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Tamás Glavosits; Árpád Száz
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Decompositions of commuting relations

Tamás Glavosits and Árpád Száz

ABSTRACT. After some preparations, we show that if R and S are full relations on the sets A and B, respectively, then $R \circ S = S \circ R$ if and only if there exist full relations R_1 and S_1 on $A \cap B$, R_2 on $A \setminus B$ and S_2 on $B \setminus A$ such that $R = R_1 \cup R_2$, $S = S_1 \cup S_2$ and $R_1 \circ S_1 = S_1 \circ R_1$.

1. A few basic facts on relations

A subset R of a product set X^2 is called a relation on X. For any $x \in X$ and $A \subset X$, the sets $R(x) = \{y \in X : (x, y) \in R\}$ and $R[A] = \bigcup_{a \in A} R(a)$ are called the images of x and A under R, respectively. If R is a relation on X, then the images $R(x)$, where $x \in X$, uniquely determine R since we have $R = \bigcup_{x \in X} \{x\} \times R(x)$. Therefore, the inverse R^{-1} of R can be defined such that $R^{-1}(x) = \{y \in X : x \in R(y)\}$ for all $x \in X$. Moreover, if R and S are relations on X, then the composition $R \circ S$ of R and S can be defined such that $(R \circ S)(x) = R\big[S(x)\big]$ for all $x \in X$. The relations R and S are said to commute with each other if $R \circ S = S \circ R$. If R is a relation on X, then the sets $\mathcal{R}_R = R[X]$ and $\mathcal{D}_R = R^{-1}[X]$ are called the range and the domain of R, respectively. If in particular $X = \mathcal{D}_F$ and $X = \mathcal{R}_R$, then we say that R is a full relation on X. In the sequel, whenever confusions seem unlikely, we shall simply write A^c and $R(A)$ in place of $X \setminus A$ and $R[A]$, respectively. Note that the latter convention may only cause some serious troubles whenever $A \subset X$ such that $A \in X$.

2. Images under commuting relations

Lemma 2.1. *If R and S are relations on X such that $R \circ S \subset S \circ R$, then*

$$R\big(S(X)\big) \subset R(X) \cap S(X).$$

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Proof. We evidently have $R\big(S(X)\big) \subset R(X)$. Moreover, it is clear that

$$R\big(S(X)\big) = (R \circ S)(X) \subset (S \circ R)(X) = S\big(R(X)\big) \subset S(X).$$

□

Lemma 2.2. *If R and S are relations on X such that $S \circ R \subset R \circ S$, then*

$$R\big(S^{-1}(X)^c\big) \subset R(X) \cap S^{-1}(X)^c.$$

Proof. We evidently have $R\big(S^{-1}(X)^c\big) \subset R(X)$. Moreover, it is clear that

$$R^{-1} \circ S^{-1} = (S \circ R)^{-1} \subset (R \circ S)^{-1} = S^{-1} \circ R^{-1}.$$

Therefore, by Lemma 2.1, we also have

$$R^{-1}\big(S^{-1}(X)\big) \subset S^{-1}(X), \quad \text{and thus} \quad R^{-1}\big(S^{-1}(X)\big) \cap S^{-1}(X)^c = \emptyset.$$

Hence, it follows that

$$S^{-1}(X) \cap R\big(S^{-1}(X)^c\big) = \emptyset, \quad \text{and thus} \quad R\big(S^{-1}(X)^c\big) \subset S^{-1}(X)^c.$$

□

Lemma 2.3. *If R and S are full relations on A and B, respectively, such that $R \circ S = S \circ R$, then*

(1) $R(A \cap B) = A \cap B$, *(2)* $R(A \setminus B) = A \setminus B$;

(3) $S(A \cap B) = A \cap B$, *(4)* $S(B \setminus A) = B \setminus A$.

Proof. By letting $X = A \cup B$ and using Lemmas 2.1 and 2.2, we can see that

$$R(A \cap B) \subset R(B) = R\big(S(X)\big) \subset R(X) \cap S(X) = A \cap B$$

and

$$R(A \setminus B) \subset R(B^c) = R\big(S^{-1}(X)^c\big) \subset R(X) \cap S^{-1}(X)^c = A \cap B^c = A \setminus B.$$

Hence, since

$$R(A \cap B) \cup R(A \setminus B) = R\big((A \cap B) \cup (A \setminus B)\big) = R(A) = A,$$

it is clear that the assertions (1) and (2) are also true.

