Acta Mathematica et Informatica Universitatis Ostraviensis

Kazimierz Szymiczek
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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 10 (2002), No. 1, 133--143

Persistent URL: http://dml.cz/dmlcz/120579

Ten problems on quadratic forms

Kazimierz Szymiczek

Abstract. Problem 1 asks for the structure of the Witt group of a general field, problems 2–6 ask questions about the relations between breaking up number fields into Witt equivalence classes and class numbers of the fields. Problem 7 is about counting Witt equivalence classes of number fields containing cyclotomic fields, and problem 8 about the number of wild primes in a Hilbert equivalence of number fields. Problems 9 and 10 are about the cokernel of the total residue homomorphism and Witt equivalence of Dedekind rings.

Introduction

In 1975 I attended the second Czech & Slovak conference on number theory in Kočovce and gave a talk *Ten problems from the algebraic theory of quadratic forms* (see [16, Chapter 7] for the complete list of problems). Five of these have been solved in the meantime, in the remaining some progress has been made but the complete solutions are still not known. The problems had very little in common with number theory and so I am not going to discuss them here. There is one annoying exception dealing with the group structure of the Witt ring of a general field which I will include in the new list of problems. All the other problems in the list below arose from papers presented at the Czech & Slovak conferences since the Kočovce conference in 1975.

By the way, it is a melancholy thought to recall that in Kočovce I was the only foreign participant. Then in the next conference in the series, also held in Kočovce in 1977, there were three foreign participants: A. Schinzel, V. G. Sprindžuk, and myself.

Received: October 17, 2001.

1991 *Mathematics Subject Classification:* 11E04, 11E12, 11E81.

Key words and phrases: Quadratic forms, Witt ring, Conner's level condition, Witt equivalence classes of number fields, tame Hilbert equivalence, Dedekind rings.

Talk at the 15th Czech & Slovak International Conference on Number Theory, Ostravice, September 3–8, 2001.

1. The group structure of the Witt ring

Let $W(F)$ be the Witt ring of quadratic forms over of a field F of characteristic not two and let X be the set of orderings of F. Then we have the total signature homomorphism

$$\sigma : W(F) \to \mathbb{Z}^X$$

(for unexplained notation and terminology see [9, p. 63] or [15, p. 128]). For any $f \in W(F)$ the image $\sigma(f) \in \mathbb{Z}^X$ is a *bounded* function from X to $\mathbb{Z}$. Indeed, if $\dim f = n$, then $|\sigma_P(f)| \leq 2n$ for all $P \in X$. Hence, by a theorem of Nöbeling, the image $\operatorname{im}\sigma$ is a free Abelian group. Since the kernel of σ is known to be the torsion subgroup $W_t(F)$ of the Witt group $W(F)$, we get the decomposition

$$W(F) = \mathbb{Z}^r \oplus W_t(F)$$

where r is the rank of the free Abelian group $\operatorname{im}\sigma$. It is well known that the torsion subgroup $W_t(F)$ is a 2-primary Abelian group. Nevertheless we do not know the answer to the following question:

Problem 1. *Is the Witt group $W(F)$ always a direct sum of cyclic groups, or equivalently, is the torsion subgroup $W_t(F)$ always a direct sum of cyclic groups ?*

When F is a nonreal field and has level s (the smallest number of terms in a representation of -1 as a sum of squares of elements of F), then $2sW(F) = 0$, i.e. the Witt group is a *bounded* Abelian group (see [8, Theorem 3.6, p. 312]). By a classical theorem of Prüfer, such a group is a direct sum of cyclic groups. A similar argument applies to formally real fields with a finite Pythagoras number. Then the torsion subgroup is bounded ([8, Theorem 3.6, p. 312]) and so a direct sum of cyclic groups.

For formally real fields one can show that when the torsion subgroup $W_t(F)$ is countable, then it is a direct sum of cyclic groups. This follows from another theorem of Prüfer saying that a countable Abelian group without elements of infinite height is a direct sum of cyclic groups (and from the fact that the Witt group has no elements of infinite height, which can be proved using the techniques of quadratic form theory).

