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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 35 (1996), No. 1, 21--24

Persistent URL: `http://dml.cz/dmlcz/120347`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **35** (1996) 21–24

On a Classification of Almost Geodesic Mappings of Affine Connection Spaces *

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(Received January 18, 1996)

Abstract

In the paper a classification of almost geodesic mappings is specified. It is proved that if an almost geodesic mapping f is simultaneously π_1 and π_2 (or π_3) then f is a mapping of affine connection spaces with preserved linear (or quadratic) complex of geodesic lines.

Key words: Almost geodesic mapping, affine connection space, classification.

1991 Mathematics Subject Classification: 53B05

The present paper is devoted to an investigation of completeness of a classification of almost geodesic mappings of affine connection spaces A_n without the torsion.

In [4, 5] the almost geodesic mappings of an affine connection space A_n were introduced and three types of them were distinguished, π_1, π_2 and π_3. We proved [1, 2] that for $n > 5$ other types of almost geodesic mappings do not exist. However, one can not exclude the case when a mapping π_τ $(\tau = 1, 2, 3)$ is simultaneously a mapping π_σ $(\sigma \neq \tau)$.

*Supported by grant No. 201/96/0227 of the Grant Agency of Czech Republic.

In this paper we characterize non-overlapping types of almost geodesic mappings. We receive the complete classification of these mappings for $n > 5$.

The curve l: $x^h = x^h(t)$ is *almost geodesic* in an affine connection space A_n if there exists a distribution E_2, complanar along l, to which the tangent vector $\lambda^h \equiv dx^h/dt$ of this curve belongs at every point. The diffeomorphism f: $A_n \to \overline{A}_n$ is *almost geodesic* if, as a result of f, every geodesic of the space A_n passes into an almost geodesic curve of the space $\overline{A}_n$.

The mapping $A_n \to \overline{A}_n$ is almost geodesic if and only if the connection deformation tensor $P^h_{ij}(x) \equiv \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ satisfies the relation [4, 5]

$$P^{[h}_{(\alpha\beta\gamma}\, P^{i}_{\delta\epsilon}\, \delta^{j]}_{\eta)} = 0,$$

where

$$P^h_{ijk} \equiv P^h_{ij,k} + P^\alpha_{ij}\, P^h_{k\alpha},$$

$\Gamma^h_{ij}(x)$ and $\overline{\Gamma}^h_{ij}(x)$ are objects of connection A_n and $\overline{A}_n$, δ^h_i is the Kronecker symbol, square and round brackets denote the alternation and symmetrization of indices without division, respectively, comma denotes the covariant derivative with respect to the connection on A_n.

N. S. Sinyukov [4, 5] defined three kinds of almost geodesic mappings, namely π_1, π_2,and π_3 which are characterized, respectively, by the conditions

$$\pi_1: \qquad P^h_{(ij,k)} + P^\alpha_{(ij}P^h_{k)\alpha} = \delta^h_{(i}a_{jk)} + b_{(i}P^h_{jk)}; \tag{1}$$

$$\pi_2: \qquad P^h_{ij} = \delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)}, \tag{2}$$

$$F^h_{(i,j)} + F^h_\alpha F^\alpha_{(i}\varphi_{j)} = \delta^h_{(i}\mu_{j)} + F^h_{(i}\sigma_{j)}; \tag{3}$$

$$\pi_3: \qquad P^h_{ij} = \delta^h_{(i}\psi_{j)} + \varphi^h\omega_{ij}, \tag{4}$$

$$\varphi^h_{,i} = \rho\delta^h_i + \varphi^h a_i, \tag{5}$$

where $a_{ij}, b_i, \psi_i, \varphi^h, \omega_{ij}, a_i, F^h_i, \rho$ are tensors of the corresponding valencies.

Under an almost geodesic mapping, only the mappings π_1, π_2 and π_3 act in the neighborhood of every point of the space A_n $(n > 5)$, exept, maybe, the set of points of measure zero [1, 2].

