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Balanced Projective Dimension of Modules

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Abstract: This semi-expository note fulfills the promise we gave some time ago of building a theory of balanced projective dimension of torsion-free modules over valuation domains, which will be used in further studies of balanced projective dimension. A complementary account of balanced projective dimension for the torsion case (for abelian p-groups) has been given by Fuchs and Hill. In addition to essential properties of balanced exact sequences, we define balanced n-fold extensions and define a module M to have the balanced projective dimension n, if there is a module C with $\mathrm{Ext}^n_{\mathcal{B}}(M, C) \neq 0$ and, for every $k \geq 1$ and every module B, $\mathrm{Ext}^{n+k}_{\mathcal{B}}(M, B) = 0$. Principal results are variations on the theme of the well-known Auslander's lemma about the dimension of the union of an ascending chain of submodules depending on the dimension of the subsequent links.

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1. Balanced exact sequences

The realm we work within is that of unitary torsion-free R-modules over a commutative valuation domain R, although a number of our results may be shown to be valid in a wider context of proper exact sequences.

Definition. *A (pure) exact sequence of R-modules $E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is called balanced exact and A is a balanced submodule of B, if every rank one module has the projective property with respect to the sequence. $E \in \mathcal{B}$ will stand for "E is a balanced exact sequence", while $A \leq_{\mathcal{B}} B$ will stand for "the module A is a balanced submodule of B"; α is a "balanced monomorphism" and β is a "balanced epimorphism".*

There are several characterizations of balanced exact sequences and we collect them below, giving first the following definitions (cf. [Dimitrić, 1992, 1993]): For an R-module M and $a \in M$, the set $\chi_M(a) = \{r \in R : a = ta_1\}$, for some $a_1 \in M$ is called the *characteristic of a in M*. The *trace* of an ideal I in an R-module M is $M(I) = \{a \in M : \exists i \in I \ \chi_I(i) \leq \chi_M(a)\}$.

Proposition 1. *A (pure) exact sequence of R-modules* $E: 0 \longrightarrow A \xrightarrow{*} B \xrightarrow{q} C \longrightarrow 0$ *is balanced if and only if any of the following statements holds:*

(1) Every completely decomposable module has projective property with respect to E.

(2) $\forall c \in C \exists b \in B$ *with* $qb = c$ *and* $\chi_C(c) = \chi_B(b)$.

(3) Every rank one module has projective property with respect to E.

(4) For every rank one module I, *the sequence* $\mathrm{Hom}(I, E) : 0 \longrightarrow \mathrm{Hom}(I, A) \longrightarrow \mathrm{Hom}(I, B) \xrightarrow{\bar{q}} \mathrm{Hom}(I, C) \longrightarrow 0$ *is (balanced) exact.*

(5) For every $r \in R$, *the sequence* $rE : 0 \longrightarrow rA \longrightarrow rB \longrightarrow rC \longrightarrow 0$ *is balanced exact.*

(6) For every rank one module I, *the sequence* $E \otimes_R I : 0 \longrightarrow A \otimes_R I \longrightarrow B \otimes_R I \xrightarrow{q \otimes id} C \otimes I \longrightarrow 0$ *is balanced exact.*

(7) For every rank one module I, *the sequence* $E(I) : 0 \longrightarrow A(I) \longrightarrow B(I) \xrightarrow{q} C(I) \longrightarrow 0$ *is balanced exact.*

Proof. Most of the claims are in [Dimitrić, 1993, Proposition 5.1] and [Dimitrić, 1992, Lemma 2.1], and equivalences with (1) and (5) are not difficult to establish. ◊

Balanced projective modules (direct sums of rank one modules) are completely decomposable modules. Direct sum of modules is balanced projective iff every component is balanced projective.

A simple, but an extremely important fact is that balanceness is preserved under push-outs and (dually) pull-backs:

Lemma 2. *(a) Given a balanced exact sequence* $E[\alpha, \beta] \in \mathcal{B}$ *and* $\gamma : B \longrightarrow C$ *(the "matching morphism"), there is a unique commutative diagram (below) with the bottom row* $E'[\alpha', \beta'] \in \mathcal{B}$. *We write* $E' = \gamma E$.

(b) Given a balanced exact sequence $E[\beta, \alpha] \in \mathcal{B}$ *and a (matching morphism)* $\gamma : B \longrightarrow C$, *there is a unique commutative diagram (shown below) with the top row* $E'[\beta', \alpha'] \in \mathcal{B}$. *We write* $E' = E\gamma$.

Proof. First build the push-out diagram CBAS. By a property of pushouts, α' is monic, since α is. Another commutative square with the same three corners A, B, C is formed by adding morphisms $0 : C \longrightarrow D$ and $\beta : A \longrightarrow D$ as its sides. By universality of the push-out construction, there is then a unique $\beta' : S \longrightarrow D$ with $\beta'\alpha' = 0, \beta'\gamma' = \beta$. Thus we have constructed the following commutative diagram:

$$\begin{array}{lccccccc}
E: & 0 \longrightarrow & B & \xrightarrow{\alpha} & A & \xrightarrow{\beta} & D & \longrightarrow 0 \\
 & & \gamma \downarrow & & \gamma' \downarrow & & \| & \\
E': & 0 \longrightarrow & C & \xrightarrow{\alpha'} & S & \xrightarrow{\beta'} & D & \longrightarrow 0
\end{array}$$

We will now prove that $\beta' \in \operatorname{coker}\alpha'$ and therefore that E' is exact. To this end let ξ be a morphism such that $\xi\alpha' = 0$. Since $\xi\alpha'\gamma = \xi\gamma'\alpha = 0$, $\xi\gamma'$ factors through $\beta \in \operatorname{coker}\alpha$ as follows: $\xi\gamma' = \delta\beta = \delta\beta'\gamma'$, for some morphisms δ. On the other hand, $\xi\alpha' = 0 = \delta\beta'\alpha'$. The universal property of the push-out at S, applied to the coterminal maps ξ and $\delta\beta'$ gives $\xi = \delta\beta'$. Every ξ with $\xi\alpha' = 0$ factors through β' and $\beta'\alpha' = 0$, hence $\beta' \in \operatorname{coker}\alpha'$. Uniqueness comes from uniqueness of push-outs.

