M}$ is a $\mathbf{A}$-subspace then*

(1) $\mathbb{A}(K)$ *is a* $\mathbf{A}$*-subspace of* $\mathbf{M}^*$

(2) $\dim \mathbb{A}(K) + \dim K = \dim \mathbf{M}$.

Lemma 15 *If $K \subseteq \mathbf{M}^*$ is a $\mathbf{A}$-subspace, then*

(1) $\mathbb{A}(K)$ *is a* $\mathbf{A}$*-subspace of* $\mathbf{M}$

(2) $\dim \mathbb{A}(K) + \dim K = \dim \mathbf{M}$.

Lemma 16

If K, L are submodules of $\mathbf{M}$, then $\mathbb{A}(K + L) = \mathbb{A}(K) \cap \mathbb{A}(L)$.

If $\mathcal{K}$, $\mathcal{L}$ are submodules of $\mathbf{M}^*$, then $\mathbb{A}(\mathcal{K} + \mathcal{L}) = \mathbb{A}(\mathcal{K}) \cap \mathbb{A}(\mathcal{L})$.

Proof The validity of this lemma is evident.

Lemma 17

If $K \subseteq \mathbf{M}$ is a $\mathbf{A}$-subspace, then $\mathbb{A}(\mathbb{A}(K)) = K$.

If $K \subseteq \mathbf{M}^*$ is a $\mathbf{A}$-subspace, then $\mathbb{A}(\mathbb{A}(K)) = K$.

Proof It is a consequence of 14 and 15.

Theorem 18 *Let K, L be $\mathbf{A}$-subspaces of $\mathbf{A}$-space $\mathbf{M}$. Then $K + L$ is an $\mathbf{A}$-subspace if and only if the $K \cap L$ is an $\mathbf{A}$-subspace and the dimensions of $\mathbf{A}$-subspaces K, L, $K \cap L$, $K + L$ fulfil the following relation:*

$$\dim(K + L) + \dim(K \cap L) = \dim K + \dim L.$$

Proof (a) Let $K \cap L$ be a $\mathbf{A}$-subspace with a basis $\langle \underline{U}_1, \dots, \underline{U}_p \rangle$. Then there exist $\underline{U}_{p+1}, \dots, \underline{U}_k$ resp. $\underline{V}_{p+1}, \dots, \underline{V}_l$ such that $\langle \underline{U}_1, \dots, \underline{U}_p, \underline{U}_{p+1}, \dots, \underline{U}_k \rangle$ is a basis of K resp. $\langle \underline{U}_1, \dots, \underline{U}_p, \underline{V}_{p+1}, \dots, \underline{V}_l \rangle$ is a basis of L.

Obviously, $[\underline{U}_1, \dots, \underline{U}_p, \underline{U}_{p+1}, \dots, \underline{U}_k, \underline{V}_{p+1}, \dots, \underline{V}_l] = K + L$. It remains to prove the linear independence of this system. Let $u_1, \dots, u_k, v_{p+1}, \dots, v_l$ be the elements of $\mathbf{A}$ such that

$$u_1\underline{U}_1 + \cdots + u_k\underline{U}_k + v_{p+1}\underline{V}_{p+1} + \cdots + v_l\underline{V}_l = \underline{o}.$$

Putting $\underline{U} = u_1\underline{U}_1 + \cdots + u_k\underline{U}_k$ we get $\underline{U} = -\left(v_{p+1}\underline{V}_{p+1} + \cdots + v_l\underline{V}_l\right)$. Hence $\underline{U} \in K \cap L$ and it can be written by $\underline{U} = \sum_{i=1}^{p} x_i\underline{U}_i$. Thus $u_{p+1} = \cdots = u_k = 0$ which implies $u_1 = \cdots = u_p = v_{p+1} = \cdots = v_l = 0$.

(b) Let $K + L$ be a $\mathbf{A}$-subspace. Then $\mathbb{A}(K)$, $\mathbb{A}(L)$, $\mathbb{A}(K + L)$ are free as well (by 14). Due to 16 and 17 we get

$$K \cap L = \mathbb{A}(\mathbb{A}(K)) \cap \mathbb{A}(\mathbb{A}(L)) = \mathbb{A}(\mathbb{A}(K) + \mathbb{A}(L)). \qquad (***)$$

Since $\mathbb{A}(K) \cap \mathbb{A}(L) = \mathbb{A}(K + L)$ is a $\mathbf{A}$-subspace, by the part (a) we obtain that $\mathbb{A}(K) + \mathbb{A}(L)$ is a $\mathbf{A}$-subspace, too. It follows from $(***)$ that $K \cap L$ is a $\mathbf{A}$-subspace. The relation between dimensions is evident.

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