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# Infinitesimal Deformations and Lie Derivative of a Non-symmetric Affine Connection Space

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**Abstract**

At the present work we consider infinitesimal deformations

$$f : x^i \to x^i + \varepsilon z^i(x^j)$$

of a space $L_N$ with non-symmetric affine connection $L^i_{jk}$, expressing the deformations of geometric magnitudes by virtue of Lie derivative. Because of non-symmetry of the connection, we use four kinds of covariant derivative to express the Lie derivative and the deformations.

**Key words:** Infinitesimal deformation, non-symmetric affine connection, Lie derivative.

**2000 Mathematics Subject Classification:** 53C25, 53A45, 53B05

## 1 Introduction

Deformations and infinitesimal deformations have been studied by many authors. We refer to [6–11] for more details and references.

Let us consider a space $L_N$ of non-symmetric affine connection $L^i_{jk}$ with the torsion tensor $T^i_{jk} = L^i_{jk} - L^i_{kj}$, at local coordinates $x^i$ $(i = 1, \dots, N)$. At the beginning we are giving some basic facts according to [4, 6, 7, 10].

**Definition 1.1** A transformation

$$f : L_N \to L_N : x = (x^1, \dots, x^N) \equiv (x^i) \to \bar{x} = (\bar{x}^1, \dots, \bar{x}^N) \equiv (\bar{x}^i),$$

where

$$\bar{x} = x + z(x)\varepsilon, \tag{1.1}$$

or in local coordinates

$$\bar{x}^i = x^i + z^i(x^j)\varepsilon, \quad i, j = 1, \dots, N, \tag{1.1$'$}$$

where $\varepsilon$ is an infinitesimal, is called *infinitesimal deformation of a space $L_N$,* determined by the vector field $z = (z^i)$, which is called *infinitesimal deformation field* (1.1).

We denote with $(i)$ local coordinate system in which the point $x$ is endowed with coordinates $x^i$, and the point $\bar{x}$ with the coordinates $\bar{x}^i$. We will also introduce *a new coordinate system* $(i')$, corresponding to the point $x = (x^i)$ new coordinates

$$x^{i'} = \bar{x}^i, \tag{1.2}$$

i.e. as new coordinates $x^{i'}$ of the point $x = (x^i)$ we choose old coordinates (at the system $(i)$) of the point $\bar{x} = (\bar{x}^i)$. Namely, at the system $(i')$ is

$$x = (x^{i'}) \overset{(1.2)}{=} (\bar{x}^i),$$

where $\overset{(1.2)}{=}$ denotes "equal according to (1.2)".

**Definition 1.2** Coordinate transformation we get based on punctual transformation $f : x \to \bar{x}$, getting for the new coordinates of the point $x$ the old coordinates of its transform $\bar{x}$, is called *dragging along punctual transformation.* New coordinates $x^{i'} = \bar{x}^i$ of the point $\bar{x}$ are called *dragged along coordinates.*

In the case of infinitesimal deformation (1.1′) coordinate transformation

$$x^{i'} = \bar{x}^i = x^i + z^i(x^1, \dots, x^N)\varepsilon \tag{1.3}$$

is called *dragging along* by $z^i\varepsilon$.

Let us consider a geometric object $\mathcal{A}$ with respect to the system $(i)$ at the point $x = (x^i) \in L_N$, denoting this with $\mathcal{A}(i, x)$.

**Definition 1.3** The point $\bar{x}$ is said to be *deformed point* of the point $x$, if (1.1) holds. Geometric object $\bar{\mathcal{A}}(i, x)$ is *deformed object* $\mathcal{A}(i, x)$ with respect to deformation (1.1), if its value at system $(i')$, at the point $x$ is equal to the value of the object $\mathcal{A}$ at the system $(i)$ at the point $\bar{x}$, i.e. if

$$\bar{\mathcal{A}}(i', x) = \mathcal{A}(i, \bar{x}). \tag{1.4}$$

**Remark 1.1** In this study of infinitesimal deformations according to (1.1′) quantities of an order higher then the first with respect to $\varepsilon$ are neglected.

We will now define some important notions of the theory of infinitesimal deformations, following from (1.1): Lie differential and Lie derivative, and in further considerations we will find them for some geometric objects.