It remains only to prove that $E' \in \mathcal{B}$. If I is a rank one module and $f : I \longrightarrow D$ is a morphism, there is a morphism $g : I \longrightarrow A$ with $\beta g = f$, because E is balanced exact. Now $\beta'(\gamma' g) = \beta g = f$.

(b) The construction is fully dual with the pull-back P etc. To show that the top row is balanced, start with a rank-one morphism $f : I \longrightarrow B$.

$$\begin{array}{llccccccccc}
E': & 0 \longrightarrow & D & \xrightarrow{\beta'} & P & \xrightarrow{\alpha'} & B & \longrightarrow & 0 \\
 & & \| & & \gamma' \downarrow & & \gamma \downarrow & & \\
E: & 0 \longrightarrow & D & \xrightarrow{\beta} & A & \xrightarrow{\alpha} & C & \longrightarrow & 0
\end{array}$$

Bottom row is balanced, thus $\exists g_1 : I \longrightarrow A$ with $\gamma f = \alpha g_1$. By universality of the pull-back construction $\exists g : I \longrightarrow P$ with $\alpha' g = f$. Uniqueness is guaranteed by uniqueness of the pull-back. $\Diamond$

This lemma is used to prove the set of self-dual properties of the class of balanced exact sequences:

Proposition 3. *Assume that $D \leq A \leq B$, for R-modules D, A, B. Then the following hold:*

- *(P1) If $D \leq A$ and $D_1 \leq A_1$ are isomorphic embeddings, then $D \leq_{\mathcal{B}} A$ implies $D_1 \leq_{\mathcal{B}} A_1$.*
- *(P2) Every direct summand of a module is its balanced submodule.*
- *(P3) If $D \leq_{\mathcal{B}} A$, $A \leq_{\mathcal{B}} B$, then $D \leq_{\mathcal{B}} B$.*
- *(P3d) If $D \leq_{\mathcal{B}} B$ and $A/D \leq_{\mathcal{B}} B/D$, then $A \leq_{\mathcal{B}} B$.*
- *(P4) If $D \leq_{\mathcal{B}} B$, then $D \leq_{\mathcal{B}} A$.*
- *(P4d) If $A \leq_{\mathcal{B}} B$, then $A/D \leq_{\mathcal{B}} B/D$.*

Proof. We give a homological proof applicable to more general classes too. Axioms (P1) and (P2) are straightforward to establish. For (P3), assume that α, α_1 are balanced monomorphisms, i.e. that the short exact sequences $C_1, E_2 \in \mathcal{B}$. Construct the commutative rectangle ABZZYX, according to Lemma 2(a). Thus, also $E_3 \in \mathcal{B}$. Fill the top boxes as indicated, where $E_1, C_3 \in \mathcal{B}$.

$$
\begin{array}{lccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
E_1: & 0 \longrightarrow & D & \xrightarrow{=} & D & \xrightarrow{0} & 0 & \longrightarrow & 0 \\
 & & \alpha_1 \downarrow & & \alpha\alpha_1 \downarrow & & 0 \downarrow & & \\
E_2: & 0 \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & Z & \longrightarrow & 0 \\
 & & \beta_1 \downarrow & & \beta_2 \downarrow & & = \downarrow & & \\
E_3: & 0 \longrightarrow & X & \xrightarrow{\gamma} & Y & \xrightarrow{\delta} & Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & C_1 & & C_2 & & C_3 & &
\end{array}
$$

We want to prove that the middle column is proper. Note that $\beta_2\alpha\alpha_1 = 0$, thus, by the middle 3×3 Lemma the middle column is exact. Now let P_3 be completely decomposable and $\phi_3 : P_3 \longrightarrow Y$ be given. Then $\delta\phi_3 : P_3 \longrightarrow Z$ and $E_2 \in \mathcal{B}$ imply the existence of $f_2 : P_3 \longrightarrow B$ with $\delta\phi_3 = \beta f_2$. Denote $\psi = \beta_2 f_2 - \phi_3$. If $\psi = 0$, we are done. Otherwise note that $\delta\psi = \beta f_2 - \delta\phi_3 = 0$. Thus $\psi = \gamma\psi_1$, for some $\psi_1 : P_3 \longrightarrow X$ (since $\gamma = \ker\delta$). Now $C_1 \in \mathcal{B}$ implies the existence of $\xi : P_3 \longrightarrow A$ with $\beta_1\xi = \psi_1$. We have $\beta_2\alpha\xi = \gamma\beta_1\xi = \gamma\psi_1 = \psi$, so $\phi_3 = \beta_2 f_2 - \psi = \beta_2(f_2 - \alpha\xi)$. This means that $f = f_2 - \alpha\xi$ is the desired map with $\beta_2 f = \phi_3$.

For (P3d), let $C_2, E_3 \in \mathcal{B}$; we need to show that $E_2 \in \mathcal{B}$ (as above we have the commutative diagram with all columns and all rows exact). To this end let P_{3d} be completely decomposable and $\phi_{3d} : P_{3d} \longrightarrow Z$. The balanced exact sequence E_3 would imply the existence of a map $f_3 : P_{3d} \longrightarrow Y$ with $\phi_{3d} = \delta f_3$ and the balanced exact sequence C_2 guarantees the existence of a map $\overline{\phi} : P_{3d} \longrightarrow B$ with $f_3 = \beta_2\overline{\phi}$. Thus $\phi_{3d} = \beta\overline{\phi}$.

