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On Multiplication of Some Generalized Functions

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Abstract

We show that if we extend the classical definition of a product of functions to a larger class of distributions, then for the distributions of the form $\frac{1}{(\cdot - i0)^\alpha}$ and $\frac{1}{(\cdot + i0)^\alpha}$, where α is complex number, we get formulas:

$$\frac{1}{(\cdot - i0)^\alpha} \cdot \frac{1}{(\cdot - i0)^\beta} = \frac{1}{(\cdot - i0)^{\alpha+\beta}}$$

and

$$\frac{1}{(\cdot + i0)^\alpha} \cdot \frac{1}{(\cdot + i0)^\beta} = \frac{1}{(\cdot + i0)^{\alpha+\beta}},$$

when α and β are complex numbers, such that $\operatorname{Re}\alpha > \frac{1}{2}$ and $\operatorname{Re}\beta > \frac{1}{2}$.

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1.

Let us consider the function $z \to z^\alpha$ for $z \in \mathbb{C} \setminus \mathbb{R}$ and $\alpha \in \mathbb{C}$ (see for example [2]) defined as follows:

$$z^\alpha := \exp\{\alpha[\ln|z| + i \arg z]\} \tag{1}$$

In the upper half plane ($\operatorname{Im} z > 0$) we take $0 < \arg z < \pi$. In the lower half plane ($\operatorname{Im} z < 0$) we take $\pi < \arg z < 2\pi$. For t in $\mathbb{R} \setminus \{0\}$ define

$$t^\alpha = \begin{cases} t^\alpha & \text{for } t > 0 \\ e^{i\alpha\pi}|t|^\alpha & \text{for } t < 0. \end{cases} \tag{2}$$

Definition 1 Let D_{L^2} denote the space of all smooth functions φ such that derivatives $\varphi^{(k)} \in L^2(\mathbb{R})$ for $k \in \mathbb{N}$.

The convergence in D_{L^2} is defined by the sequence $(\| \ \|_k)$ of norms:

$$\|\varphi\|_k := \left(\sum_{m=0}^{k} \|\varphi^{(m)}\|_{L^2}^2 \right)^{\frac{1}{2}} \qquad \text{for} \quad k = 0, 1, 2, \ldots$$

We shall denote by D'_{L^2} the space of all linear continuous functionals on D_{L^2}.

We investigate distributions of some special form:

Definition 2 For $\alpha \in \mathbb{C}$, such that $\operatorname{Re} \alpha > 0$ we define distribution $\frac{1}{(\cdot - i0)^\alpha}$ as follows:

$$\frac{1}{(\cdot - i0)^\alpha}(\varphi) := \lim_{\varepsilon \to 0+} \int_R \frac{1}{(x - i\varepsilon)^\alpha} \varphi(x) dx \qquad \text{for each } \varphi \in D_{L^2}. \tag{3}$$

Similarly:

Definition 3 For $\alpha \in \mathbb{C}$, such that $\operatorname{Re} \alpha > 0$ we define distribution $\frac{1}{(\cdot + i0)^\alpha}$ as follows:

$$\frac{1}{(\cdot + i0)^\alpha}(\varphi) := \lim_{\varepsilon \to 0+} \int_R \frac{1}{(x + i\varepsilon)^\alpha} \varphi(x) dx \qquad \text{for each } \varphi \in D_{L^2} \tag{4}$$

The existence of the above limits (3) and (4) and that these distributions belong to D'_{L^2} will be proved later.

2.

Definition 4 By S we shall denote as usually the space of infinitely differentiable functions φ on $\mathbb{R}$ such that:

$$\sup_{x \in \mathbb{R}} |x^n \varphi^{(k)}(x)| \leq c_{nk} \quad \text{for some constante } c_{nk}.$$

A convergence in S is defined by the sequence of norms

$$\|\varphi\|_{m,k} = \max_{0 \leq n \leq m} \sup_{x \in \mathbb{R}} |x^n \varphi^{(k)}(x)|.$$

Linear continuous forms defined on S are called tempered distributions. The set of tempered distributions is denoted by S'.

Denote by S'_0 the space of linear forms T defined on S by formula:

$$T(\varphi) = \sum_{k=0}^{m} \int_R x^k f_k(x)\varphi(x)dx, \quad \text{for some } m \in \mathbb{N},\ f_k \in L^2(\mathbb{R}), \text{ and every } \varphi \in S.$$

Of course these forms are tempered distributions.

The elements of the space S'_0 are called slowly increasing functions.

