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Composition of Indpendent Stochastic Variables *

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Abstract

Let a random variable and a family of random variables depending on a parameter be given. The direct substitution of the random variable for the parameter may have no sense or can have unexpected properties even if the stochastic independence is assumed. In the paper, the composition is defined in a rigorous way and it is shown that the conditional expectation of the composition with respect to the substituted random variable is expressed in the usual manner (Theorem 1). The problem is analyzed for the random variables having values in Banach spaces and the results are applied to Wiener integrals depending on a parameter.

Key words: Composition of random variables, stochastic variables with values in Banach spaces, stochastic independence.

1991 Mathematics Subject Classification: 60A05

The paper deals with the generalization of the Proposition 2.2 [1]. The sketch of the subject of this Proposition is following.

Let a family of Ito stochastic integrals (with values in a Hilbert space) $\int_0^t \sigma(s,y)\,dw(s)$ and a stochastic variable u be given. Assume that u is $\mathcal{F}_u$ measurable and $\mathcal{F}_u$ is stochastically independent of a Wiener process $w(t)$. The properties of the expression $E[\int_0^t \sigma(s,u)\,dw(s)|\mathcal{F}_u]$ (see 2.1 from [1] or Theorem 3 from the present paper) are studied.

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In the present paper the integrals are substituted by a family of stochastic variables $B(\xi)$ (with values in a Banach space) depending on a parameter ξ. In this case the condition are given such that $B(u)$ can be constructed and Theorem 2 (which corresponds to Proposition 2.2 [1]) is proved.

Let X_1, X_2 be real Banach spaces with norms $\|.\|_1, \|.\|_2$, X_3 be a real separable Banach space with a norm $\|.\|_3$ and R the real line. Let $(\Omega, \mathcal{F}, P)$ be a probability space and E denotes the mathematical expectation.

Denote $L_p(\Omega, X_2)$ the set of all Bochner integrable random variables with values in $X_2, p \geq 1$. Let $\varphi(\xi) : X_1 \to R$, $\varphi(\xi) > 0$ for $\xi \in X_1$ be Borel measurable.

Definition 1 Let $B : X_1 \to L_p(\Omega, X_2)$ be a mapping. B is called *φ-Bochner type mapping (or shortly Bochner type mapping)* if

$$K = \sup\{(\varphi(\xi))^{-1} E\|B(\xi)\|_2^p : \xi \in X_1\} < \infty, \tag{1}$$

if there exists a sequence of partitions $\{A_n^{(i)}\}_{i=1}^{\infty}$, $X_1 = \cup_i A_n^{(i)}$ (disjoint union), $A_n^{(i)}$ are Borel sets, a sequence of sets $\{\Lambda_n^{(i)}\}_{i=1}^{\infty}$, $\Lambda_n^{(i)} \in \mathcal{F}$, $P(\Lambda_n^{(i)}) = 0$ and a sequence of mappings $B_n : X_1 \to L_p(\Omega, X_2)$ such that $B_n(\xi, \omega) = B_n(\eta, \omega)$ for $\xi, \eta \in A_n^{(i)}$, $\omega \notin \Lambda_n^{(i)}$ and B_n converge to B in following sense

$$\sup\{(\varphi(\xi))^{-1} E\|B(\xi) - B_n(\xi)\|_2^p : \xi \in X_1\} \to 0 \quad \text{for } n \to \infty. \tag{2}$$

1 Definition of composition

(3) *Let $u : \Omega \to X_1$ be a random variable $\mathcal{F}_u$-measurable, $\mathcal{F}_u$—a σ-subfield of $\mathcal{F}$,*

$$E\varphi(u) < \infty.$$

Let B be a Bochner type mapping. Assume that $B_n(\xi)$ are stochastically independent of $\mathcal{F}_u$ for all ξ. Define $q_n(\omega) = B_n(u(\omega), \omega)$ for almost all ω.