For (P4), assume that $C_2 \in \mathcal{B}$; we will show that $C_1 \in \mathcal{E}$. Given a completely decomposable module P_4, $\phi_4 : P_4 \longrightarrow X$, balanceness of C_2 guarantees the existence of $f_4 : P_4 \longrightarrow B$ with $\beta_2 f_4 = \gamma\phi_4$. Note that $\beta f_4 = \delta\beta_2 f_4 = \delta\gamma\phi_4 = 0$, hence $f_4 = \alpha\phi_1$, for some $\phi_1 : P_4 \longrightarrow A$. Now $\gamma\phi_4 = \beta_2 f_4 = \beta_2\alpha\phi_1 = \gamma\beta_1\phi_1$, and γ being monic induces the following equality: $\beta_1\phi_1 = \phi_4$.

Finally, for (P4d), assume that $E_2 \in \mathcal{B}$; we will show $E_3 \in \mathcal{B}$. If P_{4d} is a completely decomposable module and a morphism $\phi_{4d} : P_{4d} \longrightarrow Z$, then there exists a $\phi_2 : P_{4d} \longrightarrow B$ with $\beta\phi_2 = \phi_{4d} = \delta(\beta_2\phi_2)$. ◇

We can show (using the Noether isomorphism theorems) that the above properties are what characterizes the relative (proper) exact sequences. One of the consequences is the 3 by 3 lemma for this class of sequences; we give an independent proof:

The balanced 3×3 Lemma. *If a commutative 3×3 diagram has middle column and row balanced exact, then, if three of the remaining four rows and columns are balanced exact, then so is the fourth.*

Proof. We omit the zero maps at both ends of all the rows and columns in the diagram below. We will prove that, given all the three columns proper, 1) If $E_1, E_2 \in \mathcal{B}$, then

$E_3 \in \mathcal{B}$; 2) If $E_3, E_2 \in \mathcal{B}$, then $E_1 \in \mathcal{B}$. First note that by the ordinary 3×3 lemma, all the rows (and columns) are short exact sequences.

$$\begin{array}{lccccc} E_1: & A_1 & \xrightarrow{\alpha_1} & B_1 & \xrightarrow{\beta_1} & C_1 \\ & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle h}\downarrow \\ E_2: & A_2 & \xrightarrow{\alpha_2} & B_2 & \xrightarrow{\beta_2} & C_2 \\ & {\scriptstyle p}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle r}\downarrow \\ E_3: & A_3 & \xrightarrow{\alpha_3} & B_3 & \xrightarrow{\beta_3} & C_3 \end{array}$$

1) Given $\phi_3 : I \longrightarrow C_3$, then $\exists \phi_2 : I \longrightarrow C_2$with $r\phi_2 = \phi_3$, since the third column is balanced. E_2 is balanced, thus $\exists \phi_1 : I \longrightarrow B_2$ with $\beta_2\phi_1 = \phi_2$. Finally $\exists \phi : I \longrightarrow B_3$, because the second column is balanced. Balanceness of E_3 follows from the following equalities: $\beta_3\phi = \beta_3 q\phi_1 = r\beta_2\phi_1 = r\phi_2 = \phi_3$. 2) Start with $f : I \longrightarrow C_1$. E_2 is balanced, so $\exists \phi_2 : I \longrightarrow B_2$ with $hf = \beta_2\phi_2$. $\beta_3 q\phi_2 = 0$, and since $\alpha_3 = \ker \beta_3$, $\exists \phi_3 : I \longrightarrow A_3$ with $q\phi_2 = \alpha_3\phi_3$. The first column is balanced, hence $\exists \psi_2$ with $p\psi_2 = \phi_3$. $\alpha_3 p\psi_2 = q\phi_2$ and $q\alpha_2\psi_2 = q\phi_2$ imply $q(\alpha_2\psi_2 - \phi_2) = 0$, and, since $g = \ker q$, $\exists \phi : I \longrightarrow B_1$ with $\alpha_2\psi_2 - \phi_2 = g\phi$. Now $f = \beta_1(-\phi)$, because $\beta_2(\alpha_2\psi_2 - \phi_2) = \beta_2 g\phi$ so $-\beta_2\phi_2 = -hf = h\beta_1\phi$; injectivity of h finishes the proof that E_1 is balanced. ◇

2. Bext functors

Having the properties established in 1., we can say that balanced exact sequences form a so-called proper class of exact sequences. A number of papers have been written on the general subject of relative (proper) exact sequences. A good treatment of the related topics is in [Mac Lane, 1963]. We give a sketch here, for we need this theory in order to define balanced dimension of modules.

The direct sum of two balanced-exact sequences $E: 0 \longrightarrow A_1 \xrightarrow{\alpha_1} B_1 \xrightarrow{\beta_1} C_1 \longrightarrow 0$ and $E: 0 \longrightarrow A_2 \xrightarrow{\alpha_2} B_2 \xrightarrow{\beta_2} C_2 \longrightarrow 0$ is the balanced-exact sequence $E: 0 \longrightarrow A_1 \oplus A_2 \xrightarrow{\alpha_1\oplus\alpha_2} B_1 \oplus B_2 \xrightarrow{\beta_1\oplus\beta_2} C_1 \oplus C_2 \longrightarrow 0$.

Two short exact sequences $E[\alpha, \beta]$ and $E[\alpha', \beta']$ with same ends A and C are said to be *(Yoneda) congruent*, if there is a morphism f with $f\alpha = \alpha'$ and $\beta' f = \beta$; by the short five lemma, every such f is an equivalence.