There is also an example of a field F where the group $W_t(F)$ is neither countable nor bounded yet it is a direct sum of cyclic groups (see [6]). Here F is the rational function field in countably many indeterminates over the field of real numbers.

2. Conner's Problem

We say that two number fields K and L are *Witt equivalent* when they have isomorphic Witt rings of quadratic forms. It is known that Witt equivalent number fields have the same degree over $\mathbb{Q}$, and that for a given positive integer n the number fields of degree n split into a finite number of classes of Witt equivalent fields. For $n > 2$ each class of Witt equivalent number fields of degree n is an infinite set.

Quadratic number fields split into 7 classes of Witt equivalence. These are represented by the fields

$$\mathbb{Q}(\sqrt{d}), \quad d = -1, \pm 2, \pm 7, \pm 17.$$

The class of $\mathbb{Q}(\sqrt{-1})$ is a singleton set, but the remaining classes are all infinite.

It is amusing to notice that 6 out of the 7 above fields have class number 1. The field $\mathbb{Q}(\sqrt{-17})$ has class number 4 and, as it turns out, it cannot be replaced with a field in the same Witt class and class number 1. For the class of fields Witt equivalent to $\mathbb{Q}(\sqrt{-17})$ consists of all quadratic fields $\mathbb{Q}(\sqrt{d})$, where $d < -1$ is a square-free integer $\equiv 7 \pmod 8$. The discriminant of such a field equals $4d$ and so has at least two distinct prime factors. Hence, according to Gauss, the class number of the field is even (for a non-real quadratic field the 2-rank of the ideal class group is $t-1$, where t is the number of distinct prime factors of the discriminant). Actually one can show that in the class of fields Witt equivalent to $\mathbb{Q}(\sqrt{-17})$ all fields have class numbers divisible by 4.

Pierre E. Conner offered the following more general version of this phenomenon. We say that a number field K satisfies the *Conner's Level Condition* (CLC, for short) if

$$s(K) = 2 \quad \text{and} \quad s(K_{\mathfrak{p}_i}) = 1, \quad i = 1, \ldots, g,$$

where $s(F)$ denotes the *level* of the field F and $K_{\mathfrak{p}_1}, \ldots, K_{\mathfrak{p}_g}$ are all of the dyadic completions of the field K. For instance, the field $\mathbb{Q}(\sqrt{-17})$ satisfies CLC. Conner's observation was the following theorem.

If a number field K satisfies CLC, *then the class number of K is even.*

Proof. Two distinct proofs have been published in [5] (see corrigendum) and [18], and here we sketch a third proof using some class field theory. The idea has been suggested to the author of [20] by an anonymous referee.

Let K satisfy CLC and set $L = K(\sqrt{-1})$. Then L is an Abelian unramified extension of K. This follows from the following observations.

1. In L all infinite primes of K split completely.
The complex infinite primes split by definition and there are no real infinite primes since $s(K) = 2$.
2. In L all dyadic primes of K split completely.
This follows from the fact that all local dyadic levels are 1. Indeed, since for a dyadic prime $\mathfrak{p}$ there is a $\beta \in K_{\mathfrak{p}}$ with $\beta^2 = -1$, the congruence $\alpha^2 \equiv -1 \pmod{\mathfrak{p}^m}$, where m is arbitrarily large, is solvable in $\mathcal{O}_K$, and this implies that $\mathfrak{p}$ splits in L (by [4], Satz 119).
3. None of the finite primes of K ramifies in L .
The dyadic primes split and the nondyadic primes are unramified by [4], Satz 118.

Now let H be the maximal Abelian unramified extension of K (the *Hilbert class field* of K). Then L is a subfield of H, hence H has even degree. On the other hand the Galois group of H/K is isomorphic to the ideal class group of K hence the class group has even order. □

By [13], Witt equivalent number fields have the same global levels and their dyadic primes can be matched up so that the local levels agree at corresponding dyadic primes. Hence, if $\mathcal{K}$ is the class of all number fields Witt equivalent to K, and if K satisfies CLC, then all the fields in the class $\mathcal{K}$ satisfy CLC. It follows that if K satisfies CLC, then the class $\mathcal{K}$ is *even* in the sense that all fields in the class $\mathcal{K}$ have even class numbers.