Let

$$\Gamma_n^{(i)} = u^{-1}(A_n^{(i)}) = \{\omega : u(\omega) \in A_n^{(i)}\}$$

and $z_n^{(i)}$ be elements of $A_n^{(i)}$ fulfilling

$$\varphi(z_n^{(i)}) < \inf\{\varphi(\xi) : \xi \in A_n^{(i)}\} + 1.$$

Let $n > m$. We can assume that $\{A_n^{(i)}\}_{i=1}^{\infty}$ is a refinement of $\{A_m^{(i)}\}_{i=1}^{\infty}$.

We have

$$E\|q_n - q_m\|_2^p = \sum_i \int_{\Gamma_n^{(i)}} \|q_n - q_m\|_2^p \, dP$$

$$= \sum_i \int_{\Gamma_n^{(i)}} \|B_n(z_n^{(i)}) - B_m(z_n^{(i)})\|_2^p \, dP$$

$$= \sum_i \int_{\Gamma_n^{(i)}} E[\|B_n(z_n^{(i)}) - B_m(z_n^{(i)})\|_2^p \,|\mathcal{F}_u]\, dP$$

$$= \sum_i \int_{\Gamma_n^{(i)}} E\|B_n(z_n^{(i)}) - B_m(z_n^{(i)})\|_2^p\, dP$$

$$= \sum_i E\|B_n(z_n^{(i)}) - B_m(z_n^{(i)})\|_2^p P(\Gamma_n^{(i)})$$

$$\leq 2^{p-1} \sum_i E\|B_n(z_n^{(i)}) - B(z_n^{(i)})\|_2^p P(\Gamma_n^{(i)}) +$$

$$+ 2^{p-1} \sum_i E\|B_m(z_n^{(i)}) - B(z_n^{(i)})\|_2^p P(\Gamma_n^{(i)})$$

$$\leq 2^p \varepsilon \sum_i \varphi(z_n^{(i)}) P(\Gamma_n^{(i)}) \leq 2^p \varepsilon \sum_i \int_{\Gamma_n^{(i)}} (\varphi(u(\omega)) + 1)\, dP$$

$$\leq 2^p \varepsilon E(\varphi(u) + 1).$$

The three last inequalities are valid for sufficiently great n, m due to (2).

In a similar manner we can derive an estimate

$$E\|q_n\|_2^p \leq 2^{p-1}(\varepsilon + K) E(\varphi(u) + 1)$$

for sufficiently great n (see (1), (2)).

Obviously the sequence $\{q_n\}$ is a Cauchy sequence in $L_p(\Omega, X_2)$ and we can define $B(u) = \lim_{n\to\infty} q_n$. The definition of $B(u)$ is independent of the choice of B_n since from two sequences $B_n^{(1)}, B_n^{(2)}$ we can compose one commom sequence. It means that $B^{(1)}(u)$ and $B^{(2)}(u)$ defined by $B_n^{(1)}$ and $B_n^{(2)}$, respectively are equivalent.

Proposition 1 *Let a Bochner type mapping $B(\xi)$ be given. Assume that a mapping u fulfils (3). If $B_n(\xi)$ are stochastically independent of $\mathcal{F}_u$ then a random variable $B(u)$ is defined so that $B(u) \in L_p(\Omega, X_2)$.*

Corollary 1 *Let $B(\xi)$ be φ-Bochner type mapping and let $\hat{B}(\xi)$ be a mapping from Ω to X_2 fulfilling $P(B(\xi) \neq \hat{B}(\xi)) = 0$ for every $\xi \in X_1$. Then the mapping $\hat{B}(\xi)$ is also φ-Bochner type.*

Assume that a random variable u fulfils the assumptions of the Proposition with respect to $B(.)$ then it fulfills the assumptions of the Proposition with respect to $\hat{B}(.)$ and $B(u), \hat{B}(u)$ are equivalent.

This follows from the fact that

$$E\|B_n(\xi) - \hat{B}(\xi)\|_2^p = E\|B_n(\xi) - B(\xi)\|_2^p$$

so that $B_n(.)$ are suitable approximations of $\hat{B}(.)$, too.

Example 1 The first example shows that the definition of $B(u)$ just by substitution has no meaning even if such mapping can be defined. Generally the mapping $T(u)$ defined by $T(u)(\omega) = B(u(\omega), \omega))$ need not be measurable.