**Definition 1.4** The magnitude $\mathcal{DA}$, the difference between deformed object $\bar{\mathcal{A}}$ and initial object $\mathcal{A}$ at the same coordinate system and at the same point with respect to (1.1′), i.e.

$$\mathcal{DA} = \bar{\mathcal{A}}(i,x) - \mathcal{A}(i,x), \tag{1.5}$$

is called *Lie difference (Lie differential)*, and the magnitude

$$\mathcal{L}_z\mathcal{A} = \lim_{\varepsilon\to 0}\frac{\mathcal{DA}}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\bar{\mathcal{A}}(i,x) - \mathcal{A}(i,x)}{\varepsilon} \tag{1.5′}$$

is *Lie derivative* of geometric object $\mathcal{A}(i,x)$ with respect to the vector field $z = (z^i(x^j))$.

Using the relation (1.5) for deformed object $\bar{\mathcal{A}}(i,x)$ we have

$$\bar{\mathcal{A}}(i,x) = \mathcal{A}(i,x) + \mathcal{DA}, \tag{1.5″}$$

and thus we can express $\bar{\mathcal{A}}$, finding previously $\mathcal{DA}$. The known main cases are:

**1.1.** According to (1.5) we have $\mathcal{D}x^i = \bar{x}^i - x^i$, i.e. for the *coordinates* we have

$$\mathcal{D}x^i = z^i(x^j)\varepsilon, \tag{1.6}$$

from where

$$\mathcal{L}_z x^i = z^i(x^j). \tag{1.6′}$$

**1.2.** For the *scalar function* $\varphi(x) \equiv \varphi(x^1,\dots,x^N)$ we have

$$\mathcal{D}\varphi(x) = \varphi_{,p}z^p(x)\varepsilon = \mathcal{L}_z\varphi(x)\varepsilon, \quad (\varphi_{,p} = \partial\varphi/\partial x^p), \tag{1.7}$$

i.e. Lie derivative of the scalar function is derivative of this function in direction of the vector field $z$.

**1.3.** For a *tensor of the kind* $(u,v)$ we get

$$\mathcal{D}t^{i_1\dots i_u}_{j_1\dots j_v} = \Big[t^{i_1\dots i_u}_{j_1\dots j_v,p}z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{,p}\binom{p}{i_\alpha}t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^p_{,j_\beta}\binom{j_\beta}{p}t^{i_1\dots i_u}_{j_1\dots j_v}\Big]\varepsilon = \mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v}\varepsilon, \tag{1.8}$$

where we denoted

$$\binom{p}{i_\alpha}t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_{\alpha-1}p i_{\alpha+1}\dots i_u}_{j_1\dots j_v}, \quad \binom{j_\beta}{p}t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_{\beta-1}p j_{\beta+1}\dots j_v}. \tag{1.9}$$

**1.4.** For the *vector* $dx^i$ we have

$$\mathcal{D}(dx^i) = \mathcal{L}_z(dx^i) = 0. \tag{1.10}$$

**1.5.** In the same way, as for the tensors, for the *connection coefficients* we have

$$\mathcal{D}L^i_{jk} = (L^i_{jk,p}z^p + z^i_{,jk} - z^i_{,p}L^p_{jk} + z^p_{,j}L^i_{pk} + z^p_{,k}L^i_{jp})\varepsilon = \mathcal{L}_z L^i_{jk}\varepsilon. \tag{1.11}$$

## 2 The Lie derivative of a tensor

**2.1.** Because of non-symmetry of connection, at $L_N$ we can consider two types of covariant derivatives for a vector and four types for general tensor. So, denoting by $\underset{\theta}{|}(\theta = 1, \dots, 4)$ derivative of the type $\theta$, we have ([1]–[3]):

$$t^{i_1\dots i_u}_{j_1\dots j_v\underset{1,2,3,4}{|}m} = t^{i_1\dots i_u}_{j_1\dots j_v,m} + \sum_{\alpha=1}^{u} L^{i_\alpha}_{\underset{mp,\,pm,\,mp}{pm}} \binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} - \sum_{\beta=1}^{v} L^{p}_{\underset{mj_\beta,\,mj_\beta,\,j_\beta m}{j_\beta m}} \binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}. \quad (2.1a\text{–}d)$$

Generally, the next theorem is in the force:

**Theorem 2.1** *Lie derivative of a tensor $t^{i_1\dots i_u}_{j_1\dots j_v}$ of the type $(u, v)$ is a tensor of the same type and can be presented in the following four ways*

$$\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} = \underset{\theta}{\mathcal{L}_z} t^{i_1\dots i_u}_{j_1\dots j_v} \equiv t^{i_1\dots i_u}_{j_1\dots j_v\underset{\theta}{|}p} z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{\underset{\theta}{|}p} \binom{p}{i_\alpha} t^{\cdots}_{\cdots} + \sum_{\beta=1}^{v} z^{p}_{\underset{\theta}{|}j_\beta} \binom{j_\beta}{p} t^{\cdots}_{\cdots}$$

$$+(-1)^{\theta-1} \sum_{\alpha=1}^{u} T^{i_\alpha}_{ps} \binom{s}{i_\alpha} t^{\cdots}_{\cdots} z^p + (-1)^{\theta-1} \sum_{\beta=1}^{v} T^{s}_{j_\beta p} \binom{j_\beta}{s} t^{\cdots}_{\cdots} z^p, \quad \theta = 1, 2; \quad (2.2a, b)$$

$$\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} = \underset{\theta}{\mathcal{L}_z} t^{i_1\dots i_u}_{j_1\dots j_v} \equiv t^{i_1\dots i_u}_{j_1\dots j_v\underset{\theta}{|}p} z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{\underset{\theta}{|}p} \binom{p}{i_\alpha} t^{\cdots}_{\cdots} + (-1)^{\theta-1} \sum_{\beta=1}^{v} z^{p}_{\underset{\theta}{|}j_\beta} \binom{j_\beta}{p} t^{\cdots}_{\cdots}$$

$$+(-1)^{\theta-1} \sum_{\alpha=1}^{u} T^{i_\alpha}_{ps} \binom{s}{i_\alpha} t^{\cdots}_{\cdots} z^p, \quad \theta = 3, 4, \quad (2.3a, b)$$

*where $\underset{\theta}{\mathcal{L}_z}$ denotes that Lie derivative $\mathcal{L}_z$ is expressed by covariant derivative of the type $\theta$, $(\underset{\theta}{|})$, $\theta = 1, \dots, 4$.*

**Proof** We will prove only the fourth case. The others can be proved in an analogous way. According to (1.8) we have

$$\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_v,p} z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{,p} \binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^{p}_{,j_\beta} \binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}, \quad (2.4)$$

and we have to express partial derivatives with respect to $\underset{4}{|}$. From $(2.1d)$ we have

$$t^{i_1\dots i_u}_{j_1\dots j_v,p} = t^{i_1\dots i_u}_{j_1\dots j_v\underset{4}{|}p} - \sum_{\alpha=1}^{u} L^{i_\alpha}_{ps} \binom{s}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} L^{s}_{j_\beta p} \binom{j_\beta}{s} t^{i_1\dots i_u}_{j_1\dots j_v}, \quad (2.5)$$

$$z^{i_\alpha}_{,p} = z^{i_\alpha}_{\underset{4}{|}p} - L^{i_\alpha}_{ps} z^s, \quad (2.6)$$

and by substituting at (2.4) we obtain:

$$\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} = [t^{i_1\dots i_u}_{j_1\dots j_v\,\underset{4}{|}\,p} - \sum_{\alpha=1}^{u} L^{i_\alpha}_{ps}\binom{s}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} L^{s}_{j_\beta p}\binom{j_\beta}{s} t^{i_1\dots i_u}_{j_1\dots j_v}] z^p$$
$$- \sum_{\alpha=1}^{u}(z^{i_\alpha}_{\underset{4}{|}p} - L^{i_\alpha}_{ps}z^s)\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v}(z^{p}_{\underset{4}{|}j_\beta} - L^{p}_{j_\beta s}z^s)\binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}$$

According to

$$- \sum_{\alpha=1}^{u} L^{i_\alpha}_{ps}\binom{s}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} z^p + \sum_{\beta=1}^{v} L^{s}_{j_\beta p}\binom{j_\beta}{s} t^{i_1\dots i_u}_{j_1\dots j_v} z^p$$
$$+ \sum_{\alpha=1}^{u} L^{i_\alpha}_{ps}\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} z^s - \sum_{\beta=1}^{v} L^{p}_{j_\beta s}\binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v} z^s$$
$$= \sum_{\alpha=1}^{u} T^{i_\alpha}_{sp}\binom{s}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} z^p,$$

the previous equation gives (2.3*b*). □

Of course, as the same magnitude at the right side at (2.4) was expressed in different ways, we have

$$\underset{\theta}{\mathcal{L}_z} = \mathcal{L}_z, \quad \theta = 1, \dots, 4. \tag{2.7}$$

**Corollary 2.1** *For the space* $\overset{0}{L}_N$ *of* symmetric connection $\overset{0}{L}{}^i_{jk}$ $(T^i_{jk} = 0)$ *we have*

$$\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} = \overset{0}{\mathcal{L}}_z t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_v;p} z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{;p}\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^{p}_{;j_\beta}\binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}, \tag{2.8}$$

*because in that case all 4 types of covariant derivatives reduce to one, which we denote by semicolon* (; ).

