Acta Mathematica et Informatica Universitatis Ostraviensis

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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 7 (1999), No. 1, 39--46

Persistent URL: `http://dml.cz/dmlcz/120546`

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Abstract: In this paper we study weakly continuous inverse operators and application in nonlinear elasticity. The first part studies properties of weakly continuous operators. The main result is theorem with conditions for weakly continuity of inverse nonlinear operator. The second part contains results, which can be applied to nonlinear differential operator of hyperelasticity.

Key Words: nonlinear functional analysis, variational calculus, optimization, weakly continuous operators, nonlinear elasticity, hyperelasticity.

Mathematics Subject Classification: 47H30

1. Introduction

The paper deals with weak continuity of inverse nonlinear operator in reflexive Banach spaces. This property is very useful in optimal control. Continous linear operator in Banach spaces is weakly continous, but there are some problems with weak continuity of continuous nonlinear operator. In section 2, we introduce basic notions for spaces and continuity - especially SC-locally weak continuity in x_0 and we study basic properties weak and SC - locally weak continuity. We formulate conditions, which imply the main result - SC-locally weak continuity for nonlinear invers operator A^{-1} in reflexive Banach space. Section 3 contains an application in mechanics of continua in hyperelasticity.

2. Abstract results

2.1. Notation

Let X, Y be Banach spaces and X^*, Y^* theirs duals. Let $B^r_X(x_0) = \{x \in X, \|x - - x_0\|_X < r\}$ and $\overline{B^r_X(x_0)}$ be its closure in X. Let D be a locally convex topological vector space and D^* its dual with "weak - star" topology. Let $A : X \to Y$ be an operator acting from the space X to Y. We shall denote the strong convergence in X ($\|u_n - u\|_X \to 0$) by $u_n \to u$ in X, weak convergence in $X(\langle b, u_n - u\rangle \to 0, \forall b \in \in X^*)$ by $u_n \rightharpoonup u$ in X and weak* convergence in $D^*(\langle u^*_n - u^*, b\rangle \to 0, \forall b \in D)$ by $u^*_n \overset{*}{\rightharpoonup} u^*$ in D^* .

Definition. (various types of continuity of operators)

Operator $A : X \to Y$ is

– *continuous* iff

$$x_n \to x \text{ in } X \quad \Rightarrow \quad Ax_n \to Ax \text{ in } Y,$$

– *strongly continuous* iff

$$x_n \rightharpoonup x \text{ in } X \quad \Rightarrow \quad Ax_n \to Ax \text{ in } Y,$$

– *weakly continuous* iff

$$x_n \rightharpoonup x \text{ in } X \quad \Rightarrow \quad Ax_n \rightharpoonup Ax \text{ in } Y,$$

– *SC locally weakly continuous in* $x_o \in X$ iff

there exist $r, s > 0$ such that

$$A : \overline{B^r_X(x_o)} \subset X \to \overline{B^s_Y(Ax_o)} \subset Y$$

holds and

$$\forall \{x_n\}_{n=1}^{\infty} \subset \overline{B^r_X(x_o)} \quad (x_n \rightharpoonup x \text{ in } X \Rightarrow Ax_n \rightharpoonup Ax \text{ in } Y),$$

– *demicontinuous* iff

$$x_n \to x \text{ in } X \Rightarrow Ax_n \rightharpoonup Ax \text{ in } Y.$$

Futher operator $A : X \to D^*$ is

– (s, w^*) *continuous* iff

$$x_n \to x \text{ in } X \Rightarrow Ax_n \overset{*}{\rightharpoonup} Ax \text{ in } D^*,$$

– (w, w^*) *continuous* iff

$$x_n \rightharpoonup x \text{ in } X \Rightarrow Ax_n \overset{*}{\rightharpoonup} Ax \text{ in } D^*.$$

Remark. It would be more precise to speak about sequentially weakly continuos operators instead of weakly continues ones.

2.2. Weakly and SC-locally weakly continuous operators

In this section we shall formulate basic properties of weakly and SC-locally weakly continuous operators x_o.

Lemma (Properties of weakly continuous operators)

(i) The set of weakly continuous operators $A : X \to Y$ forms a linear space, i.e. if A_1, A_2 are weakly contiuous and $c \in R$ then $A_1 + A_2$ and cA_1 are also weakly continuous.