Let $\Omega = (-\pi/2, \pi/2)$, $\mathcal{F}$ be Borel measurable subsets, P be Lebesgue measure multiplied by $1/\pi$. Define $u(\omega) = \tan\omega$. Assume that Γ_1 is a nonmeasurable subset of Ω. Denote Γ_2 the complement of Γ_1. For every $\xi \in R$ there exists unique ω_ξ so that $\xi = \tan\omega_\xi$. Define:

$$B(\xi, \omega_\xi) = 1 \text{ for } \omega_\xi \in \Gamma_1, \qquad B(\xi, \omega_\xi) = -1 \text{ for } \omega_\xi \in \Gamma_2,$$

$$B(\xi, \omega) = 0 \text{ for } \omega \neq \omega_\xi.$$

Certainly, $T(\omega) = 1$ for $\omega \in \Gamma_1$ and $T(\omega) = -1$ for $\omega \in \Gamma_2$. Nevertheless, the random variables $B(\xi)$ are equivalent to zero for every ξ.

The second example shows that the mapping $T(.)$ is not appropriate even if the assumptions of the Corollary are fulfilled.

Example 2 Let $B(.), u$ fulfil the assumptions of the Proposition. Define $\hat{B}$: $\hat{B}(\xi, \omega) = B(\xi, \omega) + \chi_\xi(\omega)$ where χ_ξ are the indicators of sets Λ_ξ fulfilling $P(\Lambda_\xi) = 0$. Certainly the random variables $B(\xi), \hat{B}(\xi)$ are equivalent for all $\xi \in X_1$ and due to the Corollary $\hat{B}(.), u$ fufills the assumptions, too. Nevertheless, if $P(\omega : \omega \in \Lambda_{u(\omega)}) > 0$ then $T(\omega), \hat{T}(\omega) = \hat{B}(u(\omega), \omega)$ are not equivalent.

Theorem 1 *Let $B : X_1 \to L_p(\Omega, X_2)$ be a Bochner type mapping, $u : \Omega \to X_1$ be a random variable fulfilling (3), $B_n(\xi)$ are stochastically independent of $\mathcal{F}_u$ and let $\sigma : X_1 \times X_2 \to X_3$ be a Borel measurable mapping fulfilling*

$$\|\sigma(\xi, \zeta)\|_3 \leq L(\|\zeta\|_2^p + 1) \tag{4}$$

for all $\xi \in X_1$, $\zeta \in X_2$. Set $\psi(\xi) = E\sigma(\xi, B(\xi))$ for $\xi \in X_1$.
Then

$$E[\sigma(u, B(u))|\mathcal{F}_u](.) = \psi(u)(.) \qquad P\text{-}a.e.$$

Proof First we prove a modification of Propositon 1.12 from [2].

Lemma 1 *Let $u : \Omega \to X_1$ be a random variable, $v : \Omega \to X_2$ be a random variable independent of $\mathcal{F}_u$, fulfilling*

$$E\|v\|_2^p < \infty, \tag{5}$$

$\rho : X_1 \times X_2 \to X_3$ be measurable and fulfilling

$$\|\rho(\xi, \zeta)\|_3 \leq L(\|\zeta\|_2^p + 1)$$

for all $\xi \in X_1, \zeta \in X_2$. Denote $\hat{\rho}(\xi) = E\rho(\xi, v)$ for $\xi \in X_1$.
Then

$$E[\rho(u, v))|\mathcal{F}_u](.) = \hat{\rho}(u)(.) \qquad P\text{-}a.e. \tag{6}$$

