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Compact Space-Like Submanifolds with Parallel Mean Curvature Vector of a Pseudo-Riemannian Space

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Abstract

B. Y. Chen [2] and L. Huafei [5] have studied pseudo-umbilical submanifolds. In this paper, we have generalized the compact pseudo-umbilical space-like submanifolds with parallel mean curvature in a pseudo-Riemannian space.

Key words: Pseudo-umblical submanifold, parallel mean curvature vector.

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1 Introduction

In a pseudo-Riemannian space form, space-like submanifolds with parallel mean curvature have been studied by many mathematians. Q. M. Cheng and Choi [3] proved the A complete space-like submanifold with parallel mean curvature vector of an indefinite space form $M_p^{n+p}(c)$. If the one following conditions is satisfied:

1. $c \leq 0$,
2. $c > 0$ and $n^2H^2 \geq 4(n-1)c$, then $S \leq S + K(p)$ where $K(p)$

is a constant. Later, R. Aiyama [1] proved a space-like submanifold in a Semi-Riemannian space form N with parallel non-null mean curvature vector H if M is neither minimal (i.e. maximal) nor pseudo-umbilical, then the normal connection of M in N is flat. L. Haizhong [4] discover a new theorem in the complete space-like submanifolds in de Sitter-Space with parallel mean curvature. B. Y. Chen [2] proved:

1. Let M be an n-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(c)$. Then

$$\int_M \left[nH\Delta H + n\left(c + H^2\right)S - \left(2 - \frac{1}{p}\right)S^2 - n^2H^2c\right] dv \leq 0$$

where S, H and dv denote the square of the length of h, the mean curvature of M and volume element of M, respectively.

2. Let M be an n-dimensional compact pseudo-umbilical submanifold in $N^{n+p}(c)$. If

$$nH\Delta H + n\left(c + H^2\right)S - \left(2 - \frac{1}{p}\right)S^2 - n^2H^2c \leq 0$$

then the second fundamental form is parallel and S constant.

Thus, we obtain the following generalizations of (1) and (2).

Theorem 1 *Let M be an n-dimensional compact pseudo-umbilical space-like submanifold in N. Then*

$$\int_M \left[n\left(c - 5H^2\right)S + n^2H^2c + \frac{1}{2}S^2 + n^2H^4\right] dv \leq 0, \quad for\ p > 1$$

and

$$\int_M \left[n\left(c - 2H^2\right)S + n^2H^2c + \frac{1}{2}S^2 - \frac{3}{2}n^2H^4\right] dv \leq 0, \quad for\ p > 2\ .$$

Theorem 2 *Let M be an n-dimensional compact pseudo-umbilical space-like submanifold in N. Then*

$$nH\Delta H + n\left(c - 5H^2\right)S + n^2H^2c + \frac{1}{2}S^2 + n^2H^4 \geq 0, \quad for\ p > 1$$

or

$$nH\Delta H + n\left(c - 2H^2\right)S + n^2H^2c + \frac{1}{2}S^2 - \frac{3}{2}n^2H^4 \geq 0, \quad for\ p > 2$$

then the second fundamental form is parallel and S is constant.

2 Local formulas

Let N be an $(n+p)$-dimensional pseudo-Riemannian manifold of constant curvature c, whose index is p. Let M be an n-dimensional Riemannian manifold is isometrically immersed in N. As the pseudo-Riemannian metric of N induces the Riemannian metric of M, the immersioan is called space-like. We choose a local field of pseudo-Riemannian orthonormal frames $e_1, e_2, \ldots, e_{n+p}$ in N such that, at each point of M $e_1, e_2, \ldots, e_n$ spans the tangent space of M and forms an orthonormal frame there. We make use of the following convention on the ranges of indices:

$$1 \leq A, B, C, D \leq n+p; \qquad 1 \leq i, j, k, l, m \leq n; \qquad n+1 \leq \alpha, \beta, \gamma \leq n+p.$$