In a letter to the author, dated September 8, 1989, Conner posed the following question:

Problem 2. *If the number field K does not satisfy* CLC *then is K Witt equivalent to some number field with odd class number?*

In other words, Conner asked whether any *even* class $\mathcal{K}$ of Witt equivalent number fields has to satisfy CLC. This question has been answered for Witt equivalence classes of fields of degree ≤ 4 (see [5]) and the answer is YES. Some further results are proved in [20].
We believe that the role of the prime $p = 2$ in the Conner's problem cannot be played by any other prime number. In other words we conjecture that the following statement can be proved.

Problem 3. *If a prime p divides the class numbers of all fields in a class of Witt equivalent number fields, then $p = 2$.*

There is a variant of Conner's problem involving $S-$class numbers. It turns out that for the set S of all infinite and all dyadic primes of a number field K, CLC implies that the $S-$class number of K is even (see [20, Prop. 2]). Hence, if the class $\mathcal{K}$ satisfies CLC, it is $S-$*even*, that is, all fields in $\mathcal{K}$ have even $S-$class numbers.

So it appears reasonable to ask whether any $S-$even class $\mathcal{K}$ of Witt equivalent number fields has to satisfy CLC. Even this modified question is still open. For some results towards the solution of the problem see [20].

3. Witt equivalence classes of number fields

A class of Witt equivalent number fields is said to be *even* (resp. *odd*), when every field in the class has even (resp. odd) ideal class number. A class that is neither even nor odd is said to be *mixed.*

We know that even classes do exist: Conner's level condition implies evenness of the class number. But we do not know whether odd classes of degree $n > 2$ do exist. The only exceptions are the singleton classes determined by the fields $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{-1})$. These are the only Witt classes of algebraic number fields consisting of finitely many fields. Hence our question is as follows.

Problem 4. *Do there exist infinite odd Witt equivalence classes of algebraic number fields ?*

I conjecture that odd classes do not exist in degrees $n > 2$. Here is what we know about the non-existence of odd classes.

Each class of Witt equivalent number fields of *even* degree contains a field with *even* class number (see [18]). In fact, each class of even degree contains a field with the 2-rank of the class group larger than any given number. Hence an odd class must have odd degree. For the degree 3 it is known that there are no odd classes: each of the 8 cubic classes contains a field with even class number. Actually each class contains a field with class number 2, and also a field with class number 4. Moreover, the field with class number 4 can be chosen to have prescribed class group (i.e. either cyclic or Klein 4-group). For details see [18].

There is also a more general version of the above results for fields of arbitrary degree n and p−ranks of S−class groups, where p is a prime factor of n. We quote a special case of the main result of [19].
Let $\mathcal{K}$ be an infinite class of Witt equivalent number fields of degree n and let p be a prime factor of n. Then for any positive integer r there is a field $F \in \mathcal{K}$ such that

$$\mathrm{rk}_p\, C(F) > r.$$

We consider here only the case when p is a prime factor of n since in the case when $p \nmid n$ it is not known whether there exist number fields of degree n with arbitrarily large p−ranks of class groups.

The best result known in this direction is due to S. Nakano [10]. He proved that for any pair of natural numbers $n > 1, m > 1$ there exist infinitely many number fields of degree n with the ideal class numbers divisible by m. One can conjecture that the fields of degree n with class numbers divisible by m occur in all classes of Witt equivalent number fields of degree n. So we state the following

Problem 5. *Prove that for every natural numbers $m > 1, n > 1$ each infinite class of Witt equivalent number fields of degree n contains fields with class numbers divisible by m.*