Proof of the Lemma. Due to the Proposition 1.12 from [2] equality (6) is valid for ρ which can be expressed as a finite summ $\rho(\xi,\zeta) = \sum_i \rho_i\chi_i(\xi,\zeta))$ where A_i are disjoint Borel subsets of $X_1 \times X_2$ and $\rho_i \in X_3$. Choosing positive ε there exists a partion $\{A_i\}$ of $X_1 \times X_2$ consisting of Borel disjoint sets and a sequence of $\rho_i \in X_3$ such that $\|\rho(\xi,\zeta) - \rho_i\|_3 < \varepsilon$ for $[\xi,\zeta] \in A_i$. We prove

$$E\|\rho(u,v) - \sum_i^N \rho_i\chi_{A_i}(u,v))\|_3 < 2\varepsilon$$

for sufficiently big N. Certainly, the given expression can be estimated by

$$\sum_i^N \int \chi_{A_i}(u,v)\|\rho(u,v) - \rho_i\|_3 \, dP + \sum_{i=N+1}^{\infty} \int \chi_{A_i}(u,v)\|\rho(u,v)\|_3 \, dP \le$$

$$\varepsilon + \sum_{i=N+1}^{\infty} \int \chi_{A_i}(u,v) L(1 + \|v\|_2^p) \, dP.$$

Since $E\|v\|_2^p < \infty$ the last expression converges to 0 for $N \to \infty$ and the inequality is proved. In the same way we can prove

$$\int_\Omega \|\{E\rho(\xi,v)\}_{\xi=u} - \{E\sum_i^N \rho_i\chi_{A_i}(\xi,v))\}_{\xi=u}\|_3 \, dP < 2\varepsilon$$

for sufficiently big N. The proved inequalities together with the fact that (6) is valid for finite summes of this type imply the statement of the Lemma. The existence of the conditional expectation is ensured by Proposition 1.10 [2].

Proof of Theorem 1. (i) Assume that σ is continuous in the second variable. Define $\psi_n(\xi,\eta) = E(\sigma(\xi, B_n(\eta)))$ for $\xi,\eta \in X_1$, $\{A_n^{(i)}\}$ is the partition from Definition 1 and

$$\psi_n(\xi) = \sum_i \chi_{A_n^{(i)}}(\xi)\psi_n(\xi, z_n^{(i)}),$$

$\chi_{A_n^{(i)}}$ is the indicator of the set $A_n^{(i)}$. Due to the Lemma we have

$$E[\sigma(u, B_n(\eta))|\mathcal{F}_u] = \psi_n(u, z_n^{(i)}) \quad \text{for } \eta \in A_n^{(i)}.$$

Let $\Lambda \in \mathcal{F}_u$. We have

$$\int_\Lambda E[\sigma(u,q_n)|\mathcal{F}_u] \, dP =$$

$$= \sum_i \int_{\Lambda\cap\Gamma_n^{(i)}} E[\sigma(u, B_n(z_n^{(i)}))|\mathcal{F}_u] \, dP$$

$$= \sum_i \int_{\Lambda\cap\Gamma_n^{(i)}} \psi_n(u, z_n^{(i)}) \, dP = \int_\Lambda \sum_i \chi_{A_n^{(i)}}(u)\psi_n(u, z_n^{(i)}) \, dP = \int_\Lambda \psi_n(u) \, dP \, .$$

Since $E\|q_n - B(u)\|_2^p \to 0$ for $n \to \infty$ we can choose a subsequence such that $E\|\hat{q}_n - B(u)\|_2^p < 2^{-n}$ and such that $\{\hat{q}_n\}$ converges to $B(u)$ almost surely. Certainly $\sigma(u, \hat{q}_n)$ converges almost surely to $\sigma(u, B(u))$. Further we have

$$\|\sigma(u, \hat{q}_n)\|_3 \le L(\|\hat{q}_n)\|_2^p + 1) \le 2^{p-1}L(\|\hat{q}_n - B(u)\|_2^p + \|B(u)\|_2^p + 1).$$

Let $h = 2^{p-1}L(\|B(u)\|_2^p + 1 + \sum_{n=1}^{\infty} \|\hat{q}_n - B(u)\|_2^p)$. We have $\|\sigma(u, \hat{q}_n)\|_3 \le h$ and $Eh \le 2^{p-1}(3L + LE\|B(u)\|_2^p)$. The Lebesgue theorem yields

$$\lim_{n\to\infty} \int_\Lambda E[\sigma(u, \hat{q}_n)|\mathcal{F}_u]\, dP = \lim_{n\to\infty} \int_\Lambda \sigma(u, \hat{q}_n)\, dP$$

$$= \int_\Lambda \sigma(u, B(u))\, dP = \int_\Lambda E[\sigma(u, B(u))|\mathcal{F}_u]\, dP.$$