(ii) Linear continuous operators $A : X \to Y$ are weakly continuous.

Lemma (Properties SC-locally weakly continuous operators in point x_o)

The set of SC-locally weakly continuous operators in x_o $A : X \to Y$ forms a linear space, i.e. if A_1, A_2 are SC-locally weakly continuous in x_o and $c \in R$ then $A_1 + A_2$ and cA_1 are also SC-locally weakly continuous in x_o.

2.3. SC-locally weak continuity of inverse operators

We shall deal with SC -locally weak continuity of locally inverse operator to A in 0_Y.

$$A^{-1} : Dom[A^{-1}] \subset Y \to X,$$

i.e. with the following property:

Property LWCIO: (SC-locally weakly continuous inverse operator in 0_Y) There exist $r, s > 0$ such that

$$A^{-1} : \overline{B_Y^r(0)} \subset Y \to \overline{B_X^s(0)} \subset X$$

is a weakly continuous operator i.e.

$$\forall \{y_n\}_{n=1}^{\infty} \subset \overline{B_Y^r(0)} \quad (y_n \rightharpoonup y \text{ in } Y \Rightarrow A^{-1}y_n \rightharpoonup A^{-1}y \text{ in } X)$$

Assumptions:

G-condition — general condition for spaces and operators. Let X, Y be reflexive Banach spaces and L, C be Banach spaces, D locally convex topologic vector space and D^* dual to D with "weak - star" topology and $Y \hookrightarrow L$ be a continuous imbedding. Let us consider operators:

$$\begin{aligned} &A : Dom[A] \subseteq X \to Y, 0_x \in Dom[A], A(0_x) = 0_y \\ &P : X \to C \\ &\tilde{A} : C \to D^* \\ &F : L \to D^* \end{aligned}$$

C - condition — continuity of operators

i) operator $P : X \to C$ is strongly continuous i.e.

$$x_n \rightharpoonup x \text{ in } X \Rightarrow P(x_n) \to P(x) \text{ in } C$$

ii) operator $\tilde{A} : C \to D^*$ is (s, w^*) - continuous i. e.

$$u_n \to u \text{ in } C \Rightarrow \tilde{A}(u_n) \overset{*}{\rightharpoonup} \tilde{A}(u) \text{ in } D^*$$

iii) operator $F : L \to D^*$ is (w, w^*) continuous

$$y_n \rightharpoonup y \text{ in } L \Rightarrow F(y_n) \overset{*}{\rightharpoonup} F(y) \text{ in } D^*$$

EUS - condition — existence and uniqueness of solution to $Ax = y$

$$\exists r, s > 0 \quad \forall y \in \overline{B_Y^r(0)} \quad \exists! x \in \overline{B_X^s(0)} \quad (Ax = y)$$

EGE - condition — existence of generalized equation

$$Ax = y \text{ in } Y \Rightarrow \tilde{A}(Px) = F(y) \text{ in } D^*$$

RE - condition — regularity equation

$$\forall y \in \overline{B_Y^r(0)} \quad \forall x \in \overline{B_X^s(0)} \quad (\tilde{A}(Px) = F(y) \text{ in } D^* \Rightarrow (Ax = y) \text{ in } Y)$$

Theorem 1. (SC-locally weak continuity of $\mathbf{A}^{-1}$ na $\overline{B_Y^r(0)}$)

Let *G, C, EUS, EGE, RE — condition* be satisfied. Then the operator A^{-1} is SC-locally weakly continuous in 0_Y i.e.

there exist $r, s > 0$ such that

$$A^{-1} : \overline{B_Y^r(0)} \subset Y \to \overline{B_X^s(0)} \subset X$$

and

$$\forall \{y_n\}_{n=1}^{\infty} \subset \overline{B_Y^r(0)} \quad (y_n \rightharpoonup y \text{ in } Y \Rightarrow A^{-1}y_n \rightharpoonup A^{-1}y \text{ in } X).$$