We conclude this section by discussing the occurrence of fields with class number one in Witt equivalence classes. Number fields of degree ≤ 4 split into 45 Witt equivalence classes. For fields of degrees $1, 2, 3, 4$, the numbers of classes are $1, 7, 8, 29$, respectively (see [17] for details). Of these there are exactly 3 classes satisfying CLC, one of degree 2 and two of degree 4. They are represented by the fields $\mathbb{Q}[X]/f$, where

$$f = X^2 + 17,\ X^4 + 18X^2 - 60X + 50,\ X^4 - 2X^3 - 9X^2 + 10X + 34.$$

These have class numbers $4, 2, 2$, respectively. Each of the remaining 42 classes of fields of degree ≤ 4 contains a field with class number one (see [5]). In general we cannot hope that each class not satisfying CLC contains a field with class number one. Actually we do not know whether there are infinitely many number fields of class number one. There are serious doubts about the existence of number fields of large degrees with class number one (see [11], p. 481). A more promising conjecture seems to be the following guess.

Problem 6. *There is a number N with the property that for each $n \geq N$ there is a Witt class of number fields of degree n which does not contain any field with class number one.*

As we have seen above, N, if it exists, must be at least 4.

It is also interesting to know how some particular classes of number fields are distributed in Witt equivalence classes. We consider the case of cyclotomic fields. The general theory in [13] implies that cyclotomic fields are classified up to Witt equivalence by three invariants: the degree, the level and the number of dyadic primes (see [7]). Let $w_{\text{cycl}}(2N)$ be the number of Witt equivalence classes of number fields of degree $2N$ containing some cyclotomic fields. It appears to be very difficult to find a formula for $w_{\text{cycl}}(2N)$ so we switch to the function

$$CW(x) = \sum_{2N \leq x} w_{\text{cycl}}(2N),$$

that is, the number of Witt equivalence classes generated by all cyclotomic fields of degrees $\leq x$. The following estimates have been found in [7]:

$$\frac{x}{\log x} \exp\left(\frac{1}{2}(\log\log\log x)^2\right) < CW(x) < \frac{4}{3}x$$

for all sufficiently large x. This prompts the

Problem 7. *Determine the order of magnitude of the function $CW(x)$.*

4. Tame equivalence

A *Hilbert-symbol equivalence* (HSE, for short) between two global fields K and L is a pair of maps

$$t : \dot{K}/\dot{K}^2 \to \dot{L}/\dot{L}^2, \quad T : \Omega_K \to \Omega_L,$$

where t is an isomorphism of the square class groups and T is a bijective map between the sets of all primes of K and L, preserving Hilbert symbols in the sense that

$$(a,b)_{\mathfrak{p}} = (ta, tb)_{T\mathfrak{p}} \quad \forall\ a, b \in \dot{K}/\dot{K}^2, \forall\ \mathfrak{p} \in \Omega_K.$$

Using standard results such as the Strong Hasse Principle for quadratic forms over global fields and the Harrison criterion for Witt equivalence of fields one shows that HSE implies Witt equivalence of global fields. Thus HSE is a set of local conditions for Witt equivalence of global fields. Actually, the two equivalences coincide, but the proof of the converse is not so straightforward (see [13] for a proof).

Let (t,T) be a HSE between number fields K and L. It is said to be *tame* at the finite prime $\mathfrak{p} \in \Omega_K$ if

$$\text{ord}_{\mathfrak{p}} a \equiv \text{ord}_{T\mathfrak{p}} ta \bmod 2$$

for all $a \in \dot{K}/\dot{K}^2$. Otherwise the equivalence is said to be *wild* at $\mathfrak{p}$. The set of all wild primes of an equivalence (t,T) is denoted $\text{Wild}(t,T)$.
If (t,T) is tame at every finite prime $\mathfrak{p} \in \Omega_K$, it is said to be a *tame* HSE. The tameness of a HSE amounts to the fact that, for every finite $\mathfrak{p}$, if $a \in \dot{K}$ is a unit at $\mathfrak{p}$, the square class $ta \in \dot{L}/\dot{L}^2$ contains an element $b \in \dot{L}$ which is a unit at $T\mathfrak{p}$.

It turns out that tame HSE is far more restrictive than the ordinary HSE. A. Czogała [2] classified quadratic number fields up to tame HSE. He proved that there are infinitely many tame equivalence classes in contrast to the 7 ordinary HSE classes. An interesting property of tame HSE is the preservation of 2-ranks of class groups. In [1] we have the following result.