We obtain as previously (we can assume $u = \xi$ for a while) for a subsequence $\hat{B}_n(u)$ of $B_n(u)$

$$\psi(\xi) = E\sigma(\xi, B(\xi)) = \lim_n E\sigma(\xi, \hat{B}_n(\xi))$$

$$= \lim_n \int_\Omega \sum_i \chi_{A_n^{(i)}}(\xi)\sigma(\xi, \hat{B}_n(z_n^{(i)}))\, dP$$

$$= \lim_n \sum_i \chi_{A_n^{(i)}}(\xi) E\sigma(\xi, \hat{B}_n(z_n^{(i)})) = \lim_n \hat{\psi}_n(\xi)$$

where $\hat{\psi}_n$ corresponds to $\hat{B}_n$.

Since

$$\|\sigma(\xi, B_n(z_n^{(i)}))\|_3 \le L(\|B_n(z_n^{(i)}))\|_2^p + 1)$$

we have

$$E\|\sigma(\xi, B_n(z_n^{(i)}))\|_3 \le 2^{p-1}L(1 + E\|B_n(z_n^{(i)}) - B(z_n^{(i)}))\|_2^p + E\|B(z_n^{(i)}))\|_2^p)$$

$$\le 2^{p-1}L(1 + (\varepsilon + K)\varphi(z_n^{(i)}))$$

for sufficiently great n and every i. It follows

$$\|\psi_n(u)\|_3 \le \sum_i \chi_{A_n^{(i)}}(u) 2^{p-1} L(1 + (\varepsilon + K)\varphi(z_n^{(i)}))$$

$$\le 2^{p-1}L(1 + (\varepsilon + K)\sum_i \chi_{A_n^{(i)}}(u)\varphi(z_n^{(i)}))$$

$$\le 2^{p-1}L(1 + (\varepsilon + K)\sum_i \chi_{A_n^{(i)}}(u)(\varphi(u) + 1))$$

$$\le 2^{p-1}L(1 + (\varepsilon + K)(\varphi(u) + 1))$$

for almost all ω. The Lebesgue theorem yields

$$\lim_{n\to\infty} \int_\Lambda \hat{\psi}_n(u)\, dP = \int_\Lambda \psi(u)\, dP.$$

Finally, we have

$$\int_\Lambda E[\sigma(u,B(u))|\mathcal{F}_u]\,dP = \int_\Lambda \psi(u)\,dP.$$

(ii) Let σ_n be Borel measurable mappings $X_1 \times X_2 \to X_3$ fulfilling (4),

$$\int_\Lambda \sigma_n(u,B(u))\,dP = \int_\Lambda \psi_n(u)\,dP \tag{7}$$

for $\Lambda \in \mathcal{F}_u$ where $\psi_n(\xi) = E\sigma_n(\xi, B(\xi))$. If $\sigma_n(\xi,\zeta) \to \sigma(\xi,\zeta)$ for every ξ,ζ then $\psi_n(\xi) \to \psi(\xi)$ and equality (7) is valid for σ, ψ, too. Since due to (i) the equality (7) is valid for continuous σ it is valid for all Borel measurable mappins fufilling (4).

Corollary 2 *If the assumptions of Theorem 1 are fufilled then*

$$P\{B(u) \in A\} = \int_{X_1} P\{B(\xi) \in A\}\,d\nu(\xi)$$

is valid for Borel sets A where ν is the measure on X_1 corresponding to u.