**2.2.** So, Lie derivative of a tensor at a space of symmetric affine connection can be obtained as a special case from the formulae for Lie derivative at a space of non-symmetric affine connection. We will investigate the way of presenting the Lie derivative of a tensor by covariant derivative with respect to symmetrical part $\underset{0}{L}{}^i_{jk}$ of non-symmetric connection $L^i_{jk}$. Let us consider a space $L_N$ of non-symmetric affine connection $L^i_{jk}$ and let be

$$\underset{0}{L}{}^i_{jk} = \frac{1}{2}(L^i_{jk} + L^i_{kj}), \quad T^i_{jk} = L^i_{jk} - L^i_{kj}. \tag{2.9}$$

Then

$$L^i_{jk} = \underset{0}{L}{}^i_{jk} + \frac{1}{2}T^i_{jk}. \tag{2.10}$$

The magnitudes $\underset{0}{L}{}^i_{jk}$ are the coefficients of symmetric connection associated to the connection $L^i_{jk}$, and $T^i_{jk}$ are the components of torsion tensor of connection $L^i_{jk}$. If we denote with $\underset{0}{\mathcal{L}}_z t^{i_1\dots i_u}_{j_1\dots j_v}$ the expression as on the right side at (2.8), but formed by means of $\underset{0}{L}{}^i_{jk}$ from (2.9) instead of $\overset{0}{L}{}^i_{jk}$ we have the next theorem

**Theorem 2.2** *In the non-symmetric connection space $L_N$ Lie derivative of tensor $t^{i_1\dots i_u}_{j_1\dots j_v}$ can be expressed as*

$$\begin{aligned}\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} &= \underset{\theta}{\mathcal{L}}_z t^{i_1\dots i_u}_{j_1\dots j_v} = \underset{0}{\mathcal{L}}_z t^{i_1\dots i_u}_{j_1\dots j_v}\\ &\equiv t^{i_1\dots i_u}_{j_1\dots j_v;p} z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{;p}\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^p_{;j_\beta}\binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v},\end{aligned}\tag{2.11}$$

*where the semicolon (;) denotes covariant derivative with respect to symmetric part $\underset{0}{L}{}^i_{jk}$ of the connection $L^i_{jk}$.*

**Proof** According to (2.7), we can start from any of the $\underset{\theta}{\mathcal{L}}_z$, $(\theta = 1,\dots,4)$. Let us start from $\underset{4}{\mathcal{L}}_z t^{i_1\dots i_u}_{j_1\dots j_v}$ from the equation (2.3*b*). According to (2.1, 10) we have

$$\begin{aligned}t^{i_1\dots i_u}_{j_1\dots j_v \underset{4}{|} p} &= t^{i_1\dots i_u}_{j_1\dots j_v,p} + \sum_{\alpha=1}^{u}(\underset{0}{L}{}^{i_\alpha}_{ps} + \frac{1}{2}T^{i_\alpha}_{ps})\binom{s}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v}\\ &- \sum_{\beta=1}^{v}(\underset{0}{L}{}^s_{j_\beta p} + \frac{1}{2}T^s_{j_\beta p})\binom{j_\beta}{s} t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_v;p}\\ &+ \frac{1}{2}\sum_{\alpha=1}^{u} T^{i_\alpha}_{ps}\binom{s}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} - \frac{1}{2}\sum_{\beta=1}^{v} T^s_{j_\beta p}\binom{j_\beta}{s} t^{i_1\dots i_u}_{j_1\dots j_v},\end{aligned}$$

$$z^{i_\alpha}_{\underset{4}{|} p} = z^{i_\alpha}_{,p} + L^{i_\alpha}_{ps} z^s = z^{i_\alpha}_{,p} + (\underset{0}{L}{}^{i_\alpha}_{ps} + \frac{1}{2}T^{i_\alpha}_{ps}) z^s = z^{i_\alpha}_{;p} + \frac{1}{2}T^{i_\alpha}_{ps} z^s,$$

which by substituting at (2.3*b*) gives

$$\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} = \underset{4}{\mathcal{L}}_z t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_v;p} z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{;p}\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^p_{;j_\beta}\binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v},$$

i.e. (2.11). □

**2.3** Comparing (2.2, 3) and (2.11), we can see that Lie derivative of a tensor at $L_N$ can be simpler be given by means of (2.11), i.e. with respect to covariant derivative formed by symmetrical part $\underset{0}{L}{}^i_{jk}$ of non-symmetrical connection $L^i_{jk}$.