From the assertions (1) and (2), by changing the roles of R and S, we can at once see that the assertions (3) and (4) are also true. □

3. Decompositions of commuting relations

Theorem 3.1. *If R and S are full relations on A and B, respectively, such that $R \circ S = S \circ R$, then there exist full relations R_1 and S_1 on $A \cap B$, R_2 on $A \setminus B$ and S_2 on $B \setminus A$ such that*

$$R = R_1 \cup R_2, \qquad S = S_1 \cup S_2 \qquad \textit{and} \qquad R_1 \circ S_1 = S_1 \circ R_1.$$

Proof. Define $X = A \cup B$ and

$$R_1 = R \cap (A \cap B)^2, \qquad R_2 = R \cap (A \setminus B)^2 \ ;$$
$$S_1 = S \cap (A \cap B)^2, \qquad S_2 = S \cap (B \setminus A)^2.$$

Then, by the corresponding definitions and Lemma 2.3, it is clear that

$$R_1(x) = \big(R \cap (A \cap B)^2\big)(x) = R(x) \cap (A \cap B)^2(x) = R(x) \cap (A \cap B) = R(x)$$

for all $x \in A \cap B$. Moreover, it is clear $R_1(x) = \emptyset$ for all $x \in (A \cap B)^c$. And, quite similarly, we can also see that

$$R_2(x) = R(x) \text{ for all } x \in A \setminus B \text{ and } R_2(x) = \emptyset \text{ for all } x \in (A \setminus B)^c \ ;$$
$$S_1(x) = S(x) \text{ for all } x \in A \cap B \text{ and } S_1(x) = \emptyset \text{ for all } x \in (A \cap B)^c \ ;$$
$$S_2(x) = S(x) \text{ for all } x \in B \setminus A \text{ and } S_2(x) = \emptyset \text{ for all } x \in (B \setminus A)^c.$$

Hence, it is clear that R_1, R_2, S_1 and S_2 are full relations on $A \cap B$, $A \setminus B$, $A \cap B$ and $B \setminus A$, respectively. Moreover, it is clear that

$$R(x) = R_1(x) \cup R_2(x) = (R_1 \cup R_2)(x)$$

for all $x \in X$, and thus $R = R_1 \cup R_2$. And, quite similarly, $S = S_1 \cup S_2$. On the other hand, it is clear that

$$(R_1 \circ S_2)(x) = R_1\big(S_2(x)\big) \subset R_1(B \setminus A) = \emptyset$$

for all $x \in X$, and hence $R_1 \circ S_2 = \emptyset$. Moreover, quite similarly, we can also see that $R_2 \circ S_1 = R_2 \circ S_2 = \emptyset$ and $S_1 \circ R_2 = S_2 \circ R_1 = S_2 \circ R_2 = \emptyset$. Therefore,

$$R \circ S = (R_1 \cup R_2) \circ (S_1 \cup S_2) = = R_1 \circ S_1 \cup R_1 \circ S_2 \cup R_2 \circ S_1 \cup R_2 \circ S_2 = R_1 \circ S_1, \tag{1}$$

nd quite similarly $S \circ R = S_1 \circ R_1$. Therefore, $R_1 \circ S_1 = S_1 \circ R_1$ is also true. □

Theorem 3.2. *Let R and S be full relations on A and B, respectively. Moreover, suppose that R_1, R_2, S_1 and S_2 are relations on $A \cap B$, $A \setminus B$, $A \cap B$ and $B \setminus A$, respectively, such that the assertions of Theorem 3.1 hold. Then $R \circ S = S \circ R$. Moreover, R_1, R_2, S_1 and S_2 are as in the proof of Theorem 3.1.*

Proof. From the proof of Theorem 3.1, it is clear that $R \circ S = S \circ R$. Moreover, if $x \in A \cap B$, then $x \notin A \setminus B$. Therefore, by the inclusion $R_2 \subset (A \setminus B)^2$, we have

$R_2(x) \subset (A \setminus B)^2(x) = \emptyset$, and thus $R_2(x) = \emptyset$. Hence, by the equality $R = R_1 \cup R_2$ and Lemma 2.3, it is clear that

$$R_1(x) = R_1(x) \cup R_2(x) = R(x) = R(x) \cap (A \cap B) = \big(R \cap (A \cap B)^2\big)(x).$$

Therefore, the equality $R_1 = R \cap (A \cap B)^2$ is true. The equalities $R_2 = R \cap (A \setminus B)^2$, $S_1 = S \cap (A \cap B)^2$ and $S_2 = S \cap (B \setminus A)^2$ can be proved quite similarly. □

Remark 3.3. Note that the relations R_1, R_2, S_1 and S_2, defined in the proof of Theorem 3.1, inherit several useful properties of the relations R and S. For instance, if R and S are preorders (equivalences), then R_1, R_2, S_1 and S_2 are also preorders (equivalences). To construct commuting preorders, we can note that if R_1 and S_1 are preorders on $A \cap B$ such that $R_1 \subset S_1$, and moreover R_2 and S_2 are preorders on $A \setminus B$ and $B \setminus A$, respectively, then $R = R_1 \cup R_2$ and $S = S_1 \cup S_2$ are preorders on A and B, respectively, such that $R \circ S = S \circ R$. (Necessary and sufficient conditions for equivalences to be commuting can be found in [1].)

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Institute of Mathematics and Informatics, University of Debrecen, H-4010 Debrecen, Pf.12, Hungary
E-mail address: glavositdragon.klte.hu, szazmath.klte.hu