Theorem 2 *Let X_1 be a separable real Banach space, the function φ be Borel measurable and fulfil $\inf\{\varphi(\xi) : \xi \in X_1\} > 0$. Assume that B fulfils (1) and as a mapping: $X_1 \to L_p(\Omega, X_2)$ is continuous. Then $B(.)$ is φ-Bochner type. If u is a random variable $u : \Omega \to X_1$ such that $B(\xi)$ are stochastically indpendent of $\mathcal{F}_u$ for every $\xi \in X_1$ then $B(.), u$ fulfils the assumptions of the Proposition such that $B(u)$ is defined and Theorem 1 is valid.*

Proof Let $\{\xi_n\}_1^\infty$ be a sequence dense in X_1. We construct a partition of X_1. Let a positive integer n be given.

$$A^{(1)} = \{\xi : E\|B(\xi) - B(\xi_1)\|_2^p < \delta/n\},$$

$$A^{(i+1)} = \{\xi : E\|B(\xi) - B(\xi_i)\|_2^p < \delta/n\} - \cup_{j=1}^i A^{(j)}$$

where $\delta = \inf\{\varphi(\xi)\}$. The approximations can be defined by $B_n(\xi) = B(\xi_i)$ for $\xi \in A^{(i)}$. These approximations fulfil all conditions of the Proposition.

Remark 1 The calculation of $B(.)$ by means of $T(.)$ needs much more stronger assumptions.
(H) *There exists a set $\Lambda, \Lambda \in \mathcal{F}$ such that $P(\Lambda) = 0$ and*

$$B_n(\xi,\omega) \to B(\xi,\omega), \quad n \to \infty$$

for every $\xi \in X_1$ and $\omega \notin \Lambda$.

Statement. *Let the assumptions of the Proposition be fulfilled. If additionaly the hypotheses (H) is fulflled, then*

$$B(u)(\omega) = T(\omega) \quad \textit{for a.a. } \omega.$$

Sketch ot the proof. Since $q_n(\omega)$ converge to $B(u)$ in the space $L_p(\Omega, X_2)$ there exists a subsequence $\tilde{q}_n(\omega)$ converging to $B(u)$ almost everywhere. Due to (H) for every $\xi = u(\omega), \tilde{B}_n(\xi, \omega)$ converge to $B(\xi, \omega)$ which equals $B(u(\omega), \omega)$ such that $T(\omega) = B(u)(\omega)$ for almost all ω.

2 Application to stochastic integral

If an integral

$$\int_a^b f(t, \xi)\, dw(t)$$

is continuous as a function of ξ in the sense of Theorem 2 this theorem can be applied. Otherwise the situation is much more complicated.

2.1 Wiener processes with nuclear covariant operators

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, P)$ be a probability space, H_i, $i = 1, 2, 3$ be real separable Hilbert spaces with norms $|.|_i$, $I = [0, 1]$, $\varphi : H_1 \to R$ be a positive Borel measurable funtion. Let $w(t)$, $t \in [0, 1]$ be a Wiener process with respect to $\{\mathcal{F}_t\}$ with values in H_2 and nuclear covariance operator W.

Let $\mathcal{L}(H_2, H_3)$ be a space of linear operators A from H_2 to H_3 which is equipped with the norm $||A||^2_{\mathcal{L}} = tr\{AWA^\star\}$ where tr is the trace of the given operator from H_3 to H_3.

Denote $X_1 = H_1$, $X_2 = L_p(I, \mathcal{L}(H_2, H_3)), p \geq 2$. Let $f : X_1 \to L_p(\Omega, X_2)$ be a Bochner type mapping such that the approximations $f_n(., \xi)$ fulfil: $f_n(t, \xi)h_2$ are $\mathcal{F}_t$ adapted and measurable stochastic processes for every $\xi \in H_1$, $h_2 \in H_2$.

Let $u : \Omega \to H_1$ be a random variable which is $\mathcal{F}_0$ measurable, $\{f_n(t, \xi), t \in I, \xi \in X_1\}$ are independent of $\mathcal{F}_0$ and $E\varphi(u) < \infty$.