We can now consider balanced extensions: Given R-modules C, A, denote by $\mathrm{Ext}^1_{\mathcal{B}}(C, A)$ the set (we may postulate it to be a set) of all congruence classes of balanced short exact sequences $E: 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ in $\mathcal{B}$. $\mathrm{Ext}^1_{\mathcal{B}}(C, A)$ is more than a set:

Theorem 4. *$\mathrm{Ext}^1_{\mathcal{B}}(C, A)$ is a bifunctor on the category of R-modules, contravariant in C and covariant in A. The addition, defined by the so-called Baer sum: $E_1 + E_2 = \nabla_A(E_1 \oplus E_2)\Delta_C$, makes it a bifunctor to abelian groups.*

Proof. Covariance in A follows from Lemma 2(a), and contravariance in C is a consequence of the dual (b). Thus, given γ and E, we can define $E' = \gamma E$ (or $E' = E\gamma$); these give left (right) operations on E and the (co)universality of E' implies covariance and contravariance of the relative ext functor. Proof that $\mathrm{Ext}^1_{\mathfrak{B}}(C, A)$ is an abelian group uses similar techniques. The class of the split extensions $0 \longrightarrow A \longrightarrow A \oplus C \longrightarrow C \longrightarrow 0$ is the zero element and the inverse of the class E is $(-1_A)E$. ◇

$\mathrm{Ext}^1_{\mathfrak{B}}(C, A)$ will be called the *group of balanced extensions*, and the corresponding functor the *Bext functor*. Note that [Butler and Horrocks, 1961] treat $\mathrm{Ext}^1_{\mathfrak{B}}(-,-)$ as a subfunctor of $\mathrm{Ext}^1(-,-)$, which provides for an equivalent treatment of balanced exact sequences. We now immediately have the following:

Proposition 5. *The following are equivalent for an R-module X:*

(1) X has the projective property with respect to balanced exact sequences.

(2) For every balanced epimorphism, $\beta : B \longrightarrow X$, *$\beta$ splits, i.e.* $\exists \gamma : X \longrightarrow B$ *with* $\gamma\beta = id_X$.

(3) For every R-module A, $\mathrm{Ext}^1_{\mathfrak{B}}(X, A) = 0$. ◇

We also outline relevant material related to the extension group of long balanced sequences. A long exact sequence (for every n, $\mathrm{Im}\, f_{n+1} = \mathrm{Ker}\, f_n$) $\ldots \longrightarrow X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \longrightarrow \ldots$ is *balanced exact* if every f_n may be represented by a composition $\alpha_n\beta_n$, where α_n is a balanced monomorphism and β_n is a balanced epimorphism (equivalently, if every $\mathrm{Im}\, f_{n+1}$ is balanced in X_n). In particular, a balanced exact sequence

$$S:\ 0 \longrightarrow A \xrightarrow{\alpha} X_n \xrightarrow{\alpha_n} \ldots \xrightarrow{\alpha_2} X_1 \xrightarrow{\gamma} C \longrightarrow 0$$

is called an *n-fold balanced exact extension of A by C*. We will refer to it also as an *n-fold balanced extension starting at A and ending at C*.

Given an n-fold balanced exact sequence, $S_1 :\ 0 \longrightarrow A \longrightarrow X_n \longrightarrow \ldots \longrightarrow X_1 \xrightarrow{\beta} C \longrightarrow 0$ starting at A and ending at C, and an m-fold balanced exact sequence, $S_2 :\ 0 \longrightarrow C \xrightarrow{\gamma} Y_m \longrightarrow \ldots \longrightarrow Y_1 \longrightarrow D \longrightarrow 0$ starting at C and ending at D, then the *Yoneda composition* is the following $(n+m)$-fold balanced exact sequence starting at A and ending at D: $S_1 \circ S_2 :\ 0 \longrightarrow A \longrightarrow X_n \longrightarrow \ldots \longrightarrow X_1 \xrightarrow{\gamma\beta} Y_m \longrightarrow \ldots \longrightarrow Y_1 \longrightarrow D \longrightarrow 0$; composition is in fact the "splicing" of the two sequences. This operation is clearly associative and non-commutative. Every n-fold balanced extension may be written as the Yoneda composition of n short balanced exact sequences: $S = E_n \circ \ldots \circ E_1$. If S is as above, take $E_n :\ 0 \longrightarrow A \xrightarrow{\alpha} X_n \longrightarrow \mathrm{Im}\, \alpha = \mathrm{Ker}\, \alpha_n \longrightarrow 0$, and so on.

We now define composition with matching morphisms: If $\beta : A \longrightarrow B$ (or $\beta : B \longrightarrow C$) is a matching morphism, then $\beta S = \beta(E_n \circ \ldots \circ E_1) = (\beta E_n) \circ E_{n-1} \ldots \circ E_1$ (or $S\beta = (E_n \circ \ldots \circ E_1)\beta = E_n \circ \ldots \circ (E_1\beta)$. Note that in general $(E\beta) \circ E_1 \neq E \circ (\beta E_1))$, and this could be corrected by defining o congruence relation on (long) balanced exact sequences in a way that it is the weakest reflexive, symmetric and transitive relation satisfying $(S\beta) \circ S_1 \equiv S \circ (\beta S_1)$, as well as including the Yoneda congruence relation for short balanced exact sequences. Assume that two n-fold balanced exact sequences S, S', both starting at A and ending at C, have the following (unique) factorizations into short balanced exact sequences: $S = E_n \circ \ldots \circ E_1$ and $S' = E'_n \circ \ldots \circ E'_1$. Then S and S' are said to be *congruent n-fold extensions*, if, for every i, either E_i is congruent to E'_i, or two successive factors in S are replaced by the corresponding factor in S' using the rules: $(E\gamma) \circ F \equiv E \circ (\gamma F)$ and $E \circ (\gamma F) \equiv (E\gamma) \circ F$ (here E, F are short balanced exact sequences and γ is any matching morphism).

