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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 39 (2000), No. 1, 67--71

Persistent URL: `http://dml.cz/dmlcz/120417`

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **39** (2000) 67–71

Galois Triangle Theory for Direct Summands of Modules *

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(Received September 13, 1999)

Abstract

The purpose of this paper is to find an analogon of the Galois triangle theory for direct summands of modules over arbitrary unitary commutative rings.

Key words: Module, direct summand of module, ring of endomorphisms, anihilator, isomorphism of ordered sets.

1991 Mathematics Subject Classification: 13C99, 06A15

Let an arbitrary unitary commutative ring $\mathbf{A}$ be given and let $\mathbf{M}$ be an $\mathbf{A}$-module. The aim of this paper is (by using the properties of projections) to find 1-1 correspondences between the ordered set of direct summands of $\mathbf{M}$ and the ordered set of left (right) principal ideal of the ring of endomorphisms of $\mathbf{M}$ generated by an idempotent element.

The solution of this problem is well known f.e. in the case $\mathbf{M}$ is a vector space [1] or totally reducible module [3] (the set of direct summands of $\mathbf{M}$ is equal to the set of all submodules in this cases).

Notation 1

1.1. By $\mathbf{P}$ we will denote the ring of endomorphisms on $\mathbf{M}$ i.e. $\mathbf{P} = \operatorname{End}\mathbf{M}$, the composition of $f, g \in \mathbf{P}$ we will define by $(fg)(\mathbf{x}) = g(f(\mathbf{x}))$.

*Supported by the grant of the Palacký University 1999 "Rozvoj algebraických metod v geometrii a uspořádaných množinách"

1.2. By $L(\mathbf{P})$ we will denote the set of the all left ideals of $\mathbf{P}$ generated by an idempotent element and by $R(\mathbf{P})$ the set of the all right ideals of $\mathbf{P}$ generated by an idempotent element.

1.3. By $U(\mathbf{M})$ we will denote the set of the all direct summands of the $\mathbf{A}$-module $\mathbf{M}$.

1.4. For every $S \in U(\mathbf{M})$ let us denote

$$\mathbf{N}(S) = \{f \in \mathbf{P}; \forall \mathbf{x} \in S : f(\mathbf{x}) = \mathbf{o}\},$$
$$\mathbf{Q}(S) = \{f \in \mathbf{P}; \forall \mathbf{x} \in \mathbf{M} : f(\mathbf{x}) \in S\}.$$

(Equivalently, $\mathbf{N}(S) = \{f \in \mathbf{P}; S \subseteq \operatorname{Ker} f\}$, $\mathbf{Q}(S) = \{f \in \mathbf{P}; \operatorname{Im} f \subseteq S\}$).

1.5. For every $J \in R(\mathbf{P})$ let us denote

$$\mathbf{K}(J) = \{\mathbf{x} \in \mathbf{M}; \forall f \in J : f(\mathbf{x}) = \mathbf{o}\},$$
$$\mathbf{L}(J) = \{f \in \mathbf{P}; \forall g \in J : fg = o\}.$$

1.6. For every $J \in L(\mathbf{P})$ let us denote

$$\mathbf{H}(J) = \{\mathbf{x} \in \mathbf{M}; \exists f_1, \ldots, f_r \in J, \exists \mathbf{y}_1, \ldots, \mathbf{y}_r \in \mathbf{M} : \mathbf{x} = \sum_{i=1}^{r} f_i(\mathbf{y}_i)\},$$
$$\mathbf{R}(J) = \{f \in \mathbf{P}; \forall g \in J : gf = o\}.$$

Remark 2 Every idempotent element of $\mathbf{P}$ is usually called a *projection.*

It is well known (see [?]), that $f \in \mathbf{P}$ is a projection if and only if there exist elements $U, V \in U(\mathbf{M})$ such that:

(1) $\mathbf{M} = U \oplus V$,

(2) $\forall \mathbf{x} \in \mathbf{M}, \mathbf{x} = \mathbf{x}_U + \mathbf{x}_V, \mathbf{x}_U \in U, \mathbf{x}_V \in V : f(\mathbf{x}) = \mathbf{x}_U$.

Therefore f is called a *projection* $\mathbf{M}$ *onto* U *parallely* V. Clearly, $\operatorname{Im} f = U$, $f = V$ and $f|U$ is an identity mapping.

Lemma 3 *For any $f, g \in \mathbf{P}$, f is a projection, the following holds:*

$$\operatorname{Ker} f \subseteq \operatorname{Ker} g \iff fg = g.$$

Proof Let $\operatorname{Ker} f \subseteq \operatorname{Ker} g$. Let us consider an arbitrary element $\mathbf{x}$ of $\mathbf{M}$. It may be written by $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{x}_1 \in \operatorname{Im} f$, $\mathbf{x}_2 \in \operatorname{Ker} f$.

Then we have

$$g(\mathbf{x}) = g(\mathbf{x}_1) + g(\mathbf{x}_2) = g(\mathbf{x}_1) = g(f(\mathbf{x}_1)) = g(f(\mathbf{x})) = (fg)(\mathbf{x}).$$

The reverse implication is clear.