If we use at the same time different kinds of covariant derivative at the right side at (2.2, 3) with respect to $L^i_{jk}$, we can write this equations in the more condensed form (analogously to (2.11)). In connection with this the next theorem is in the force

**Theorem 2.3** *The Lie derivative of the tensor of type $(u,v)$ can be expressed using covariant derivatives with respect to non-symmetric connection $L^i_{jk}$ in the next way*

$$\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} = t^{i_1\dots i_u}_{j_1\dots j_v \underset{\lambda}{|} p} z^p - \sum_{\alpha=1}^{u} z^{i_\alpha}_{\underset{\mu}{|} p} \binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + \sum_{\beta=1}^{v} z^{p}_{\underset{\nu}{|} j_\beta} \binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}, \qquad (2.12a\text{–}d)$$

*where* $(\lambda,\mu,\nu)\in\{(1,2,2),(2,1,1),(3,4,3),(4,3,4)\}$.

**Proof** We will prove only the first case, the others can be proved analogously. Let us start from $(2.2a)$. We have

$$z^{i_\alpha}_{\underset{1}{|} p}\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} = (z^{i_\alpha}_{,p} + L^{i_\alpha}_{sp} z^s)\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} = (z^{i_\alpha}_{,p} + L^{i_\alpha}_{ps} z^s - L^{i_\alpha}_{ps} z^s$$
$$+ L^{i_\alpha}_{sp} z^s)\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} = z^{i_\alpha}_{\underset{2}{|} p}\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + T^{i_\alpha}_{sp} z^s \binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v}$$

and analogously

$$z^{p}_{\underset{1}{|} j_\beta}\binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v} = z^{p}_{\underset{2}{|} j_\beta}\binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v} + T^{s}_{p j_\beta}\binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v} z^p.$$

Substituting this at $(2.2a)$ it follows that

$$\mathcal{L}_z t^{i_1\dots i_u}_{j_1\dots j_v} =$$
$$= t^{i_1\dots i_u}_{j_1\dots j_v \underset{1}{|} p} z^p - \sum_{\alpha=1}^{u} [z^{i_\alpha}_{\underset{2}{|} p}\binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} + T^{i_\alpha}_{sp} z^s \binom{p}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v}] + \sum_{\beta=1}^{v} [z^{p}_{\underset{2}{|} j_\beta}\binom{j_\beta}{p} t^{i_1\dots i_u}_{j_1\dots j_v}$$
$$+ T^{s}_{p j_\beta}\binom{j_\beta}{s} t^{i_1\dots i_u}_{j_1\dots j_v} z^p] + \sum_{\alpha=1}^{u} T^{i_\alpha}_{ps}\binom{s}{i_\alpha} t^{i_1\dots i_u}_{j_1\dots j_v} z^p + \sum_{\beta=1}^{v} T^{s}_{j_\beta p}\binom{j_\beta}{s} t^{i_1\dots i_u}_{j_1\dots j_v} z^p,$$

from where we obtain (2.12) for $(\lambda,\mu,\nu)=(1,2,2)$. □

## 3 Lie derivative of the connection

**3.1** On the base of (1.11) for the Lie derivative of the connection we have

$$\mathcal{L}_z L^i_{jk} = z^i_{,jk} + L^i_{jk,p} z^p - z^i_{,p} L^p_{jk} + z^p_{,j} L^i_{pk} + z^p_{,k} L^i_{jp}. \qquad (3.1)$$

As it was proved at [10] Lie derivative can be written in the next way

$$\mathcal{L}_z L^i_{jk} = z^i_{\underset{1}{|} jk} + \underset{1}{R}{}^i_{jkp} z^p + T^i_{jp,k} z^p + L^s_{jk} T^i_{ps} z^p + L^i_{sk} T^s_{jp} z^p + T^i_{jp} z^p_{,k}, \qquad (3.2)$$

$$\mathcal{L}_z L^i_{jk} = \underset{1}{\mathcal{L}}_z L^i_{jk} \equiv z^i_{\underset{1}{|} jk} + \underset{1}{R}{}^i_{jkp} z^p + (T^i_{jp} z^p)_{\underset{1}{|} k}, \qquad (3.2')$$

$$\mathcal{L}_z L^i_{jk} = \underset{2}{\mathcal{L}}_z L^i_{jk} = z^i_{\underset{2}{|}\, jk} + \underset{2}{R}{}^i_{jkp} z^p + T^i_{pj\underset{2}{|}\, k} z^p + T^i_{pk} z^p_{\underset{2}{|}\, j}$$
$$+ T^p_{kj} z^i_{\underset{2}{|}\, p} + T^i_{jk\underset{2}{|}\, p} z^p + (T^i_{sj} T^s_{kp} + T^i_{sk} T^s_{pj} + T^i_{sp} T^s_{jk}) z^p, \tag{3.3}$$