We shall agree that repeated indices are summed over the respective ranges. Let $\omega_1, \omega_2, \ldots, \omega_{n+p}$ be its dual frame field so that the pseudo-Riemannian metric of N is given by

$$ds_N^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \epsilon_A \omega_A^2$$

where $\epsilon_i = 1$ for $1 \leq i \leq n$ and $\epsilon_\alpha = -1$ for $n+1 \leq \alpha \leq n+p$. Then the structure equations of N are given by

$$d\omega_A = \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{CD} \epsilon_C \epsilon_D K_{ABCD} \omega_C \wedge \omega_D, \tag{2.1}$$

$$K_{ABCD} = c\,(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

The restrict these forms to M. Then

$$\omega_\alpha = 0, \qquad \text{for} \quad n+1 \leq \alpha \leq n+p \tag{2.2}$$

and the Riemannian metric of M is written as

$$ds_M^2 = \sum_i \omega_i^2.$$

We may put

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha \tag{2.3}$$

Then h_{ij}^α are the components of the second fundamental form of M. From (2.1), we obtain the structure equations of M

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{kl} R_{ijkl} \omega_k \wedge \omega_l \,, \tag{2.4}$$

and the Gauss formula

$$R_{ijkl} = c\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right) - \sum_\alpha \left(h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{il} h^\alpha_{jk}\right),$$

$$R_{\alpha\beta ij} = \sum_k \left(h^\alpha_{ki} h^\beta_{kj} - h^\alpha_{kj} h^\beta_{ki}\right). \tag{2.5}$$

We also have the structure equations of the normal bundle of M:

$$d\omega_\alpha = -\sum_\beta \omega_{\alpha\beta} \wedge \omega_\beta$$

$$d\omega_{\alpha\beta} = -\sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\beta} - \frac{1}{2} \sum_{ij} R_{\alpha\beta ij} \omega_i \wedge \omega_j \,. \tag{2.6}$$

Let h^α_{ijk} denote the covariant derivative of h^α_{ij} so that

$$\sum_k h^\alpha_{ijk} \omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ik} \omega_{kj} + \sum_k h^\alpha_{kj} \omega_{ki} - \sum_\beta h^\beta_{ij} \omega_{\beta\alpha} \,. \tag{2.7}$$

Then we have $h^\alpha_{ijk} = h^\alpha_{ikj}$. Next take the exterior derivative of (2.7) and define the second covariant derivative of h^α_{ij} by

$$\sum_l h^\alpha_{ijkl} \omega_l = dh^\alpha_{ijk} + \sum_l h^\alpha_{ijl} \omega_{lk} + \sum_l h^\alpha_{ilk} \omega_{lj} + \sum_l h^\alpha_{ljk} \omega_{li} - \sum h^\alpha_{ijk} \omega_{\beta\alpha} \,. \tag{2.8}$$

Then we have obtain the Ricci formula

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{im} R_{mjkl} + \sum_m h^\alpha_{mj} R_{mikl} + \sum_\beta h^\beta_{ij} R_{\alpha\beta kl} \,. \tag{2.9}$$

We call

$$h = \sum_{ij\alpha} h^\alpha_{ij} \omega_i \omega_j e_\alpha$$

the second fundamental form of the immersed manifold M.

$$\zeta = \frac{1}{n} \sum_\alpha tr H_\alpha e_\alpha$$

and

$$H = \sqrt{\frac{1}{n} \sum_\alpha \left(\text{tr}\, H_\alpha\right)^2}$$

denote the mean curvature vector and the mean curvature of M, respectively. Here tr is trace of the matrix $H_\alpha = \left(h_{ij}^\alpha\right)$. The square of the length of the second fundamental form of M in N is given by

$$S = \sum_{ij\alpha} \left(h_{ij}^\alpha\right)^2 .$$

Now, let e_{n+p} be parallel to ζ. Then we get

$$\operatorname{tr} H_{n+p} = nH, \qquad \operatorname{tr} H_\alpha = 0, \qquad \alpha \neq n+p. \tag{2.10}$$