Denote by $\mathrm{Ext}^n_{\mathcal{B}}(C, A)$ the set (and we have made it an axiom earlier that this is a set) of all the congruence classes of n-fold balanced exact extensions of A by C. By convention, we set $\mathrm{Ext}^0_{\mathcal{B}}(C, A) = \mathrm{Hom}_{\mathcal{E}}(C, A)$. In fact, $\mathrm{Ext}^n_{\mathcal{B}}(-,-) : R\mathbf{Mod} \times \mathbf{RMod} \longrightarrow \mathbf{Sets}$ is a bifunctor, contravariant in the first argument and covariant in the second.

The *direct sum of n-fold extensions* is defined in a natural way, to respect the congruence relation and in a way to enable defining the abelian group structure on the Bext set:

Theorem 6. $\mathrm{Ext}^n_{\mathcal{B}}(C, A)$ *is an abelian group with respect to the following Baer sum addition:* $\sigma_1 + \sigma_2 = \nabla_A(\sigma_1 \oplus \sigma_2)\Delta_C$. *The inverse (opposite) of* S *is represented by* $(-1_A)S$, *and the zero extension is the congruence class of* $0 \longrightarrow A \xrightarrow{1} A \longrightarrow 0 \longrightarrow \ldots \longrightarrow 0 \longrightarrow C \xrightarrow{1} C \longrightarrow 0.$ ◇

Congruent long balanced exact sequences are also congruent as $R\mathbf{Mod}$ sequences, therefore there is the induced natural transformation of bifunctors $\mathrm{Ext}^n_{\mathcal{B}}(C, A) \longrightarrow \mathrm{Ext}^n_{R\mathbf{Mod}}(C, A)$. This transformation is a monomorphism, for $n = 1$, but not always so, for $n > 1$.

The following long exact sequences are extremely useful in computational matters and otherwise:

Theorem 7. *If* $\mathcal{B}$ *denotes the proper class of balanced exact sequences of torsion-free modules, and* $E : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0 \in \mathcal{B}$, *then, for every object* M *in* $R\mathbf{Mod}$, *we have the following exact sequences:*

$$(1) \qquad 0 \longrightarrow \mathrm{Hom}_{\mathcal{B}}(M, A) \xrightarrow{\alpha_0} \mathrm{Hom}_{\mathcal{B}}(M, B) \xrightarrow{\beta_0} \mathrm{Hom}_{\mathcal{B}}(M, C) \xrightarrow{E_0} \ldots \longrightarrow$$
$$\mathrm{Ext}^n_{\mathcal{B}}(M, C) \xrightarrow{E_n} \mathrm{Ext}^{n+1}_{\mathcal{B}}(M, A) \xrightarrow{\alpha_{n+1}} \mathrm{Ext}^{n+1}_{\mathcal{B}}(M, B) \xrightarrow{\beta_{n+1}} \mathrm{Ext}^{n+1}_{\mathcal{B}}(M, C) \xrightarrow{E_{n+1}} \ldots$$

$$(2) \qquad 0 \longrightarrow \mathrm{Hom}_{\mathcal{B}}(C, M) \xrightarrow{\alpha^0} \mathrm{Hom}_{\mathcal{B}}(B, M) \xrightarrow{\beta^0} \mathrm{Hom}_{\mathcal{B}}(A, M) \xrightarrow{E^0} \ldots \longrightarrow$$
$$\mathrm{Ext}^n_{\mathcal{B}}(A, M) \xrightarrow{E^n} \mathrm{Ext}^{n+1}_{\mathcal{B}}(C, M) \xrightarrow{\alpha^{n+1}} \mathrm{Ext}^{n+1}_{\mathcal{B}}(B, M) \xrightarrow{\beta^{n+1}} \mathrm{Ext}^{n+1}_{\mathcal{B}}(A, M) \xrightarrow{E^{n+1}} \ldots$$

The participating morphisms are defined as follows: $\alpha_n(clsS) = cls(\alpha S)$, $\beta_n(clsS) = cls(\beta S)$, $\alpha^n(clsS) = cls(S\alpha), \beta_n(clsS) = cls(S\beta)$ *and the connecting morphisms are acting as follows:* $E_n(clsS) = cls(E \circ S)$ *and* $E^n(clsS) = (-1)^n cls(S \circ E)$ $\diamondsuit$

We have constructed the iterated bext functor $\mathrm{Ext}^n_{\mathcal{B}}(-,-)$ without reference to balanced projectives. By Proposition 5.2 [Dimitrić, 1993] there are enough balanced projectives. Namely, given any R-module M, set $P = \oplus_{a\in M}\langle a\rangle_*$ and let $q(\oplus b) = \sum b$, for $\oplus b \in P$. This gives rise to the *canonical projective cover* $0 \longrightarrow N \xrightarrow{*} P \xrightarrow{q} M \longrightarrow 0$. As is often the case, the (iterated) ext functors are constructed via projective resolutions. We outline now this alternative construction.

Given an object C and the class $\mathcal{B}$ of (short) balanced sequences, a $\mathcal{B}$-*complex over* C, denoted by $\epsilon : X \longrightarrow C$ is a chain complex of the form $\ldots \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_1 \longrightarrow X_0 \xrightarrow{\epsilon} C \longrightarrow 0$, $n \in \mathbf{N}$. This complex is called a *balanced resolution of* C, if the sequence is exact and all the maps are balanced (compositions of balanced monomorphism and balanced epimorphism). It is a *balanced projective resolution of* C, if in addition, every X_n is a balanced projective object (i.e. completely decomposable).

It can be shown [Mac Lane, 1963] that the cohomologies do not depend on the resolutions used, namely if X, X' are two balanced projective resolutions of C, and A is any object, then $H^n(X,A) \cong H^n(X',A)$ depends only on C and A.

It is easy to establish that $\mathrm{Ext}^0_{\mathcal{B}}(C,A) = \mathrm{Hom}_{\mathcal{E}}(C,A) \cong H^0(X,A)$, where $X_1 \longrightarrow X_0 \longrightarrow C \longrightarrow$ is a balanced projective resolution of C. A more general result is also valid. Note first that any balanced n-fold extension of A by C may be regarded as a resolution of C, since it may be extended to the left of A by zeros.