Proof: Let $r, s > 0$ be constants from EUS — the condition which implies existence of a locally inverse operator A^{-1}

$$A^{-1} : \overline{B_Y^r(0)} \subset Y \to \overline{B_X^s(0)} \subset X.$$

Now we will show that A^{-1} is locally weakly continuous in 0_y. Let $\{y_n\}_{n=1}^{\infty} \subset \subset \overline{B_Y^r(0)}$ and $y_n \rightharpoonup y$ in Y. Hence Y is a reflexive Banach space and $\overline{B_Y^r(0)} \subset Y$ is a convex and closed set,which means also weakly closed set in Y. This implies that $y \in \overline{B_Y^r(0)} \subset Y$. If $x_n = A^{-1}y_n$, then $x_n \in \overline{B_X^s(0)} \subset X$, $(Ax_n = y_n)$. As X is a reflexive and $\overline{B_X^s(0)} \subset X$ is weakly compact, then there exist a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ such that $x_{n_j} \rightharpoonup x \in \overline{B_X^s(0)}$ in X. Now we show that $A^{-1}y = x, (Ax = = y)$ and all subsequences $\{x_{n_j}\}_{j=1}^{\infty}$ weakly converge to x in X. If $Ax_{n_j} = y_{n_j}$ for all $j \in$ N,then EGE - condition implies that $\tilde{A}(Px_{n_j}) = F(y_{n_j})$. If $x_{n_j} \rightharpoonup \rightharpoonup x \in \overline{B_X^s(0)}$ in X and P is strongly continuous (C - condition) implies that $P(x_{n_j}) \to P(x)$ in C. If $y_{n_j} \rightharpoonup y$ in Y and Y is continuous imbedding to L, it implies $y_{n_j} \rightharpoonup y$ in L. From C - condition implies that $F(y_{n_j}) \overset{*}{\rightharpoonup} F(y)$ in D^*. The equations $\tilde{A}((Px_{n_j}) = F(y_{n_j})$ and C - condition implies that $\tilde{A}(Px) = F(y)$ in D^*. RR- condition and $y \in \overline{B_Y^r(0)} \subset Y$ and $x \in \overline{B_X^s(0)}$ in X implies that $Ax = y$ in Y and $A^{-1}y = x$. By contradiction we prove that all subsequences $\{x_{n_i}\}_{i=1}^{\infty}$ converge to x. Let $x_{n_i} \rightharpoonup \tilde{x} \neq x$ in X, then we can prove analogously as before, that $A\tilde{x} = y$ and EUS condition implies that $\tilde{x} = x$, which is a contradiction. □

3. Application in hyperelasticity

In the last section we use the main result to the proof the SC- weak continuity of inverse operator of hyperelasticity. In the first part of the section we will formulate G, C, EUS, EGE, RE conditions in hyperelasticity and sketch proofs. At the end we formulate Theorem 2 about SC - locally weak continuity for inverse operator to the hyperelasticity operator.

3.1. Assumptions

Let $p > 3, \Omega \subset \mathbf{R}^3, \partial\Omega \in C^2$, $\mathbf{S}^3$ - be a space symmetri matrices of the type 3×3
G - condition - general condition for spaces and operators.

Let $X = \mathbf{V}^p = \mathbf{W}^{2,p}(\Omega) \bigcap \mathbf{W}_0^{1,p}(\Omega) = (W^{2,p}(\Omega))^3 \bigcap (W_0^{1,p}(\Omega))^3$, $Y = \mathbf{L}_p(\Omega) = (L_p(\Omega))^3$, $C = \mathbf{C}(\overline{\Omega}) = (C(\overline{\Omega}))^{3\times 3}$, $L = \mathbf{L}_1(\Omega) = (L_1(\Omega))^3$. All spaces have standard norms. $D = \mathcal{D} = (C_0^\infty(\Omega))^3$, $D^* = \mathcal{D}^* = ((C_0^\infty(\Omega))^3)^*$. The topology is a product of standard topology of $C_0^\infty(\Omega)$. [1]

Assumptions for hyperelasticity operator: Body is represented by a three dimensional bounded domain $\Omega \subset \mathbf{R}_3$, with the boundary $\partial\Omega \in C^2$. We assume that the material is homogeneous and isotropic and the reference configuration is *a natural state*. For modeling of the hyperelasticity operator we will use Green–Saint–Venant strain tensor $\mathbf{E}$ and second Piola–Kirchhoff stress tensor Σ. The relation between Green–Saint–Venant strain tensor $\mathbf{E}$ and displacement $\mathbf{u}$ is