It is natural to presume that the affinor F^h_i of the mapping π_2 satisfies $F^h_i \not\equiv \rho\delta^h_i + \varphi^h a_i$ and $\varphi^h\omega_{ij} \not\equiv 0$ for the mapping π_3. Then $\pi_2 \cap \pi_3 = \emptyset$. Indeed, let us suppose, that a mapping is simultaneously π_2 and π_3. Then (2) and (4) imply

$$\delta^h_{(i}\psi_{j)} + F^h_{(i}\varphi_{j)} = \delta^h_{(i}\overset{*}{\psi}_{j)} + \varphi^h\omega_{ij}. \tag{6}$$

Since $\varphi_i \not\equiv 0$ then there exists a vector ϵ^i such that $\epsilon^\alpha\varphi_\alpha = 1$. Contracting (6) with $\epsilon^i\epsilon^j$ we get

$$F^h_\alpha\epsilon^\alpha = \alpha\epsilon^h + \beta\varphi^h,$$

where α, β are functions.

By the help of the above formula and after contracting (6) with ϵ^j we have

$$F_i^h = \rho\delta_i^h + \varphi^h a_i$$

which was required to prove.

Theorem 1 *If an almost geodesic mapping f is simultaneously π_1 and π_2 then f is a mapping of an affine connection space with preserving a linear complex of geodesic lines.*

Proof Let a mapping f be an almost geodesic mapping of types π_1 and π_2 simultaneously. After substituting (2) in (1) and taking into account (3) one finds

$$\delta^h_{(i}A_{jk)} + F^h_{(i}B_{jk)} = 0 \tag{7}$$

where $B_{jk} \equiv \varphi_{(j,k)} - \varphi_{(j}\theta_{k)}$, A_{jk}, θ_k are tensors. Equation (7) implies $A_{jk} \equiv 0$ and $B_{jk} \equiv 0$.

The construction of these tensors shows that relation

$$\varphi_{(i,j)} = \varphi_{(i}\theta_{j)} \tag{8}$$

is correct.

A mapping π_2 such that (2), (3) and (8) hold, is, evidently, a mapping π_1. On the other hand, equations (2) and (8) characterize mappings preserving a linear complex of geodesic lines [3]. The theorem is proved.

Theorem 2 *If an almost geodesic mapping f is simultaneously π_1 and π_3 then f is a mapping of an affine connection space which preserves a quadratic complex of geodesic lines.*

Proof Let a mapping f be an almost geodesic mapping of types π_1 and π_3 simultaneously. After substituting (4) in (1) and taking into account (5) we obtain

$$\delta^h_{(i}A_{jk)} + \varphi^h B_{ijk} = 0, \tag{9}$$

where $B_{ijk} \equiv \omega_{(ij,k)} - a_{(i}\omega_{jk)}$, A_{jk}, a_i are tensors. From (9) we have $A_{jk} \equiv 0$ and $B_{ijk} \equiv 0$.

From here we get

$$\omega_{(ij,k)} = a_{(i}\omega_{jk)}. \tag{10}$$

Mappings π_3 given by (4), (5) and satisfying conditions (10) are π_1 mappings.

On the other hand, equations (4) and (10) characterize mappings preserving a quadratic complex of geodesic lines [3]. The theorem is proved.

In a natural way, there are distiguished mappings $\pi_{12} = \pi_1 \cap \pi_2$ and $\pi_{13} = \pi_1 \cap \pi_3$.

As we have already noted, mappings π_{12} preserve a linear complex of geodesic lines and these mappings are characterized by equations (2), (3) and (8).

Mappings π_{13} preserve a quadratic complex of geodesic lines and are characterized by equations (4), (5) and (10).

Theorem 3 *The space A_n $(n > 5)$, except, maybe, the set of measure zero, is divided into open domains. In each of them one of the following six mappings acts: geodesic,* π_{12}, π_{13}, $\pi_1 \setminus \{\pi_2 \cup \pi_3\}$, $\pi_2 \setminus \pi_1$, $\pi_3 \setminus \pi_1$.

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