Assume that a mapping $\sigma : H_1 \times H_3 \to X_3$ fulfils the conditions of Theorem 1. Denote

$$\psi(\xi) = E\sigma(\xi, \int_0^1 f(t, \xi)\, dw(t)).$$

Theorem 3 *Let the previous assumptions be fulfiled then the integral* $I(u) = \int_0^1 f(t, u)\, dw$ *exists and*

$$E[\sigma(u, I(u))|\mathcal{F}_0](.) = \psi(u)(.) \qquad P\textit{-a.e.}$$

is fulfilled.

Proof First, we have to prove that integrals $I_n(u) = \int_0^1 f_n(t,u)\,dw$ and $I(u) = \int_0^1 f(t,u)\,dw$ exist. In the second step we prove $I(u) = B(u)$ where $B(\xi) = \int_0^1 f(t,\xi)\,dw(t)$ i.e. that the usual definition of the integral and the definition given in the first section give the same result and that $B(u)$ is Bochner type. The last step is the application of Theorem 1.

Let $A_n^{(i)}$ be the partition given by the Bochner property of f. Choose $z_n^{(i)}$ from $A_n^{(i)}$ as in the previous sections. Since

$$f_n(t,u) = \sum_i f_n(t,z_n^{(i)})\chi_{A_n^{(i)}}(u) \tag{8}$$

and u is $\mathcal{F}_0$ measurable we conclude that the process $f_n(t,u)h_2$ is $\mathcal{F}_t$ adapted and measurable for every $h_2 \in H_2$. Using the inequality

$$\{\int_0^1 [E\|f\|_{\mathcal{L}}^p]^{2/p}\,dt\}^{p/2} \le \int_0^1 E\|f\|_{\mathcal{L}}^p\,dt$$

the existence of $I_n(u)$ is ensured by the inequality

$$S = \int_0^1 E[tr\, f_n(t,u)Wf_n^\star(t,u)]^{p/2}\,dt < \infty$$

(see the inequality from [4] Prop. 1.9 or [3] Lemma 1). Certainly we have

$$S = \int_0^1 E[\sum_i tr\, f_n(t,z_n^{(i)})Wf_n^\star(t,z_n^{(i)})\chi_{A_n^{(i)}}(u)]^{p/2}\,dt.$$

Since

$$\{\sum_i tr\, f_n(t,z_n^{(i)})Wf_n^\star(t,z_n^{(i)})[\chi_{A_n^{(i)}}(u)]^{2/p}] \times [\chi_{A_n^{(i)}}(u)]^{1-2/p}\}^{p/2} \le$$

$$\le \sum_i [tr\, f_n(t,z_n^{(i)})Wf_n^\star(t,z_n^{(i)})]^{p/2}[\chi_{A_n^{(i)}}(u)] \times [\sum_j \chi_{A_n^{(j)}}(u)]^{p/2-1}$$

$$\le \sum_i [tr\, f_n(t,z_n^{(i)})Wf_n^\star(t,z_n^{(i)})]^{p/2}[\chi_{A_n^{(i)}}(u)]$$

we have

$$S \le \int_0^1 \sum_i E[tr\, f_n(t,z_n^{(i)})Wf_n^\star(t,z_n^{(i)})]^{p/2}[\chi_{A_n^{(i)}}(u)]\,dt$$

and since

$$K' = \sup_n \sup_\xi \varphi(\xi)^{-1}E\int_0^1 \|f_n(t,\xi)\|_{\mathcal{L}}^p\,dt < \infty$$

(consider (2), the facts that $f_n(.,\xi) \in L_p(\Omega, X_2)$ and $f_n(.,\xi)$ are stochastically independet of u) we deduce similarly as in previous sections

$$S \le K'\int_0^1 \sum E\varphi(z_n^{(i)})\chi_{A_n^{(i)}}(u)\,dt \le K'E(\varphi(u)+1) < \infty.$$

