Tran Quoc Chien

Semistochastic decomposition scheme in mathematical programming and game theory

Kybernetika, Vol. 27 (1991), No. 6, 535--541

Persistent URL: http://dml.cz/dmlcz/124295

KYBERNETIKA — VOLUME 27 (1991), NUMBER 6

SEMISTOCHASTIC DECOMPOSITION SCHEME IN MATHEMATICAL PROGRAMMING AND GAME THEORY

TRAN QUOC CHIEN

This work deals with the optimization problem $\sup_{x \in X} \inf_{y \in Y} S(x, y)$, which plays, undoubtedly a crucial part in mathematical programming and game theory. On the basis of probability theory, a solving approach — semistochastic decomposition is proposed. This method is finite with probability 1 without any hypothesis about convexity or differentiability as it is usually required in traditional methods.

1. INTRODUCTION

Decomposition method was used first in linear programming by Dantzig and Wolfe [1]. The decomposition idea was then further developed by numerous authors (see Benders [2], Geoffrion [3], Van Roy [4], Kornai and Liptak [5] and references therein). In [6, 7] it was shown that decomposition methods can be unified in a general scheme of the following problem

$$s^* = \sup_{x \in X} \inf_{y \in Y} S(x, y), \tag{1}$$

where X and Y are nonempty sets and $S(x, y)$ is a real function on $X \times Y$.

Problem (1) is also the main problem in game theory [18]. However, existing iterative algorithms for solving problem (1) (see [7—12]) converge only under certain hypotheses about convexity and differentiability. As regards the solving scheme in (6), no convergence criterion has been there established except the trivial case when Y is finite.

On the other hand, using duality theory one can convert optimization problems of the form

$$f^* = \inf_{z \in Z} f(z) \tag{2}$$

where Z is a set and $f(z)$ is a real-valued function on Z, to the ones of form (1), where problems

$$\inf_{y \in Y} S(x, y)$$

are relaxed problems (see [13–16]). This fact enables us to apply duality theory for solving optimization problems of the form (2). Therefore, it is desirable to work up an efficient method for solving the most general problem (1).

2. SEMISTOCHASTIC DECOMPOSITION SCHEME

First let us define some necessary notions. Given a positive number ε, $z_\varepsilon \in Z$ is called an ε-optimal solution of problem (2) if

$$|f(z_\varepsilon) - f^*| \leqq \varepsilon\,, \tag{3}$$

$(x_\varepsilon, v_\varepsilon) \in X \times Y$ is called an ε-optimal solution of problem (1) if

$$|S(x_\varepsilon, v_\varepsilon) - \inf_{y \in Y} S(x_\varepsilon, y)| \leqq \varepsilon \tag{4}$$

(i.e. y_ε is an ε-optimal solution of problem $\inf_{y \in Y} S(x_\varepsilon, y)$)

and

$$|S(x_\varepsilon, y_\varepsilon) - s^*| \leqq \varepsilon\,. \tag{5}$$

Henceforth, instead of strictly optimal solutions we shall work with ε-optimal solutions, where ε is a given precision.

Further suppose that Y is a subset of $\mathbb{R}^r$ and v is an r-dimensional random vector the values of which can be generated by a generator.

The semistochastic decomposition scheme consists of the following steps.

Initiation: Choose a precision $\varepsilon > 0$, positive parameters ε_1, ε_2, ε_3 and a nonempty set $\bar{Y} \subset Y$.

Iteration: $k = 0, 1, 2, \ldots$

(i) Solve the master problem

$$s' = \sup_{x \in X} \inf_{y \in \bar{Y}} S(x, y)\,. \tag{6}$$

Let $(x^*, \bar{y})$ be an ε_1-optimal solution of problem (6).

(ii) Solve the subproblem

$$s'' = \inf_{y \in Y} S(x^*, y)\,. \tag{7}$$

Let y^* be an ε_2-optimal solution of (7). If

$$|S(x^*, \bar{y}) - S(x^*, y^*)| \leqq \varepsilon_3\,, \tag{8}$$

stop, (x^*, y^*) is the desirable solution. Otherwise, generate a vector value of the random vector v. Denote the just generated vector by v_{k+1}, set

$$\bar{Y} := \bar{Y} \cup \{y^*, v_{k+1}\} \tag{9}$$

and go to iteration $k + 1$.

Remark 1. It is easily seen that

$$s' \geqq s^* \geqq s''. \tag{10}$$

Furthermore, s' decreases whilst s'' generally is not monotoneous.

