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A Characterization of Congruence Kernels in Pseudocomplemented Semilattices

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Abstract

We set two necessary and sufficient conditions such that any subset of a pseudocomplemented semilattice which satisfies these conditions is a kernel (i.e. a 0-class) of some congruence.

Key words: Pseudocomplemented semilattice, congruence kernel.

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By a *pseudocomplemented semilattice* is meant an algebra $\mathcal{S} = (S; \wedge, ^*, 0)$ of type $\langle 2, 1, 0\rangle$ such that $(S; \wedge)$ is a $\wedge$-semilattice with the least element 0 (with respect to the induced order) and a^* denotes the *pseudocomplement* of $a \in S$, i.e. a^* is the greatest element of the set $\{b \in S;\ a \wedge b = 0\}$. It is well-known that the class of all pseudocomplemented semilattices forms a variety, see e.g. [3].

Let $\mathcal{S} = (S; \wedge, ^*, 0)$ be a pseudocomplemented semilattice and R be a binary relation on S. Denote by $[0]_R = \{x \in S;\ \langle x, 0\rangle \in R\}$. Especially, if R is a congruence on $\mathcal{S}$, $[0]_R$ is called a *congruence kernel* (of R). The aim of our paper is to characterize congruence kernels in pseudocomplemented semilattices. Let us note that a similar characterization (containing three conditions) was recently settled by P. Agliano and A. Ursini [1] based on the concept of ideal in universal algebra. Although our result is similar, our reasoning is based on different concepts.

Recal from [1] that a variety $\mathcal{V}$ is *subtractive* if there exists a binary term $s(x,y)$ of $\mathcal{V}$ such that $\mathcal{V}$ satisfies the identities $s(x,x) = 0$ and $s(x,0) = x$. Of course, the variety of pseudocomplemented semilattices is subtractive since $s(x,y) = x \wedge y^*$.

Moreover, $\mathcal{V}$ is subtractive if and only if for each $\mathcal{A} \in \mathcal{V}$ and every $\Theta, \Phi \in Con\,\mathcal{A}$ it holds that $[0]_{\Theta\cdot\Phi} = [0]_{\Phi\cdot\Theta}$; algebras satisfying the later condition are usually called *permutable at 0*.

Let us recall the following result (Lemma 2 of [2]):

Lemma *Let $\mathcal{V}$ be a subtractive variety, $\mathcal{A} = (A,F) \in \mathcal{V}$ and R be a reflexive and compatible binary relation on A. Let $\Theta(R)$ be the least congruence on A containing R. Then*

$$[0]_R = [0]_{\Theta(R)}.$$

We are able to formulate our result.

Theorem *Let $\mathcal{S} = (S; \wedge, ^*, 0)$ be a pseudocomplemented semilattice and $\emptyset \neq I \subseteq S$. Then I is a kernel of some congruence on $\mathcal{S}$ if and only if I satisfies the following two conditions:*

(i) if $x \in I$ and $a \in S$ then $x \wedge a \in I$

(ii) if $x, y \in I$ then $(x^* \wedge y^*)^* \in I$.

Before the proof, let us note that (i) yields $0 \in I$ and $x \in I, a \leq x \Rightarrow a \in I$ and (ii) yields $x \in I \Rightarrow x^{**} \in I$ (taking $x = y$).

Proof If $I = [0]_\Theta$ for some $\Theta \in Con\,\mathcal{A}$ and $x, y \in I, a \in S$ then $\langle x, 0\rangle \in \Theta$ implies $\langle x \wedge a, 0\rangle = \langle x \wedge a, 0 \wedge a\rangle \in \Theta$ proving (i); moreover, $\langle x, 0\rangle \in \Theta$ and $\langle y, 0\rangle \in \Theta$ thus also $\langle (x^* \wedge y^*)^*, 0\rangle = \langle (x^* \wedge y^*)^*, (0^* \wedge 0^*)^*\rangle \in \Theta$ proving (ii).

Conversely, suppose that $\emptyset \neq I \subseteq S$ satisfies (i) and (ii) and define R on S by setting

$$\langle x, y\rangle \in R \text{ iff } x \wedge y^* \in I \text{ and } y \wedge x^* \in I.$$

Clearly, R is reflexive. Prove compatibility of R:

if $\langle a, b\rangle \in R$ then $a \wedge b^* \in I$ and $b \wedge a^* \in I$ thus, by (ii), also $(a \wedge b^*)^{**} \in I$ and $(b \wedge a^*)^{**} \in I$, i.e. also $a^{**} \wedge b^* \in I$ and $b^{**} \wedge a^* \in I$ proving $\langle a^*, b^*\rangle \in R$.

Suppose $\langle a, b\rangle \in R$ and $\langle c, d\rangle \in R$. Then $a \wedge b^* \in I, b \wedge a^* \in I, c \wedge d^* \in I$ and $d \wedge c^* \in I$ thus, by (ii), also

$$((a \wedge b^*)^* \wedge (c \wedge d^*)^*)^* \in I \text{ and } ((b \wedge a^*)^* \wedge (d \wedge c^*)^*)^* \in I. \qquad (*)$$

We have

$$a \wedge c \wedge (a \wedge b^*)^* \wedge (c \wedge d^*)^* \wedge b^* = (a \wedge b^*) \wedge (a \wedge b^*)^* \wedge c \wedge (c \wedge d^*)^* = 0 \wedge c \wedge (c \wedge d^*)^* = 0,$$

and, analogously,

$$a \wedge c \wedge (a \wedge b^*)^* \wedge (c \wedge d^*)^* \wedge d^* = 0.$$

Thus

$$a \wedge c \wedge (a \wedge b^*)^* \wedge (c \wedge d^*)^* \leq b^{**} \quad \text{and} \quad a \wedge c \wedge (a \wedge b^*)^* \wedge (c \wedge d^*)^* \leq d^{**}$$

whence

$$a \wedge c \wedge (a \wedge b^*)^* \wedge (c \wedge d^*)^* \leq b^{**} \wedge d^{**} = (b \wedge d)^{**}$$

which yields

$$a \wedge c \wedge (a \wedge b^*)^* \wedge (c \wedge d^*)^* \wedge (b \wedge d)^* = 0.$$

It gives

$$a \wedge c \wedge (b \wedge d)^* \leq ((a \wedge b^*)^* \wedge (c \wedge d^*)^*)^*$$

which, together with $(*)$ and (i) yields

$$(a \wedge c) \wedge (b \wedge d)^* \in I.$$

Analogously, we can show

$$(b \wedge d) \wedge (a \wedge c)^* \in I$$

thus $\langle a \wedge c, b \wedge d \rangle \in R$. Hence, R is reflexive and compatible relation on $\mathcal{S}$.

If $a \in I$ then also $a \wedge 0^* = a \in I$ and $0 \wedge a^* = 0 \in I$ showing $\langle a, 0 \rangle \in R$, i.e. $a \in [0]_R$. If $a \in [0]_R$ then $\langle a, 0 \rangle \in R$ and hence $a = a \wedge 0^* \in I$. We have shown $[0]_R = I$. By the Lemma, $I = [0]_{\Theta(R)}$, i.e. I is a congruence kernel. □

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