The Laplacian Δh_{ij}^α of second fundamental form h_{ij}^α is defined by

$$\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha .$$

Using the same method as in [6], we have

$$\Delta h_{ij}^\alpha = \sum_k h_{kkij}^\alpha + \sum_{mk} h_{im}^\alpha R_{mkjk} + \sum_{mk} h_{mk}^\alpha R_{mijk} + \sum_{\alpha\beta k} h_{ik}^\beta R_{\alpha\beta jk} .$$

By a simple calculation we have

$$\frac{1}{2}\Delta S = \sum_{ijk\alpha} \left(h_{ijk}^\alpha\right)^2 + \sum_{ij\alpha} \left(h_{ij}^\alpha\right) \Delta h_{ij}^\alpha$$

or

$$\begin{aligned} \frac{1}{2}\Delta S = & \sum_{ijk\alpha} \left(h_{ijk}^\alpha\right)^2 + \sum_{ijk\alpha} h_{ij}^\alpha h_{kkij}^\alpha + \sum_{ijkm\alpha} h_{ij}^\alpha h_{im}^\alpha R_{mkjk} \\ & + \sum_{ijkm\alpha} h_{ij}^\alpha h_{mk}^\alpha R_{mijk} + \sum_{ijk\alpha\beta} h_{ij}^\alpha h_{ik}^\beta R_{\alpha\beta jk} \\ = & \sum_{ijk\alpha} \left(h_{ijk}^\alpha\right)^2 + nH\Delta H + n\left(c - H^2\right) S + n^2 H^2 c \\ & + \sum_{\alpha\beta} \left(\operatorname{tr} H_\alpha H_\beta\right)^2 + \sum_{\alpha\beta} \operatorname{tr}\left(H_\alpha H_\beta - H_\beta H_\alpha\right)^2 . \end{aligned} \tag{2.11}$$

Definition 1 A space-like submanifold M is said pseudo-umbilic, if it is umbilic with respect to direction of mean curvature vector h, that is

$$h_{ij}^{n+p} = H\delta_{ij} \tag{2.12}$$

In order to prove our Theorems, we need following lemmas.

3 Proofs of Theorems

Lemma 1 [5] *Let H_i $(i \geq 2)$ be symetric $(n \times n)$-matrices, $s_i = \operatorname{tr} H_i^2$ and $S = \sum_i s_i$. Then*

$$\sum_{ij} \operatorname{tr}(H_iH_j - H_jH_i)^2 - \sum_{ij} (trH_iH_j)^2 \geq -\frac{3}{2}S^2 \tag{3.1}$$

and the equality holds if and only if all $H_i = 0$ or there exist two of H_i different from zero. Morever, if $H_1 \neq 0$, $H_2 \neq 0$, $H_i = 0$, $(i \neq 1, 2)$ then $s_1 \neq s_2$ and there exists on ortogonal $(n \times n)$-matrix T such that

$$TH_1^{'}T = \sqrt{\frac{s_1}{2}} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \qquad TH_2^{'}T = \sqrt{\frac{s_2}{2}} \begin{pmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

Lemma 2 [5] *When $p > 2$,*

$$\sum_{\alpha\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 - \sum_{\alpha\beta} (\operatorname{tr} H_\alpha H_\beta)^2 \geq -\frac{3}{2}S^2 + 3nH^2S - \frac{5}{2}n^2H^4. \tag{3.2}$$

Lemma 3 *When $p > 1$,*

$$\sum_{\alpha\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 + \sum_{\alpha\beta} (\operatorname{tr} H_\alpha H_\beta)^2 \geq -\frac{1}{2}S^2 - 4nH^2S + n^2H^4. \tag{3.3}$$

Proof From (3.1), we have

$$\sum_{\alpha\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 + \sum_{\alpha\beta} (\operatorname{tr} H_\alpha H_\beta)^2 \geq -\frac{3}{2}S^2 + 2\sum_{\alpha\beta} (\operatorname{tr} H_\alpha H_\beta)^2. \tag{3.4}$$