If K and L are tamely Hilbert-symbol equivalent, then

$$\text{rk}_2\, C(K) = \text{rk}_2\, C(L) \quad \text{and} \quad \text{rk}_2\, C^+(K) = \text{rk}_2\, C^+(L).$$

Thus tamely equivaient number fields háve class groups of the samé 2-rank and also narrow class groups of the samé 2-rank. As an example consider the quadratic fields $K = \mathbb{Q}(\sqrt{2})$ and $L = \mathbb{Q}(\sqrt{3})$. They are known to be Hilbert-symbol equivaient but they are not tamely equivaient. Although each has class number 1, the narrow class number of K is 1 while the narrow class number of L is 2. Thus, the 2-ranks of the narrow class groups are not equal, so every Hilbert-symbol equivalence between K and L must háve at least one wild prime.

The main result of [1] gives a lower bound for the number of wild primes in a HSE in terms of the 2-ranks of class groups, A speciál čase reads as follows. Let (t, T) be a Hilbert-symbol equivalence between number fields K and L with finite wild set Wild(t,T). Further, let

$$d(K,L) = \max\{|rk_2 C(K) - rk_2 C(L)|, \ |rk_2 C^+(K) - rk_2 C^+(L)|\}.$$

Then

$$d(K,L) < \#\mathrm{Wild}(t,T).$$

Thus, in particular, when the equivalence is tame the set Wild(t,T) is empty, so that $d(K, L) = 0$ and so the 2-ranks coincide. The number $d(K, L)$ is thus a lower bound on the size of the wild set for any Hilbert-symbol equivalence between K and L. This raises the

Problém 8. *Does $d(K, L)$ equal the minimal number of wild primes that can occur in a Hilbert-symbol equivalence between K and L ?*

5. Dedekind rings

We begin with the Knebusch-Milnor sequence for Witt groups of a Dedekind domain O and its field of fractions K. This contains condensed information on quadratic forms over O and K and on the arithmetic of the ring O. We gather here known examples of computation of the cokernel of the total residue homomorphism and ask some natural questions about the cokernel of the total residue homomorphism for Dedekind rings.

Let K be the field of fractions of a Dedekind ring O. In 1978 Knebusch proved that the natural homomorphism of Witt rings

$$W(O) \to W(K)$$

is injective. Milnor extended the result in the following way.

We will refer to the maximal ideals $p \subset O$ as primes of K. Then, with p running through the primes of K we háve the following exact sequence for the additive Witt groups

$$(5.1) \qquad 0 \to W(O) \to W(K) \xrightarrow{\partial} \bigoplus_p W(O/p)$$

where d is the *total residue homomorphism.*

We recall now the definition of d. For each prime p of K we consider the *second residue class homomorphism*

$$d_p : W(K) \to W(O/p)$$

as defined in [9, pp. 85–86, 91]. This can be defined only after fixing a prime element π in K for the $\mathfrak{p}$-adic valuation on K. Then every element $\alpha \in W(K)$ can be written as

$$\alpha = \langle a_1, \ldots, a_k, b_1\pi, \ldots, b_m\pi \rangle,$$

where a_i, b_j are units in the $\mathfrak{p}$-adic valuation ring in K. Then we set

$$\partial_{\mathfrak{p}}(\alpha) := \langle \bar{b}_1, \ldots, \bar{b}_m \rangle \in W(\overline{K}),$$

where $\bar{b}$ is the canonical image of the $\mathfrak{p}$–adic unit b in the residue class field $\overline{K}$ of the $\mathfrak{p}$-adic valuation on K. Since $\overline{K}$ can be identified with $\mathcal{O}/\mathfrak{p}$ this construction yields for each maximal ideal $\mathfrak{p}$ of $\mathcal{O}$ the group homomorphism $\partial_{\mathfrak{p}} : W(K) \to W(\mathcal{O}/\mathfrak{p})$. For any fixed $\alpha \in W(K)$ we have $\partial_{\mathfrak{p}}(\alpha) = 0$ for almost all primes $\mathfrak{p}$. Hence we can aggregate these residue homomorphisms into one total residue homomorphism

$$\partial : W(K) \longrightarrow \coprod_{\mathfrak{p}} W(\mathcal{O}/\mathfrak{p}), \quad \partial(\alpha) = (\partial_{\mathfrak{p}}(\alpha)).$$

The sequence (5.1) is said to be the *Knebusch-Milnor exact sequence*. We direct the reader to [9, Ch. IV] for the background and proofs.