We conclude that the integrals $I_n(u)$ exist and due to (8)

$$I_n(u) = \int_0^1 f_n(t,u)\,dw = \sum \chi_{A_n^{(i)}}(u) \int_0^1 f_n(t, z_n^{(i)})\,dw = B_n(u)$$

where

$$B_n(\xi) = \int_0^1 f_n(t,\xi)\,dw.$$

$f(t,u)$ is defined in the first section. Since $f_n(t,u)$ converges to $f(t,u)$ in the space $L_p(\Omega, X_2)$, the processes $f(t,u)h_2$ are $\mathcal{F}_t$ adapted measurable for every $\xi \in H_1$ and fulfils

$$E\int_0^1 \|f(t,u)\|_{\mathcal{L}}^p\,dt < \infty.$$

Further

$$E\int_0^1 \|(f(t,u) - f_n(t,u)\|_{\mathcal{L}}^p\,dt \to 0.$$

It follows that $I(u)$ exists and the well-known inequality (see [3])

$$E\|\int_0^1 (f(t,u) - f_n(t,u))\,dw\|_2^p \le C(p)\{\int_0^1 [E\|f(t,u) - f_n(t,u)\|_{\mathcal{L}}^p]^{2/p}\,dt\}^{p/2}$$

$$\le C(p)\int_0^1 E\|f_n - f\|_{\mathcal{L}}^p dt$$

is valid where the constant $C(p)$ depends only on p. It follows $I_n(u) \to I(u)$ and due to definition of $B(u)$ we have $B_n(u) \to B(u)$ in the space $L_p(\Omega, H_3)$. We proved $I(u) = B(u)$ and the statement follows from Theorem 1.

2.2 Cylindrical Wiener processes

Assume now that $w(t), t \in T$ is a cylindrical Wiener process with values in H_2. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_t, P)$, H_i, $i = 1,2,3$, $I, \varphi, u, \psi, \sigma$ be the same as in the case of the Wiener process with nuclear covariant operator.

Let $\mathcal{K}(H_2, H_3)$ be a space of linear operators A from H_2 to H_3 which is equipped with the Hilbert–Schmidt norm $\|A\|_{\mathcal{K}}^2 = tr\{AA^*\}$ where tr is the trace of the given operator from H_3 to H_3. Let f also fulfil the conditions from the previous section ($\mathcal{L}$ is substituted by $\mathcal{K}$).

Denote $X_1 = H_1$, $X_2 = L_p(I, \mathcal{K}(H_2, H_3))$, $p \ge 2$.

Theorem 4 *Let the assumptions of this section be fulfiled then the integral $J(u) = \int_0^1 f(t,u)\,dw$ exists and*

$$E[\sigma(u, J(u))|\mathcal{F}_0](.) = \psi(u)(.) \qquad P\text{-}a.e.$$

is valid.

Let $J_n(u) = \int_0^1 f_n(t,u)\,dw$. We need to prove the existence of the integral $J_n(u)$. To do this it is sufficient to prove

$$\int [E\|f(t)\|_{\mathcal{K}}^p]^{2/p}\,dt < \infty. \tag{9}$$

Since

$$\{\int_0^1 [E\|f(t)\|_{\mathcal{K}}^p]^{2/p}\,dt\}^{p/2} \leq \int E[tr\, f(t)f^*(t)]^{p/2}\,dt,$$

(and due to inequality from [5] Prop. 1.3 or [3] Lemma 1) the proof of (9) is analogous to the proof of $S < \infty$. The rest ot the proof of Theorem 4 is almost the same as the proof of Theorem 3 we only use the space $\mathcal{K}(H_2, H_3)$ instead of $\mathcal{L}(H_2, H_3)$.

Corollary 3 *Let H_4 be a real Hilbert space. Let $g : R \times H_4 \to H_3$ be continuous and Lipschitz continuous in the second variable. Assume that $z(t,\xi)$ are $\mathcal{F}_t$ adapted and measurable stochastic processes for every $\xi \in H_1$ which form Bochner type mapping as a mapping*

$$X_1 \to L_p(\Omega, L_p(I, H_4))).$$

Denote $B(\xi) = \int_0^1 g(t, z(t,\xi))\,dw(t)$. If σ fulfil the condition of Theorem 1, u is $\mathcal{F}_0$ measurable, the processes $w(t), z(t,\xi)$ are stochastically independent of $\mathcal{F}_0$ then $B(\xi)$ are Bochner type and Theorems 1 and 3 can be applied.

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