Theorem 8. *If* C, A *are* R*-modules and* $\epsilon : X \longrightarrow C$ *is a balanced projective resolution of* C*, then there is an isomorphism* $\xi_n : \mathrm{Ext}^n_{\mathcal{B}}(C,A) \cong H^n(X,A)$*, natural in* A *for every* $n = 0,1,\ldots$ $\diamondsuit$

3. Balanced projective dimension

The *balanced homological dimension* of an R-module A is denoted and defined as $\mathrm{hd}_{\mathcal{B}}A = \sup\{n : \mathrm{Ext}^n_{\mathcal{B}}(A,-) \not\equiv 0\}$. Thus, $\mathrm{hd}_{\mathcal{B}}A = n$ iff $\forall B$ and all $k \geq 1$, $\mathrm{Ext}^{n+k}_{\mathcal{B}}(A,B) = 0$ and there exists a module C with $\mathrm{Ext}^n_{\mathcal{B}}(A,C) \neq 0$. Note that we allow the homological dimension to be infinite, but at this stage we do not distinguish among different infinite ordinals. To accommodate the above definition, we set $\mathrm{hd}0 = -\infty$ and add the axiom $-\infty + \infty = -\infty$ to the usual set of conventions in dealing with addition of the ∞ symbols.

We also define the *balanced global dimension of the category* $R\mathbf{Mod}$ to be

$$\mathrm{gl.dim}_{\mathcal{B}}R = \sup\{n : \mathrm{Ext}^n_{\mathcal{B}}(-,-) \not\equiv 0\} = \sup\{\ \mathrm{hd}_{\mathcal{B}}A : A \in R\mathbf{Mod}\}.$$

Lemma 9. *Let* $E :\ 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *be a balanced exact sequence. Then*

(1) If $\mathrm{hd}_{\mathcal{B}}A <$ $\mathrm{hd}_{\mathcal{B}}B$*, then* $\mathrm{hd}_{\mathcal{B}}C =$ $\mathrm{hd}_{\mathcal{B}}B$

(2) If $\mathrm{hd}_{\mathcal{B}}A =$ $\mathrm{hd}_{\mathcal{B}}B$*, then* $\mathrm{hd}_{\mathcal{B}}C \leq 1 +$ $\mathrm{hd}_{\mathcal{B}}B$.

(3) If $\mathrm{hd}_{\mathcal{B}}A >$ $\mathrm{hd}_{\mathcal{B}}B$*, then* $\mathrm{hd}_{\mathcal{B}}C = 1 +$ $\mathrm{hd}_{\mathcal{B}}A$.

Proof. This is a consequence of Theorem 7 $\diamondsuit$

Corollary 10. *Assume that* $0 \longrightarrow A \longrightarrow X_n \longrightarrow \ldots \longrightarrow X_1 \longrightarrow C \longrightarrow 0$ *is a balanced* n*-fold extension of* A *by* C*. If, for all* $i = 1, \ldots, n$, $\mathrm{hd}_{\mathcal{B}}X_i \leq$ $\mathrm{hd}_{\mathcal{B}}A$*, then* $\mathrm{hd}_{\mathcal{B}}C \leq n +$ $\mathrm{hd}_{\mathcal{B}}A$*; if, moreover,* $\mathrm{hd}_{\mathcal{B}}X_i <$ $\mathrm{hd}_{\mathcal{B}}A$*, for all* i*, then* $\mathrm{hd}_{\mathcal{B}}C = n +$ $\mathrm{hd}_{\mathcal{B}}A$*. If, in addition,* $A = C$*, then* $\mathrm{hd}_{\mathcal{B}}A = \infty$. $\diamondsuit$

Lemma 11. *For an* R*-module* B*, if in the long balanced sequences* $S: \ 0 \longrightarrow C_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_0 \longrightarrow C \longrightarrow 0$*, all* X_i, $i = 0, \ldots, n-1$ *are balanced projective, then* $\mathrm{Ext}^{n+1}_{\mathcal{B}}(C, B) \cong \mathrm{Ext}^{1}_{\mathcal{B}}(C_n, B)$

Proof. Decompose S into short balanced sequences $E_i : \ 0 \longrightarrow C_{i+1} \longrightarrow X_i \longrightarrow C_i \longrightarrow 0$: $S = E_{n-1} \circ \ldots \circ E_0$. Each of them induces the following long exact sequences, by Theorem 7: $\mathrm{Ext}^{k}_{\mathcal{B}}(X_i, B) \longrightarrow \mathrm{Ext}^{k}_{\mathcal{B}}(C_{i+1}, B) \stackrel{E_i^k}{\longrightarrow} \mathrm{Ext}^{k+1}_{\mathcal{B}}(C_{i+1}, B) \longrightarrow \mathrm{Ext}^{k+1}_{\mathcal{B}}(X_i, B)$. Since all X_i are balanced projective, the end terms in this sequence are zero by Proposition 5, thus E_i^k are isomorphisms for all i and $k \geq 1$. We can now form the iterated connecting morphism $S^n = E_0^n \circ \ldots \circ E_{n-1}^n$, which will be the desired isomorphism. $\diamondsuit$

Theorem 12. *The following statements are equivalent:*

(1) $\mathrm{hd}_{\mathcal{B}}C = n$.

(2) If in the long balanced sequences $S: \ 0 \longrightarrow C_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow X_0 \longrightarrow C \longrightarrow 0$*, all* X_i, $i = 0, \ldots, n-1$ *are balanced projective, then* C_n *is likewise balanced projective.*

(3) C *has a balanced projective resolution of length* n*:* $0 \longrightarrow X_n \longrightarrow X_{n-1} \longrightarrow \ldots \longrightarrow C \longrightarrow 0$.