Lemma 4 *For any $f, g \in \mathbf{P}$, f is a projection, the following holds:*

$$\operatorname{Im} g \subseteq \operatorname{Im} f \iff gf = g.$$

Proof Evidently, $gf = g$ implies $\operatorname{Im} g \subseteq \operatorname{Im} f$.

Let $\operatorname{Im} g \subseteq \operatorname{Im} f$. As $f|\operatorname{Im} f = id$ we may write

$$(gf)(\mathbf{x}) = f(g(\mathbf{x})) = g(\mathbf{x}), \quad \text{for any } \mathbf{x} \in \mathbf{M}.$$

Proposition 5 *For any submodule $S \in U(\mathbf{M})$ the following hold:*
(1) $\mathbf{N}(S) \in R(\mathbf{P})$,
(2) $\mathbf{Q}(S) \in L(\mathbf{P})$.

Proof Firstly, let us prove the part (1).

Let us consider an $S \in U(\mathbf{M})$. It follows from the definition of the operator $\mathbf{N}$ than $\mathbf{N}(\mathrm{S})$ is a right ideal of $\mathbf{P}$.

Since $S \in U(\mathbf{M})$ we have a $T \in U(\mathbf{M})$ such that $\mathbf{M} = S \oplus T$. Using remark 2 we obtain the existence of a projection f $\mathbf{M}$ onto T parallely $S = \operatorname{Ker} f$.

Let an arbitrary endomorphism $g \in \mathbf{N}(S)$ be given. It yields $S \subseteq \operatorname{Ker} g$ which implies (by lemma 3) that $g = fg$. Thus $\mathbf{N}(S) \subseteq f\mathbf{P}$. Since the reverse inclusion is evident we have $\mathbf{N}(S) \in R(\mathbf{P})$.

The part (2) may be proved by an analogical way (due to lemma 4).

Proposition 6 *If $J \in R(\mathbf{P})$ then $\mathbf{K}(J) \in U(\mathbf{M})$.*

Proof Let f be the idempotent generator of J, $J = f\mathbf{P}$. Using the definition of the operator $\mathbf{K}$ we have $\mathbf{x} \in \mathbf{K}(J)$ iff $(fp)(\mathbf{x}) = \mathbf{o}$ for every $p \in \mathbf{P}$, which is equivalent to $f(\mathbf{x}) = \mathbf{o}$. It yields $\operatorname{Ker} f = \mathbf{K}(J)$.

As f is a projection the kernel of it is an element of $U(\mathbf{M})$.

Proposition 7 *If $J \in L(\mathbf{P})$ then $\mathbf{H}(J) \in U(\mathbf{M})$.*

Proof If we denote by f the idempotent generator of J then the submodule $S = \operatorname{Im} f$ belongs to $U(\mathbf{M})$.

Arbitrary $\mathbf{x} \in \mathbf{H}(J)$ may be written as follows ($f_i \in J$ for any $i = 1, \ldots, r$):

$$\mathbf{x} = \sum_{i=1}^{r} f_i(\mathbf{y}_i) = \sum_{i=1}^{r} (g_i f)(\mathbf{y}_i) = \sum_{i=1}^{r} f(g_i(\mathbf{y}_i)),$$

which gives that $\mathbf{H}(J) \subseteq S$.

The reverse inclusion follows from the definition of $\mathbf{H}$, immediately.

It follows from the propositions 5, 6, 7 that it is possible to investigate the compositions $\mathbf{KN}$, $\mathbf{NK}$, $\mathbf{HQ}$ and $\mathbf{QH}$.

Proposition 8 *If $S \in U(\mathbf{M})$ then the following hold:*
(1) $\mathbf{K}(\mathbf{N}(S)) = S$,
(2) $\mathbf{H}(\mathbf{Q}(S)) = S$.

Proof Firstly, let us prove the part (1).

If $S \in U(\mathbf{M})$ then there exists a projection f such that $\operatorname{Ker} f = S$ and moreover $\mathbf{N}(S)) = f\mathbf{P}$ (see the proof of proposition 5). Using this fact and the definition of $\mathbf{K}$ we may write $\mathbf{K}(\mathbf{N}(S)) = \operatorname{Ker} f = S$ (see the proof of 6).

Now, let us prove the (2).

It follows from the lemma 4 that for any $S \in U(\mathbf{M})$ we have a projection g such that $\operatorname{Im} g = S$ and $\mathbf{Q}(S) = \mathbf{P}g$. Since every element from $\mathbf{H}(\mathbf{P}g)$ may be written by

$$\mathbf{x} = \sum_{i=1}^{r}(f_i g)(\mathbf{y}_i) = \sum_{i=1}^{r} g(f_i(\mathbf{y}_i)), \quad f_1, \ldots, f_r \in \mathbf{P}$$

it follows from this $\mathbf{H}(\mathbf{P}g) = \operatorname{Im} g$. It means $\mathbf{H}(\mathbf{Q}(S)) = S$, consequently.