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u}^\top + \nabla\mathbf{u} + \nabla\mathbf{u}^\top \cdot \nabla\mathbf{u}), \tag{1}$$

The response function

$$\Sigma = \check{\Sigma}(\mathbf{E}), \tag{2}$$

which is a relation between Green-Saint-Venant strain tensor $\mathbf{E}$ and second Piola-Kirchhoff stress tensor Σ, has the follouing properties:

i) The response function

$$\check{\Sigma} : \mathbf{W}(0) \subset \mathbf{S}^3 \to \mathbf{S}^3,$$

$\mathbf{W}(0)$ is an open neighbourhood of zero in $\mathbf{S}^3$.

ii) The relation between stress and strain tensors is

$$\Sigma = \check{\Sigma}(\mathbf{E}) = \lambda(tr\mathbf{E})I + 2\mu\mathbf{E} + O(||\mathbf{E}||^2_{\mathbf{S}^3}), \tag{3}$$

where $\lambda > 0$, $\mu > 0$ are *Lamé constants*.

iii) The body is *a natural state*, which means that $\mathbf{0} = \check{\Sigma}(\mathbf{E}(0))$.

iv) The response function is twice continuously differentiable, i.e. $\check{\Sigma} \in C^2$.

A is a hyperelasticity operator

$$A : Dom[A] \subset \mathbf{V}^p \to \mathbf{L}_p(\Omega)$$

$$A : \mathbf{u} \mapsto -\mathrm{div}\{(\mathbf{I} + \nabla\mathbf{u})\check{\Sigma}(\mathbf{E}(\mathbf{u}))\}$$

P is a projection operator

$$P : \mathbf{V}^p \to \mathbf{C}(\overline{\Omega})$$

$$P : \mathbf{u} \mapsto \nabla\mathbf{u}$$

$\tilde{A}$ - operator of a generalized equation

$$\tilde{A} : \mathbf{C}(\overline{\Omega}) \to \mathcal{D}^*$$

$$\tilde{A}(\mathbf{v}) : \mathcal{D} \to \mathbf{R}$$

$$\tilde{A}(\mathbf{v}) : \varphi \mapsto \int_{\Omega} (\mathbf{I} + \mathbf{v}) \check{\Sigma}(\mathbf{S}(\mathbf{v})) : \nabla \varphi d\mu$$

$$\mathbf{S} : \mathbf{C}(\overline{\Omega}) \to \mathbf{C}(\overline{\Omega}), \mathbf{S} : \mathbf{v} \mapsto \frac{1}{2}(\mathbf{v}^{\top} + \mathbf{v} + \mathbf{v}^{\top} \cdot \mathbf{v})$$

$$\mathbf{E}(\mathbf{u}) = \mathbf{S}(\nabla \mathbf{u})$$

$$F : L \to \mathcal{D}^*$$

$$F(\mathbf{y}) : \mathcal{D} \to \mathbf{R}$$

$$F(\mathbf{y}) : \varphi \mapsto \int_{\Omega} \mathbf{y} \cdot \varphi d\mu$$

C - condition — continuity of operators

i) Operator $P = \mathrm{id} \circ \nabla$, where $\nabla : \mathbf{V}^{\mathrm{p}} \to (W^{1,p}(\Omega))^{3\times 3}, \nabla : \mathbf{u} \to \nabla \mathbf{u}$ is linearly continuous and weakly continuous , id: $(W^{1,p}(\Omega))^{3\times 3} \to (C(\overline{\Omega}))^{3\times 3}$ is strongly continuous from Rellich - Kondrašov embedding theorem and that implies P is strongly continuous.