Definition 5 The Fourier transform F for $\Lambda \in S'$ is defined as follows:

$$F\Lambda(\varphi) := \Lambda(F\varphi) \qquad \text{for } \varphi \in S.$$

It is known that the Fourier transformation is a one-to-one mapping D'_{L^2} on S'_0, it means

$$F(D'_{L^2}) = S'_0 \qquad \text{and} \qquad F^{-1}(S'_0) = D'_{L^2} \tag{5}$$

and the following theorem (see for example in the book of Beltrami and Wohlers [1] th. 1.36, p. 43) holds:

Theorem 1 *Let $U, V \in D'_{L^2}$. Then $U * V$ exists and*

$$F(U * V) = FU \cdot FV, \tag{6}$$

or if $U, V \in S'_0$ then

$$F(U \cdot V) = \frac{1}{2\pi} FU * FV. \tag{7}$$

It means that for $U, V \in S'_0$, we have

$$U \cdot V = \frac{1}{2\pi} F^{-1}(FU * FV), \tag{8}$$

where F^{-1} denotes the inverse Fourier transformation.

This formula may be used to defining a product of other distributions.

Definition 6 By a product of elements $U, V \in D'_{L^2}$ we understand;

$$U \cdot V := \frac{1}{2\pi} F^{-1}(FU * FV), \tag{9}$$

if the right side of (9) is meaningful. (Compare [5], p. 106).

We shall now give the definition of the Carleman transform for slowly increasing functions:

Definition 7 Let f be in S'_0. Put

$$\widehat{F}f(z) = \begin{cases} \int_0^\infty f(t)e^{itz}dt & \text{if } \operatorname{Im} z > 0, \\ -\int_{-\infty}^0 f(t)e^{itz}dt & \text{if } \operatorname{Im} z < 0. \end{cases} \tag{10}$$

Similarly we define the inverse Carleman transform:

$$\widehat{F}^{-1}f(z) = \begin{cases} \frac{1}{2\pi}\int_{-\infty}^0 f(t)e^{-itz}dt & \text{if } \operatorname{Im} z > 0, \\ -\frac{1}{2\pi}\int_0^\infty f(t)e^{-itz}dt & \text{if } \operatorname{Im} z < 0. \end{cases} \tag{11}$$

We shall base on the theorem which gives posibility to determine inverse Fourier transform $F^{-1}f$ for $f \in S_0'$.

Theorem 2 (analogy of th. 4 in [3]) *If $f \in S_0'$, then*

$$(F^{-1}f)(\varphi) = \lim_{\varepsilon \to 0+} \int_{-\infty}^{\infty} [\widehat{F}^{-1}f(x+i\varepsilon) - \widehat{F}^{-1}f(x-i\varepsilon)]\varphi(x)dx \qquad \text{for } \varphi \in D_{L^2}. \tag{12}$$

3.

Let us consider the functions $(\cdot)_+^\alpha$ and $(\cdot)_-^\alpha$ on $\mathbb{R}$ for Re $\alpha > -1$, defined as follows:

$$(\cdot)_+^\alpha = H \cdot (\cdot)^\alpha \tag{13}$$

$$(\cdot)_-^\alpha = \tilde{H} \cdot (\cdot)^\alpha \tag{14}$$

where H denotes the Heaviside step function and $\tilde{H}(x) := H(-x)$. (Compare [4], p. 67). Notice that $(\cdot)_+^\alpha$ and $(\cdot)_-^\alpha$ belong to S_0' when Re $\alpha > -\frac{1}{2}$.

We shall now show

Lemma 1 *The inverse Fourier transform of $(\cdot)_+^\alpha$ has a form:*

$$F^{-1}(\cdot)_+^\alpha = \frac{1}{2\pi} i^{-(\alpha+1)}\Gamma(\alpha+1)\frac{1}{(\cdot - i0)^{\alpha+1}} \tag{15}$$

for $\alpha \in \mathbb{C}$ such that Re $\alpha > -\frac{1}{2}$, where Γ is Euler Γ-function defined as $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

Proof We calculate $F^{-1}(\cdot)_+^\alpha$ by means of formula (12):

$$F^{-1}(\cdot)_+^\alpha(\varphi) = \lim_{\varepsilon \to 0+} \int_R [\widehat{F}^{-1}(\cdot)_+^\alpha(x+i\varepsilon) - \widehat{F}^{-1}(\cdot)_+^\alpha(x-i\varepsilon)]\varphi(x)dx \qquad \text{for } \varphi \in D_{L^2}. \tag{16}$$

Note that

$$\widehat{F}^{-1}(\cdot)_+^\alpha(z) = \begin{cases} -\frac{1}{2\pi}(iz)^{-(\alpha+1)}\Gamma(\alpha+1) & \text{for Im}\, z < 0, \\ 0 & \text{for Im}\, z > 0. \end{cases} \tag{17}$$