Proposition 9
(1) If $J \in L(\mathbf{P})$ then $\mathbf{Q}(\mathbf{H}(J)) = J$.
(2) If $J \in R(\mathbf{P})$ then $\mathbf{N}(\mathbf{K}(J)) = J$.

Proof
Ad (1): Let f be an idempotent generator of J. Then $\mathbf{H}(J) = \operatorname{Im} f$ (see the proof of 7). Respecting the fact g is an element of $\mathbf{Q}(\operatorname{Im} f)$ iff $\operatorname{Im} g \subseteq \operatorname{Im} f$ we (according to 4) have $g \in \mathbf{P}f = J$.

Ad (2): Let f be an idempotent generator of J. Then $\mathbf{K}(J) = \operatorname{Ker} f$ (see the proof of 6). Using the lemma 3 we obtain that $g \in \mathrm{N}(\operatorname{Ker} f)$ iff $g \in f\mathbf{P} = J$.

Proposition 10
(1) If $J \in R(\mathbf{P})$ then $\mathbf{L}(J) = \mathbf{Q}(\mathbf{K}(J))$.
(2) If $J \in L(\mathbf{P})$ then $\mathbf{R}(J) = \mathbf{N}(\mathbf{H}(J))$.

Proof Firstly, we prove the part (1).

Let $J = f\mathbf{P}$, Then $\mathbf{K}(J) = \operatorname{Ker} f$ and therefore $\mathbf{Q}(\mathbf{K}(J)) = \mathbf{P}g$ where g is a projection with $\operatorname{Im} g = \operatorname{Ker} f$ (see proofs of 6 and 8).

Let us consider arbitrary elements $q \in \mathbf{Q}(\mathbf{K}(J))$, $j \in J$ which means that $q = pg$ and $j = fr$, where $p, r \in \mathbf{P}$. For any $\mathbf{x} \in \mathbf{M}$ we may write

$$(qj)(\mathbf{x}) = r(f(g(p(\mathbf{x})) = \mathbf{o},$$

since $\operatorname{Im} g = \operatorname{Ker} f$. Thus $\mathbf{Q}(\mathbf{K}(J)) \subseteq \mathbf{L}(J)$.

If $h \in \mathbf{L}(J)$ then $hf = o$. It implies that $Imh \subseteq \operatorname{Ker} f$. Using the fact $\operatorname{Ker} f = \operatorname{Im} g$ and lemma 4 we obtain $h = hg$. It implies $h \in \mathbf{P}g = \mathbf{Q}(\mathbf{K}(J))$.

The second part may be proved analogously.

It is easy to derive that for every $U, S \in U(\mathbf{P})$, $J, K \subseteq \mathbf{P}$ hold
(1) $J \subseteq K \Rightarrow \ \mathbf{K}(J) \supseteq \mathbf{K}(K), \mathbf{H}(J) \subseteq \mathbf{H}(K), \mathbf{R}(J) \supseteq \mathbf{R}(K), \mathbf{L}(J) \supseteq \mathbf{L}(K)$,
(2) $U \subseteq S \Rightarrow \ \mathbf{N}(U) \supseteq \mathbf{N}(S), \mathbf{Q}(U) \subseteq \mathbf{Q}(S)$.

Now, if we consider operators $\mathbf{N}$, $\mathbf{K}$, $\mathbf{Q}$, $\mathbf{H}$, $\mathbf{L}$, $\mathbf{R}$ as mappings of corresponding ordered sets then the following theorem follows clearly from propositions 8, 9 and 10.

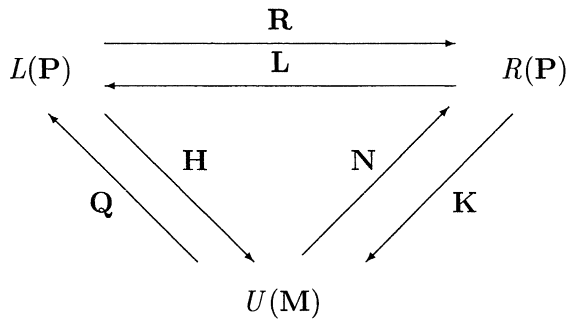
Theorem 11

(1) Operators $\mathbf{N}$ *and* $\mathbf{K}$ *are mutually inverse antiisomorphisms of ordered sets* $(U(\mathbf{M}), \subseteq)$ *and* $(R(\mathbf{P}), \subseteq)$.

(2) Operators $\mathbf{Q}$ *and* $\mathbf{H}$ *are mutually inverse isomorphisms of ordered sets* $(U(\mathbf{M}), \subseteq)$ *and* $(L(\mathbf{P}), \subseteq)$.

(3) Operators $\mathbf{L}$ *and* $\mathbf{R}$ *are mutually inverse antiisomorphisms of ordered sets* $(R(\mathbf{P}), \subseteq)$ *and* $(L(\mathbf{P}), \subseteq)$.

(4) The following diagram is commutative



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