The sequence (5.1) can be extended to the right in some important special cases. When $\mathcal{O} = k[X]$ is the ring of polynomials in one indeterminate over a field k and $K = k(X)$ is the field of rational functions over k, the cokernel is known to be the zero group (Milnor's theorem, see [15, p. 211]).
When $\mathcal{O}$ is the ring of integers of a number field K and $C = C(K)$ is the ideal class group, Milnor proves that with a suitable choice of the homomorphism λ the sequence

$$0 \to W(\mathcal{O}) \xrightarrow{i} W(K) \xrightarrow{\partial} \coprod_{\mathfrak{p}} W(\mathcal{O}/\mathfrak{p}) \xrightarrow{\lambda} C/C^2 \to 0 \tag{5.2}$$

is exact. In other words, the cokernel of the total residue homomorphism ∂ is the finite elementary Abelian 2–group C/C^2. This result is stated in [9, pp. 93–94], see also [15, p. 227].

Milnor and Husemoller [9, p. 94] give another example of computation of the cokernel of ∂ in the case when $\mathcal{O}$ is the ring of polynomial functions on the circle:

$$\mathcal{O} = \mathbb{R}[X, Y]/(X^2 + Y^2 - 1). \tag{5.3}$$

They sketch a proof that

$$\operatorname{coker} \partial = \mathbb{Z}$$

so that this time the cokernel turns out to be the infinite cyclic group.

Observe that the field of fractions K of the ring $\mathcal{O}$ in (5.3) is isomorphic to the rational function field $\mathbb{R}(X)$. Roughly speaking, the circle is a rational curve, hence its function field is the rational function field. An elementary and direct argument is as follows. We have

$$K = \mathbb{R}(x, y) \quad \text{where} \quad x = X + I, \ y = Y + I, \ I = (X^2 + Y^2 - 1).$$

Here the generators satisfy the relation $x^2 + y^2 = 1$. Write $t := \frac{y}{1+x}$. Then $t \in K$, hence $\mathbb{R}(t) \subseteq K$. On the other hand

$$1 - x^2 = y^2 = t^2(1 + x)^2,$$

hence it follows that $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$ belong to $\mathbb{R}(t)$. This proves that $K = \mathbb{R}(t)$ is isomorphic to the rational function field $\mathbb{R}(X)$.

Thus both $\mathcal{O}$ and $\mathbb{R}[X]$ have $\mathbb{R}(X)$ as the field of fractions but the cokernels of the total residue homomorphisms defined on $\mathbb{R}(X)$ are $\mathbb{Z}$ and 0, respectively. Observe that the kernels of the total residue homomorphisms also differ for $\mathcal{O}$ and $\mathbb{R}[X]$. In the first case we can identify the kernel (which is the additive group of the ring $W\mathcal{O}$) as isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$. This follows from the computation of the Witt group of conics in [12, Theorem 6.1]. In the second case the kernel of the total residue homomorphism is the additive group $W\mathbb{R}[X] \cong W\mathbb{R} \cong \mathbb{Z}$ (see [15, p. 211]).