Proof. Note first that for $n = 0$, this is essentially Proposition 5.

(1)⇒(2) Since $\mathrm{Ext}^{n+1}_{\mathcal{B}}(C, B) = 0$, for every B, Lemma 11. guarantees that

$$\mathrm{Ext}^{1}_{\mathcal{B}}(C_n, B) = 0,$$

for every B, thus proving that C_n is $\mathcal{B}$-projective, by Proposition 5.

(2)⇒(3) This follows from the assumption that there are enough balanced projectives, thus there is at least one projective resolution.

(3)⇒(1) Again by Lemma 11., $\mathrm{Ext}^{n+1}_{\mathcal{B}}(C, B) \cong \mathrm{Ext}^{1}_{\mathcal{B}}(C_n, B) = 0$. $\diamondsuit$

This theorem is the reason that balanced homological dimension is also called *balanced projective dimension* (denoted by bpd). We also define *balanced injective dimension* of an object A as follows: $\mathrm{id}_{\mathcal{B}}A = \sup\{n : \mathrm{Ext}^{n}_{\mathcal{B}}(-, A) \not\equiv 0\}$.

Note. If the quotient field Q of R is generated by $\aleph_n$ elements, it is well known that the projective dimension of Q is $n+1$; on the other hand $\operatorname{bpd} Q = 0$, since Q is balanced projective. $gen\, M$ and $rk\, M$ denote a generating set of minimal cardinality and a maximal independent set of M or their cardinalities respectively. Defining an (equivalence) relation $\sim$ on M by $a \sim b$ iff there are $r, s \in R$ with $ra = sb$, the equivalence classes are rank one pure submodules generated by a single element: $\langle a \rangle_*$. $sep\, M$ denotes the set of those equivalence classes or the cardinality of that set; clearly $rk\, M \leq sep\, M$. Given an R-module M, the *canonical projective cover* of M is a balanced exact sequence $0 \longrightarrow N \longrightarrow P \xrightarrow{q} M \longrightarrow 0$ with $P = \oplus_{a \in M} \langle a \rangle_*$ and $q(\oplus b) = \sum b$. We will call this the *separable projective cover*, if we only take $a \in sep\, M$. A pure ascending chain of R-modules M: $0 = M_0 \leq_* M_1 \leq_* \ldots \leq_* M_i \leq_* \ldots \leq_* M$, $i \leq \alpha$ is a *continuous filtration*, if for every limit ordinal $\alpha \leq \beta$, $M_\beta = \cup_{i<\beta} M_i$. This is a *slice filtration* if M_i is a direct summand of M_{i+1}, for every i. If $M = \{x_i\}_{i<\alpha}$, we get the *rank filtration* of M by setting $M_\beta = \langle \{x_i\}_{i<\beta} \rangle_*$

We need the following easy and useful

Lemma 13. *For an ordinal τ, assume that $0 \longrightarrow A_i \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$, $i \in \tau$ is a direct system of balanced exact sequences determined by continuous ascending chains of R-modules respectivelly A_i, B_i, C_i, $i \in \tau$, with every module pure in its subsequent link. Then the direct limit*

$$0 \longrightarrow \varinjlim A_i \longrightarrow \varinjlim B_i \longrightarrow \varinjlim C_i \longrightarrow 0$$

is also a balanced exact sequence.

Proof. The limit sequence is exact, because $\lim_\rightarrow$ is an exact functor. If $f : I \longrightarrow \cup C_i = \lim_\rightarrow C_i$ is a morphism from a rank one module, then $\exists i_0$ with $f(I) \subseteq C_{i_0}$, because of purity. Now the proof follows from the fact that the exact sequence with index i_0 is balanced (and properties of $\lim_\rightarrow$). $\diamondsuit$

We are interested in determining how balanced projective dimension is affected by ascending unions of modules. As a model to start at we modify the classical result of Auslander who addresses this question in the context of ordinary projective dimension of modules:

Theorem 14. *Assume that the following is a continuous ascending chain of pure submodules, where every link is balanced in the subsequent one (α a limit ordinal):*

$$0 = M_0 \leq_{\mathcal{B}} \ldots \leq_{\mathcal{B}} M_\beta \leq_{\mathcal{B}} M_{\beta+1} \leq_{\mathcal{B}} \ldots \leq_{\mathcal{B}} M = \cup_{\beta<\alpha} M_\beta \,.$$

If, for every β, $\operatorname{bpd} M_{\beta+1}/M_\beta \leq n$, then $\operatorname{bpd} M \leq n$.

Proof. If $n = \infty$, there is nothing to prove. We prove the claim by induction on finite n. If $n = 0$ then $M_{\beta+1}/M_\beta$ is balanced projective and the exact sequence $0 \longrightarrow M_\beta \longrightarrow M_{\beta+1} \longrightarrow M_{\beta+1}/M_\beta \longrightarrow 0$ splits, hence $M_{\beta+1} = M_\beta \oplus P_\beta$ $(*)$, where $P_\beta \cong M_{\beta+1}/M_\beta$ is balanced projective. Claim that $M_\beta = \oplus_{\gamma<\beta} P_\gamma$, for all $\beta \leq \alpha$

(thus, all M_β, including M are balanced projective). We do it inductively on β: If β is a successor ordinal this is obvious, in view of (*); if β is a limit ordinal, then the claim follows from the fact that the union (limit) of a slice filtration is again a slice filtration.