$$\mathcal{L}_z L^i_{jk} = \underset{3}{\mathcal{L}}_z L^i_{jk} \equiv z^i_{\underset{3}{|}\, jk} + \underset{3}{R}{}^i_{jkp} z^p - T^p_{jk} z^i_{\underset{3}{|}\, p} + T^i_{jp} z^p_{\underset{3}{|}\, k}, \tag{3.4}$$

$$\mathcal{L}_z L^i_{jk} = \underset{4}{\mathcal{L}}_z L^i_{jk} \equiv z^i_{\underset{4}{|}\, jk} + \underset{4}{R}{}^i_{jkp} z^p + (T^i_{pj\underset{4}{|}\, k} + T^i_{sj} T^s_{pk} + T^i_{sk} T^s_{pj}) z^p + T^i_{pk} z^p_{\underset{4}{|}\, j}, \tag{3.5}$$

where [1–3]

$$\underset{1}{R}{}^i_{jkp} = L^i_{jk,p} - L^i_{jp,k} + L^s_{jk} L^i_{sp} - L^s_{jp} L^i_{sk} \tag{3.6}$$

$$\underset{2}{R}{}^i_{jkp} = L^i_{kj,p} - L^i_{pj,k} + L^s_{kj} L^i_{ps} - L^s_{pj} L^i_{ks} \tag{3.7}$$

$$\underset{3}{R}{}^i_{jkp} = L^i_{jk,p} - L^i_{pj,k} + L^s_{jk} L^i_{ps} - L^s_{pj} L^i_{sk} + L^s_{pk} T^i_{sj} \tag{3.8}$$

$$\underset{4}{R}{}^i_{jkp} = L^i_{jk,p} - L^i_{pj,k} + L^s_{jk} L^i_{ps} - L^s_{pj} L^i_{sk} + L^s_{kp} T^i_{sj} \tag{3.9}$$

are curvature tensors of the space $L_N$.

**3.2.** We have proved at Theorem 2.3. that the Lie derivative of a tensor can be expressed more concise by using several types of covariant derivatives at $L_N$ simmultaneously. It is the same case for the Lie derivative of the conecction. Namely, the next theorem is in force.

**Theorem 3.1** *The Lie derivative of non-symmetric connection $L^i_{jk}$ is a tensor of the type $(1,2)$ and can be expressed with respect to covariant derivatives by equations (3.2–5), as well as by*

$$\mathcal{L}_z L^i_{jk} = z^i_{\underset{2}{|}\, j \underset{1}{|}\, k} + \underset{1}{R}{}^i_{jkp} z^p. \tag{3.10}$$

$$\mathcal{L}_z L^i_{jk} = z^i_{\underset{1}{|}\, k \underset{2}{|}\, j} + \underset{2}{R}{}^i_{kjp} z^p. \tag{3.11}$$

**Proof** The equations (3.2–5), (3.10) are proved at [10]. We will here prove (3.11).

Starting from the equation

$$z^i_{\underset{1}{|}\, j} = z^i_{,j} + L^i_{pj} z^p, \tag{3.12}$$

we get

$$z^i_{\underset{1}{|}\, j \underset{2}{|}\, k} = (z^i_{\underset{1}{|}\, j})_{,k} + L^i_{ks} z^s_{\underset{1}{|}\, j} - L^s_{kj} z^i_{\underset{1}{|}\, s}$$
$$= z^i_{,jk} + L^i_{pj,k} z^p + L^i_{pj} z^p_{,k} + L^i_{ks}(z^s_{,j} + L^s_{pj} z^p) - L^s_{kj}(z^i_{,s} + L^i_{ps} z^p).$$

From here we substitute $z^i_{,jk}$ at (3.3) and we have

$$\mathcal{L}_z L^i_{jk} =$$
$$= z^i_{\underset{1}{|}j\underset{2}{|}k} + (L^i_{kj,p} - L^i_{pj,k} + L^s_{kj}L^i_{ps} - L^s_{pj}L^i_{ks})z^p + T^i_{jk,p}z^p - z^i_{,p}T^p_{jk} + z^p_{,j}T^i_{pk} + z^p_{,k}T^i_{jp}.$$