Yet another example of computation of the cokernel of the total residue homomorphism comes from Pfister [14] who studies the Milnor sequence in the case when $\mathcal{O} = k[X, \sqrt{aX^2+b}]$ is a quadratic extension of the polynomial ring $k[X]$ such that for $K = k(X, \sqrt{aX^2+b})$, the field of fractions, K/k is a function field of genus 0 but is not a rational function field. This assumption is equivalent to the condition that the 2-fold Pfister form $\langle 1, -a, -b, ab\rangle$ be anisotropic over k. Pfister [14, Theorem 5] proves that in that case

$$\operatorname{coker}\partial \cong \operatorname{ann}\langle 1, -a, -b, ab\rangle / \operatorname{ann}\langle 1, -a\rangle =: A$$

where the annihilators are taken in the Witt ring $W(k)$. Using this result one can construct a Dedekind domain $\mathcal{O}$ with the cokernel of $\partial_{\mathcal{O}}$ containing torsion elements of any given 2-power order. Indeed if we choose $a, b \in k$ so that not only $\langle 1, -a, -b, ab\rangle$ but also $\varphi := \langle 1, -a, b, -ab\rangle$ is anisotropic, then the coset

$$\psi = \langle 1, b\rangle + \operatorname{ann}\langle 1, -a\rangle \in A$$

is a nonzero element of the cokernel. Moreover

$$2^n\psi = 0 \in A \iff 2^n\varphi = 0 \in W(k).$$

Thus we need an example of a field k with the property that there is a 2-fold Pfister form φ over k having exactly the order 2^n in the Witt group $W(k)$. One possible choice is to consider

$$k = \mathbb{R}(t_1, t_2, \dots, t_{2^n}), \quad n \geq 2,$$

the rational function field in $2^n \geq 4$ variables over $\mathbb{R}$, and to take

$$a = t_2^2 + \cdots + t_{2^n}^2, \quad b = t_1.$$

Then the forms $\langle 1, -a, \pm b, \mp ab\rangle$ are anisotropic over k. This follows from the argument below showing that $2^{n-1}\langle 1, -a, \pm b, \mp ab\rangle \neq 0$ in $W(k)$.

First observe that $2^n\varphi = 0$ in $W(k)$, since $2^n\langle 1, -a\rangle = 0$. On the other hand, suppose that $2^{n-1}\varphi = 0$. Then $-b$ is represented by $2^{n-1}\langle 1, -a\rangle$ over k. Set $F = \mathbb{R}(t_2, \dots, t_{2^n})$ so that $k = F(t_1)$. By Cassels-Pfister theorem ([8, p. 256]), $-b = -t_1$ is represented by the form already over the polynomial ring $F[t_1]$. Hence there are polynomials $u_i, v_i \in F[t_1]$ satisfying

$$-t_1 = \sum_{i=1}^{2^{n-1}} u_i^2 - a \sum_{i=1}^{2^{n-1}} v_i^2.$$

Clearly, not all u_i, v_i can be divisible by t_1, hence substituting $t_1 = 0$ in the identity we get a representation of a as the quotient of two sums of 2^{n-1} squares over the field F. Since sums of 2^{n-1} squares in a field form a subgroup of the multiplicative group of the field, it follows that a itself is a sum of 2^{n-1} squares in F. Since $2^{n-1} < 2^n - 1$, this contradicts a theorem of Cassels (see [8, Cor. 2.4, p. 262]). Hence φ has exactly the order 2^n in the group $W(k)$.

It would be too demanding to ask for the description of the cokernels of the total residue homomorphisms $\partial_{\mathcal{O}}$ for all Dedekind rings $\mathcal{O}$. Instead we concentrate on possible cokernels:

Problem 9. *Characterize the class of Abelian groups which occur as cokernels of total residue homomorphisms $\partial_{\mathcal{O}}$ for all Dedekind rings $\mathcal{O}$.*

We end the list of open problems with a general question about Witt equivalence of Dedekind rings. Very little is known about the structure of Witt rings of Dedekind domains. In case of rings of S-integers in global fields some sufficient conditions for the isomorphism of Witt rings are proved in [3] and as a result examples are given of Witt equivalent Dedekind rings in number fields of non-equal degrees. An explicit computation of Witt groups of some Dedekind domains can be found in [12] where the Witt groups of rings of polynomial functions on conics are determined. In the lack of any substantial hints how the general theory could look like we state flatly

Problem 10. *Classify Dedekind rings up to isomorphism of their Witt rings.*

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INSTYTUT MATEMATYKI, UNIWERSYTET ŚLĄSKI, KATOWICE, POLAND
E-mail address: szymiczek@ux2.math.us.edu.pl