On the other hand, by a simple calculation we have

$$2\sum_{\alpha\beta} (\operatorname{tr} H_\alpha H_\beta)^2 \geq 2S^2 - 4nH^2S + n^2H^4. \tag{3.5}$$

Using (3.5) in (3.4), we obtain (3.3). □

Lemma 4 *When $p > 2$,*

$$\sum_{\alpha\beta} \operatorname{tr}(H_\alpha H_\beta - H_\beta H_\alpha)^2 + \sum_{\alpha\beta} (\operatorname{tr} H_\alpha H_\beta)^2 \geq +\frac{1}{2}S^2 - nH^2S - \frac{3}{2}n^2H^4. \tag{3.6}$$

Proof From Lemma 2 and (3.5), it can seen easily (3.6). □

Using (2.12) we can get

$$\sum_{ijk\alpha} \left(h_{ijk}^{\alpha}\right)^2 \geq \sum_{ik} \left(h_{iik}^{n+p}\right)^2. \tag{3.7}$$

It is obvious that

$$\frac{1}{2}n\Delta H^2 = nH\Delta H + \sum_{ik} \left(h_{iik}^{n+p}\right)^2. \tag{3.8}$$

Therefore, using Lemma 3, (3.7) and (3.8) when $p > 1$ by (2.11) we have

$$\begin{aligned}
\frac{1}{2}\Delta S &\geq \sum_{ijk} \left(h_{ijk}^{\alpha}\right)^2 + nH\Delta H + n\left(c - H^2\right)S + n^2H^2c + \frac{1}{2}S^2 \\
&\quad - 4nH^2S + n^2H^4 \\
&\geq \sum_{ik} \left(h_{iik}^{n+p}\right)^2 + nH\Delta H + n\left(c - 5H^2\right)S + \frac{1}{2}S^2 + n^2H^2c + n^2H^4 \\
&= \frac{1}{2}n\Delta H^2 + n\left(c - 5H^2\right)S + n^2H^2c + \frac{1}{2}S^2 + n^2H^4.
\end{aligned} \tag{3.9}$$

Since M is compact, form (3.9) we have

$$\int_M \left[n\left(c - 5H^2\right)S + n^2H^2c + \frac{1}{2}S^2 + n^2H^4\right] dv \leq 0.$$

On the other hand, from the first inequality of (3.9), we have that if

$$nH\Delta H + n\left(c - 5H^2\right)S + n^2H^2c + \frac{1}{2}S^2 + n^2H^4 \geq 0 \tag{3.10}$$

and M is compact, then the second fundamental from h_{ij}^{α} is parallel and S is constant.

On the other hand, when $p > 2$ using Lemma 4, (3.7) and (3.8) from (2.11) we get

$$\begin{aligned}
\frac{1}{2}\Delta S &\geq \sum_{ijk\alpha} \left(h_{ijk}^{\alpha}\right)^2 + nH\Delta H + n\left(c - H^2\right)S + n^2H^2c + \frac{1}{2}S^2 \\
&\quad - n^2H^2S - \frac{3}{2}n^2H^4 \\
&\geq \frac{1}{2}n\Delta H^2 + n\left(c - 2H^2\right)S + n^2H^2S + \frac{1}{2}S^2 - \frac{3}{2}n^2H^4.
\end{aligned} \tag{3.11}$$

Thus, when M is compact by (3.11) we obtain

$$\int_M \left[n\left(c - 2H^2\right)S + \frac{1}{2}S^2 + n^2H^2c - \frac{3}{2}n^2H^4\right] dv \leq 0.$$

From the first inequality of (3.11), we see that if

$$nH\Delta H + n\left(c - 2H^2\right)S + n^2H^2c + \frac{1}{2}S^2 - \frac{3}{2}n^2H^4 \geq 0 \tag{3.12}$$

then the second fundamental form h_{ij}^{α} is parallel and S is constant.

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