According to (3.7) and (1.8) this equation becomes

$$\mathcal{L}_z L^i_{jk} = z^i_{\underset{1}{|}j\underset{2}{|}k} + \underset{2}{R}{}^i_{jkp}z^p + \mathcal{L}_z T^i_{jk}. \tag{3.13}$$

From here

$$\mathcal{L}_z(L^i_{jk} - T^i_{jk}) \overset{(3.7)}{=} \mathcal{L}_z(L^i_{jk} - L^i_{jk} + L^i_{kj}) = \mathcal{L}_z L^i_{kj} = z^i_{\underset{1}{|}j\underset{2}{|}k} + \underset{2}{R}{}^i_{jkp}z^p,$$

i.e. $(j \leftrightarrow k)$ we get (3.11). □

The difference between (3.11) and (3.10) gives

$$0 = z^i_{\underset{1}{|}k\underset{2}{|}j} - z^i_{\underset{2}{|}j\underset{1}{|}k} + (\underset{2}{R}{}^i_{kjp} - \underset{1}{R}{}^i_{jkp})z^p,$$

i.e.

$$z^i_{\underset{1}{|}j\underset{2}{|}k} - z^i_{\underset{2}{|}k\underset{1}{|}j} = (\underset{1}{R}{}^i_{kjp} - \underset{2}{R}{}^i_{jkp})z^p = \underset{3}{R}{}^i_{pjk}z^p, \tag{3.14}$$

as from (3.6, 7, 8) we have

$$\underset{2}{R}{}^i_{kjp} - \underset{1}{R}{}^i_{jkp} = \underset{3}{R}{}^i_{pjk}. \tag{3.15}$$

The equation (3.14) is one of the Ricci type identities at $L_N$ (see [1], [3]).

**3.3.** Comparing (2.8) and (2.11), we can see that the Lie derivative of a tensor at space $\overset{0}{L}_N$ of symmetric connection $\overset{0}{L}{}^i_{jk}$ and Lie derivative at the space $L_N$ of non-symmetric connection $L^i_{jk}$ are expressed in the same way: with respect to given symmetric connection $\overset{0}{L}{}^i_{jk}$ in the first case, and in the second with respect to the symmetric part $\underset{0}{L}{}^i_{jk}$ of non-symmetric connection $L^i_{jk}$.

We will here consider an analogous problem in a case of a connection (that is not a tensor). At the space $\overset{0}{L}_N$ of symmetric connection $\overset{0}{L}{}^i_{jk}$, by reason of $T^i_{jk} = 0$ all the cases of expresses for the Lie derivative considered before, reduce to

$$\mathcal{L}_z \overset{0}{L}{}^i_{jk} = z^i_{;jk} + \overset{0}{R}{}^i_{jkp}z^p, \tag{3.16}$$

where $\overset{0}{R}{}^i_{jkp}$ is curvature tensor, generated by $\overset{0}{L}{}^i_{jk}$. Let us examine a space $L_N$ of non-symmetric affine connection $L^i_{jk}$, where $\underset{0}{L}{}^i_{jk}$, $T^i_{jk}$ are given by (2.9).

The main purpose is to express $\mathcal{L}_z L^i_{jk}$ (3.2′) by covariant derivatives with respect to $\underset{0}{L}{}^i_{jk}$, and $\underset{1}{R}{}^i_{jkp}$ by $\underset{0}{R}{}^i_{jkp}$, formed by $\underset{0}{L}{}^i_{jk}$. We have

$$z^i_{\underset{1}{|}j} = z^i_{,j} + L^i_{pj} z^p = z^i_{,j} + (\underset{0}{L}{}^i_{pj} + \frac{1}{2} T^i_{pj}) z^p = z^i_{;j} + \frac{1}{2} T^i_{pj} z^p, \tag{3.17}$$

$$z^i_{\underset{1}{|}jk} = (z^i_{;j})_{\underset{1}{|}k} + \frac{1}{2}(T^i_{pj} z^p)_{\underset{1}{|}k} = (z^i_{;j})_{,k} + L^i_{sk} z^s_{;j} - L^s_{jk} z^i_{;s} + \frac{1}{2}(T^i_{pj} z^p)_{\underset{1}{|}k}$$

$$= z^i_{;jk} + \frac{1}{2}[T^i_{sk} z^s_{;j} - T^s_{jk} z^i_{;s} + (T^i_{pj} z^p)_{\underset{1}{|}k}] \tag{3.18}$$