Using it to (16) we have

$$F^{-1}(\cdot)_+^\alpha(\varphi) = \lim_{\varepsilon \to 0+} \int_R \frac{1}{2\pi}[i(x-i\varepsilon)]^{-(\alpha+1)}\Gamma(\alpha+1)\varphi(x)dx$$

$$= \frac{1}{2\pi} i^{-(\alpha+1)}\Gamma(\alpha+1) \lim_{\varepsilon \to 0+} \int_R \frac{1}{(x-i\varepsilon)^{\alpha+1}}\varphi(x)dx \quad \text{for } \varphi \in D_{L^2}.$$

This proves the existence of the above limit. By (5) and definition 2 distribution $\frac{1}{(\cdot - i0)^\alpha}$ belong to D'_{L^2} when $\operatorname{Re}\alpha > \frac{1}{2}$.

Lemma 1 is proved.

Since the function $(\cdot)_+^\alpha$ has the support in $[0, \infty)$, therefore there exist the convolution product $(\cdot)_+^\alpha * (\cdot)_+^\beta$. We shall now show: □

Lemma 2 *The following equality holds:*

$$\frac{(\cdot)_+^{\alpha-1}}{\Gamma(\alpha)} * \frac{(\cdot)_+^{\beta-1}}{\Gamma(\beta)} = \frac{(\cdot)_+^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \tag{18}$$

for complex numbers α and β such that Re $\alpha > 0$ and Re $\beta > 0$.

Proof For $t > 0$ we have

$$\left[\frac{(\cdot)_+^{\alpha-1}}{\Gamma(\alpha)} * \frac{(\cdot)_+^{\beta-1}}{\Gamma(\beta)}\right](t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t (t-u)^{\alpha-1} \cdot u^{\beta-1} du. \tag{19}$$

By substitution $t - u = tw$ we obtain:

$$\begin{aligned} \left[\frac{(\cdot)_+^{\alpha-1}}{\Gamma(\alpha)} * \frac{(\cdot)_+^{\beta-1}}{\Gamma(\beta)}\right](t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^\alpha w^{\alpha-1} t^{\beta-1} (1-w)^{\beta-1} dw \\ &= \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 w^{\alpha-1}(1-w)^{\beta-1} dw = \frac{(\cdot)_+^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}(t), \end{aligned} \tag{20}$$

on virtue of

$$\int_0^1 w^{\alpha-1}(1-w)^{\beta-1} dw = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{21}$$

when $\operatorname{Re}\alpha > 0$ and $\operatorname{Re}\beta > 0$. Of course (18) is true for $t \leq 0$, too.

Now we are ready to formulate our main theorem. □

Theorem 3 *The following relation holds:*

$$\frac{1}{(\cdot - i0)^\alpha} \cdot \frac{1}{(\cdot - i0)^\beta} = \frac{1}{(\cdot - i0)^{\alpha+\beta}}, \tag{22}$$

for each complex numbers α and β, such that

$$\operatorname{Re}\alpha > \frac{1}{2} \qquad \textit{and} \qquad \operatorname{Re}\beta > \frac{1}{2}.$$

Proof From lemma 1 and lemma 2 we have

$$F\frac{1}{(\cdot - i0)^\alpha} * F\frac{1}{(\cdot - i0)^\beta} = 2\pi F\frac{1}{(\cdot - i0)^{\alpha+\beta}} \tag{23}$$

when $\operatorname{Re}\alpha > \frac{1}{2}$ and $\operatorname{Re}\beta > \frac{1}{2}$.

So by the definition 6 we obtain:

$$\frac{1}{(\cdot - i0)^{\alpha}} \cdot \frac{1}{(\cdot - i0)^{\beta}} = \frac{1}{2\pi} F^{-1}\left[F\frac{1}{(\cdot - i0)^{\alpha}} * F\frac{1}{(\cdot - i0)^{\beta}}\right]$$
$$= F^{-1}F\frac{1}{(\cdot - i0)^{\alpha+\beta}} = \frac{1}{(\cdot - i0)^{\alpha+\beta}}$$

for α and β such that $\operatorname{Re}\alpha > \frac{1}{2}$ and $\operatorname{Re}\beta > \frac{1}{2}$. □

Similar relation for distributions $\frac{1}{(\cdot + i0)^{\alpha}}$ for $\alpha \in \mathbb{C}$, such that $\operatorname{Re}\alpha > \frac{1}{2}$, can be proved.

Another equivalent definitions of distributions $\frac{1}{(\cdot - i0)^{\alpha}}$ and $\frac{1}{(\cdot + i0)^{\alpha}}$ for $\alpha \in \mathbb{C}$ are given in [4].

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