Assume now the inductive step that the claim is true for all numbers $< n$ and prove it for n; to that end consider the canonical (or separable) balanced projective covers $0 \longrightarrow N_\beta \longrightarrow P_\beta \xrightarrow{q_\beta} M_\beta \longrightarrow 0$, $\beta < \alpha$, $P_\beta = \oplus_{a \in M_\beta} \langle a \rangle_*$, $q_\beta(\oplus b) = \sum b$, hence $q_\beta = q_{\beta+1} \mid P_\beta$. Note that this is a direct system of balanced exact sequences, since $N_\beta \leq_{\mathcal{B}} P_\beta \leq_{\mathcal{B}} P_{\beta+1}$, hence $N_\beta \leq_{\mathcal{B}} N_{\beta+1}$, by Proposition 3.; use Lemma 13. to arrive at the canonical balanced exact sequence $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$, with $N = \cup_{\beta<\alpha} N_\beta$ with N_β balanced in $N_{\beta+1}$ and $P = \cup_{\beta<\alpha} P_\beta$ balanced projective (the same applies to every limit ordinal $< \alpha$). Thus far we have not taken advantage of the fact that the links M_β are balanced. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & N_\beta & \longrightarrow & P_\beta & \longrightarrow & M_\beta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & N_{\beta+1} & \longrightarrow & P_{\beta+1} & \longrightarrow & M_{\beta+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & N_{\beta+1}/N_\beta & \longrightarrow & P_{\beta+1}/P_\beta & \longrightarrow & M_{\beta+1}/M_\beta & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

with all the columns and the top two rows balanced, hence the bottom row must be balanced, by balanced 3×3 lemma. By the assumption, $\operatorname{bpd} M_{\beta+1}/M_\beta \leq n$, hence application of Corollary 10. to the bottom row yields $\operatorname{bpd} N_{\beta+1}/N_\beta \leq n-1$. By the inductive hypothesis we conclude $\operatorname{bpd} N \leq n-1$ and applying Lemma 9. to the balanced exact sequence $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$, we get $\operatorname{bpd} M \leq n$. ◇

An application of Lemma 9. yields the following:

Corollary 15. *If the conditions are same as in Theorem 14., except that the assumption on the links is:* $\operatorname{bpd} M_\beta \leq n$, *for every* β, *then the conclusion is that* $\operatorname{bpd} M \leq n+1$. ◇

We can strengthen this however, replacing the balanceness condition by mere purity:

Corollary 16. *Assume that the following is a continuous ascending chain of pure submodules:*

$$0 = M_0 \leq_* \ldots \leq_* M_\beta \leq_* M_{\beta+1} \leq_* \cdots \leq_* M = \cup_{\beta<\alpha} M_\beta .$$

If, for every β, $\operatorname{bpd} M_\beta \leq n$, *then* $\operatorname{bpd} M \leq n+1$.

Proof. As in the proof of Theorem 14., we arrive at the 3×3 commutative diagram, except we cannot assume that the third column is balanced exact. Balanced exactness of the first row, the assumption and Lemma 9. force $\operatorname{bpd} N_\beta \leq n-1$, $n \geq 1$. If $n = 0$, then the top row splits and all N_β are balanced projective and direct summands in the subsequent links hence their union N is balanced projective, thus $\operatorname{bpd} M \leq 1$, from the balanced exact sequence $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$. If we assume the statement true for dimension $< n$, then applying Corollary 15. yields $\operatorname{bpd} N \leq n$. This again implies $\operatorname{bpd} M \leq n+1$. ◇

A similar argument was attributed by [Fuchs and Hill, 1986] to [Simmons, 1983], in a context of cyclic purity. An immediate consequence is that pure continuous ascending union of completely decomposable modules has balanced projective dimension at most 1. The following is an analogue of Proposition 8., [Dimitrić and Fuchs, 1987]:

Proposition 17. *If M is a rank $\aleph_n$ pure submodule of a completely decomposable module P, then* $\operatorname{bpd} M \leq n$

Proof. Countable pure submodules of completely decomposable modules over valuation domains are again completely decomposable, thus the claim is verified for $n = 0$. Without loss of generality P is also of rank $\aleph_n$ and is the union of the slice filtration P_α, where each P_α is of rank at most $\aleph_{n-1}$. If $M_\alpha = M \cap P_\alpha$, then M_α also has rank at most $\aleph_{n-1}$, thus inductivelly, by Theorem 16 we get the desired result. ◇

Corollary 18. *Let $|R| \leq \aleph_n$, and let M be an $\aleph_n$-generated R-module. Then* $\operatorname{bpd} M \leq n+1$

Proof. Consider the rank filtration M_β of M, and the canonical balanced projective covers $0 \longrightarrow N_\beta \longrightarrow P_\beta \longrightarrow M_\beta \longrightarrow 0$, just as in the proof of Theorem 14. M_β will be also at most $\aleph_n$-generated, because they are pure submodules, thus the rank of $P_{\beta+1}/P_\beta$ is at most $\aleph_n$, since the cardinality of R is bound by the same cardinal. Applying Proposition 17. we arrive at $\operatorname{bpd} N_{\beta+1}/N_\beta \leq n$, hence $\operatorname{bpd} N \leq n$ by Theorem 14. The result follows from applying Lemma 9. to the balanced exact sequence $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$. ◇

Proposition 19. *For $n \geq 2$, the following are equivalent:*

(1) $\operatorname{gl.dim}_{\mathcal{B}} R \leq n$.

(2) Balanced submodules of balanced projective modules have balanced projective dimension at most $n-2$.

Proof. (1)⇒(2) If P is a balanced projective module and $0 \longrightarrow N \longrightarrow P \longrightarrow P/N \longrightarrow 0$ a balanced exact sequence, $\operatorname{bpd} P/N \leq n-1$, by (1) and $\operatorname{bpd} N \leq n-2$, by Lemma 9. (2)⇒(1) For any R-module M, let $0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$ be a canonical balanced projective cover. By the assumption, $\operatorname{bpd} N \leq n-2$, hence $\operatorname{bpd} M \leq n-1$ (for every module M). ◇

The results in this paper were also presented at The International Conference on Abelian Groups and Modules, Colorado Springs, August 7–12, 1995. In a sequel to this paper we will explore further properties of balanced projective dimension and in particular the balanced global dimension.

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