According to (2.7) at [3] we have

$$\underset{1}{R}{}^i_{jkp} = \underset{0}{R}{}^i_{jkp} + \frac{1}{2} T^i_{jk;p} - \frac{1}{2} T^i_{jp;k} + \frac{1}{4} T^s_{jk} T^i_{sp} - \frac{1}{4} T^s_{jp} T^i_{sk}, \tag{3.19}$$

and substituting (3.18,19) at (3.2′) we obtain

$$\mathcal{L}_z L^i_{jk} = z^i_{;jk} + \frac{1}{2}[T^i_{sk} z^s_{;j} - T^s_{jk} z^i_{;s} + (T^i_{pj} z^p)_{\underset{1}{|}k}]$$
$$+ \underset{0}{R}{}^i_{jkp} z^p + \frac{1}{2}[T^i_{jk;p} - T^i_{jp;k} + \frac{1}{2} T^s_{jk} T^i_{sp} - \frac{1}{2} T^s_{jp} T^i_{sk}] z^p + (T^i_{jp} z^p)_{\underset{1}{|}k}.$$

Based on (2.11), we get

$$\mathcal{L}_z L^i_{jk} = \underset{0}{\mathcal{L}}_z L^i_{jk} \equiv z^i_{;jk} + \underset{0}{R}{}^i_{jkp} z^p + \frac{1}{2} \underset{0}{\mathcal{L}}_z T^i_{jk}. \tag{3.20}$$

As

$$\underset{0}{\mathcal{L}}_z (L^i_{jk} - \frac{1}{2} T^i_{jk}) = \underset{0}{\mathcal{L}}_z (L^i_{jk} - \frac{1}{2} L^i_{jk} + \frac{1}{2} L^i_{kj}) = \underset{0}{\mathcal{L}}_z (\frac{1}{2} L^i_{jk} + \frac{1}{2} L^i_{kj}) = \underset{0}{\mathcal{L}}_z \underset{0}{L}{}^i_{jk},$$

from (3.20) we have

$$\mathcal{L}_z \underset{0}{L}{}^i_{jk} = \underset{0}{\mathcal{L}}_z \underset{0}{L}{}^i_{jk} = z^i_{;jk} + \underset{0}{R}{}^i_{jkp} z^p. \tag{3.21}$$

Based on the pointed facts follows

**Theorem 3.2** *Lie derivative of non-symmetric connection $L^i_{jk}$ can be given by the equation (3.20), where covariant derivative denoted by; and curvature tensor $\underset{0}{R}{}^i_{jkp}$ are formed with respect to symmetric part $\underset{0}{L}{}^i_{jk}$ of the connection $L^i_{jk}$, and $\underset{0}{\mathcal{L}}_z T^i_{jk}$ is expressed according to (2.11) with respect to $\underset{0}{L}{}^i_{jk}$. The Lie derivative of symmetric part of connection is given according to (3.21) i.e. it is the same as for symmetric connection (equation (3.16)).*

## References

[1] Minčić, S. M.: *Ricci identities in the space of non-symmetric affine connexion.* Matematički vesnik **10(25)**, 2 (1973), 161–172.

[2] Minčić, S. M.: *New commutation formulas in the non-symetric affine connexion space.* Publ. Inst. Math. (Beograd) (N.S) **22,(36)** (1977), 189–199.

[3] Minčić, S. M.: *Independent curvature tensors and pseudotensors of spaces with non-symmetric affine connexion.* Coll. math. soc. János Bolyai, 31. Dif. geom., Budapest (Hungary), 1979, 445–460.

[4] Stojanović, R.: Osnovi diferencijalne geometrije. *Gradjevinska knjiga, Beograd,* 1963.

[5] Schouten, J. A.: Ricci Calculus. *Springer Verlag, Berlin–Gotingen–Heildelberg,* 1954

[6] Yano, K.: *Sur la theorie des deformations infinitesimales.* Journal of Fac. of Sci. Univ. of Tokyo **6** (1949), 1–75.

[7] Yano, K.: The Theory of Lie Derivatives and its Applications. *N-Holland Publ. Co., Amsterdam,* 1957.

[8] Ivanova-Karatopraklieva, I., Sabitov, I. Kh.: *Surface deformation.* J. Math. Sci., New York **70**, 2 (1994), 1685–1716.

[9] Ivanova-Karatopraklieva, I., Sabitov, I. Kh.: *Bending of surfaces II.* J. Math. Sci., New York **74**, 3 (1995), 997–1043.

[10] Minčić, S. M., Velimirović, L. S., Stanković, M. S.: *Infinitesimal Deformations of a Non-symmetric Affine Connection Space.* Filomat **15** (2001).

[11] Mikes, J.: *Holomorphically projective mappings and their generalizations.* J. Math. Sci., New York **89**, 3 (1998), 1334–1353.