ii)

$$\begin{aligned}
\mathbf{u}_n \to \mathbf{u} \text{ in } (C(\overline{\Omega}))^{3\times 3} &\Rightarrow \mathbf{S}(\mathbf{u}_n) \to \mathbf{S}(\mathbf{u}) \text{ in } (C(\overline{\Omega}))^{3\times 3} \Rightarrow \\
&\Rightarrow \check{\Sigma}(\mathbf{S}(\mathbf{u}_n)) \to \check{\Sigma}(\mathbf{S}(\mathbf{u})) \text{ in } (C(\overline{\Omega}))^{3\times 3} \Rightarrow \\
&\Rightarrow (\mathbf{I} + \mathbf{u}_n)\check{\Sigma}(\mathbf{S}(\mathbf{u}_n)) \to (\mathbf{I} + \mathbf{u})\check{\Sigma}(\mathbf{S}(\mathbf{u})) \text{ in } (C(\overline{\Omega}))^{3\times 3} \Rightarrow \\
&\Rightarrow (\mathbf{I} + \mathbf{u}_n)\check{\Sigma}(\mathbf{S}(\mathbf{u}_n)) \rightrightarrows (\mathbf{I} + \mathbf{u})\check{\Sigma}(\mathbf{S}(\mathbf{u})) \text{ in } \overline{\Omega} \Rightarrow \\
&\Rightarrow \forall \varphi \in D (\int_{\Omega} (\mathbf{I} + \mathbf{u}_n)\check{\Sigma}(\mathbf{S}(\mathbf{u}_n)) : \nabla \varphi \mathbf{d}\mu \to \\
&\qquad \to \int_{\Omega} (\mathbf{I} + \mathbf{u})\check{\Sigma}(\mathbf{S}(\mathbf{u})) : \nabla \varphi \mathbf{d}\mu) \Rightarrow \\
&\Rightarrow \tilde{A}(\mathbf{u}_n) \stackrel{*}{\rightharpoonup} \tilde{A}(\mathbf{u}) \text{ in } D^*
\end{aligned}$$

iii) $\mathbf{y}_n \rightharpoonup \mathbf{y}$ in $L \Rightarrow \forall \varphi \in D \int_{\Omega} \mathbf{y}_n \cdot \varphi d\mu \to \int_{\Omega} \mathbf{y} \cdot \varphi d\mu \Rightarrow F(\mathbf{y}_n) \stackrel{*}{\rightharpoonup} F(\mathbf{y})$ in D^* H

EUS - condition — existence and uniqueness of solutions to $Ax = y$.

The proof of this condition follows from local inversion theorem, and it can be found it in [1]. Under smoothness assumptions on response function $\check{\Sigma}$, it can be shown that the nonlinear operator $A : Dom[A] \subset \mathbf{V}^{\mathrm{p}} \to \mathbf{L}_{\mathrm{p}}(\Omega)$ and A is continously differentiable . The key result there is that Sobolev spaces $W^{1,p}(\Omega)$ forms an algebra for $p > 3$ and continuously differentciable response function $\check{\Sigma}$ maps $\mathbf{W}^{1,p}(\Omega)$ to $\mathbf{W}^{1,p}(\Omega)$. The assumption that the reference configuration is in natural state implies that $A(0_x) = 0_y$. The equation $A'(0_x)u = f$ is precisely boundary value problem of linearized elasticity that implies that $A'(0_x)$ is an isomorphism between the spaces $\mathbf{V}^{\mathrm{p}}$ and $\mathbf{L}_{\mathrm{p}}(\Omega)$. All assumptions of local inversion theorem are fulfilled and this implies existence of continuous locally inverse operator.

EGE - condition — existence of generalized equation.

It is the application of Green's formulas.

RE - condition — regularity equation.

$$\mathbf{u} \in \mathbf{W}^{2,p}(\Omega) \Rightarrow \nabla\mathbf{u} \in \mathbf{W}^{1,p}(\Omega)$$
$$\nabla\mathbf{u} \in \mathbf{W}^{1,p}(\Omega) \text{ and } \check{\Sigma}(\mathbf{E}(\mathbf{u})) \in \mathbf{W}^{1,p}(\Omega) \Rightarrow (\mathbf{I} + \nabla\mathbf{u})\check{\Sigma}(\mathbf{E}(\mathbf{u})) \in \mathbf{W}^{1,p}(\Omega)$$

We can use Green's formulas and basic properties of distributions which imply RE - condition.

3.2. The main result.

Now we can formulate the main result in application.

Theorem 2. The inverse hyperelasticity operator is SC-locally weakly continuous in 0_y

Proof: It is the application of Theorem 1.

Remarks. This property is very useful in optimal control.

Acknowledgment. I would like to thank to MUDr. Lenka Karpetova for her motivation to finish this paper.

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Received: August 1, 1999