Remark 2. Suppose that the procedure terminates. Then we have

$$|s' - S(x^*, \bar{y})| \leqq \varepsilon_1 \tag{11}$$

$$|s'' - S(x^*, y^*)| \leqq \varepsilon_2 \tag{12}$$

and

$$|S(x^*, \bar{y}) - S(x^*, y^*)| \leqq \varepsilon_3 . \tag{8}$$

From (8) and (11) it follows

$$|s' - S(x^*, y^*)| \leqq \varepsilon_1 + \varepsilon_3 . \tag{13}$$

Combining (10), (12) and (13) we have

$$-\varepsilon_2 \leqq s^* - S(x^*, y^*) \leqq \varepsilon_1 + \varepsilon_3 . \tag{14}$$

Consequently, if

$$\max\{\varepsilon_2, \varepsilon_1 + \varepsilon_3\} \leqq \varepsilon \tag{15}$$

from (12) and (14) it follows that (x^*, y^*) is an ε-optimal solution of problem (1).

In the sequel we denote by $\mathsf{P}(E)$ the probability of an event E.

Lemma 1. Suppose that a subset $M \subset Y$ satisfies

$$\mathsf{P}\{v \in M\} > 0 .$$

Then

$$\mathsf{P}\{\exists k : v_k \in M\} = 1 . \tag{16}$$

Proof. The assertion immediately follows from the law of large numbers [17]. □

Remark 3. (16) is equivalent to

$$\mathsf{P}\{v_k \notin M \;\; \forall k\} = 0 . \tag{17}$$

For $y \in Y$ and $\delta > 0$ we denote

$$B_y^\delta = \{y' \in Y : \|y' - y\| \leqq \delta\} \cap Y .$$

Lemma 2. Suppose that $\delta > 0$ and Y is bounded. Let E denote the event that for all $n \in N$ the set $\{v_1, \ldots, v_n\}$ is not a δ-net of Y. Further let F denote the event

$$\exists \bar{y} \in Y \quad \forall n \in N : v_n \notin B_{\bar{y}}^{\delta/2} .$$

Then we have

$$E \subset F .$$

Proof. If $\{v_1, \ldots, v_n\}$ is not a δ-net of Y, there exist $y_n \in Y$ such that

$$v_k \notin B^{\delta}_{y_n} \quad \forall k = 1, \ldots, n .$$

Therefore, if E occurs, there exists a sequence $\{y_n\} \subset Y$ such that

$$\forall n \in N \quad \forall k = 1, \ldots, n\colon v_k \notin B^{\delta}_{y_n} .$$

Since Y is bounded, there exists a cluster point y of the sequence $\{y_n\}$. It is easily seen that

$$v_n \notin B^{3/4\delta}_{y} \quad \forall n . \tag{18}$$

If $y \in Y$, set $\bar{y} = y$ and we are done. Otherwise choose $\bar{y} \in Y$ such that

$$\|\bar{y} - y\| \leqq \tfrac{1}{4}\delta . \tag{19}$$

From (18) and (19) it follows

$$v_n \notin B^{\delta/2}_{\bar{y}} \quad \forall n$$

which means that the event F also occurs. □

Lemma 3. Suppose that $\delta > 0$ and Y is bounded. Let $\{y_1, \ldots, y_m\}$ be a $\frac{1}{4}\delta$-net of Y and let F_k, $k = 1, \ldots, m$, denote the event

$$v_n \notin B^{\delta/4}_{y_k} \quad \forall n .$$

Then

$$F \subset F_1 + F_2 + \ldots + F_m ,$$

where F is the event defined in Lemma 2.

Proof. Suppose that the event F occurs, i.e. there exists $\bar{y} \in Y$ such that

$$v_n \notin B^{\delta/2}_{\bar{y}} \quad \forall n .$$

Since $\{y_1, \ldots, y_m\}$ is a $\frac{1}{4}\delta$-net of Y, there exists y_k, $1 \leqq k \leqq m$, such that

$$\|y_k - \bar{y}\| \leqq \tfrac{1}{4}\delta .$$

It is easily seen that

$$v_n \notin B^{\delta/4}_{y_k} \quad \forall n$$

which means that the event F_k also occurs. □

Lemma 4. Suppose that Y is bounded and the random vector v satisfies

$$\forall \delta > 0 \quad \forall y \in Y\colon \mathsf{P}\{v \in B^{\delta}_{y}\} > 0 . \tag{20}$$

Then

$$\mathsf{P}(E) = 0 ,$$

where E is the event defined in Lemma 2.

Proof. Since Y is bounded, there exists a $\frac{1}{4}\delta$-net $\{y_1, \ldots, y_m\}$ of Y. In virtue of

Lemma 2 and Lemma 3 we have

$$E \subset F_1 + F_2 + \ldots + F_m$$

where F_k, $k = 1, \ldots, m$, are the events defined in Lemma 3. By Lemma 1 we have $\mathsf{P}(F_k) = 0$ $\forall k = 1, \ldots, m$. Consequently, our assertion follows from the inequality

$$\mathsf{P}(E) \leqq \mathsf{P}(F_1) + \mathsf{P}(F_2) + \ldots + \mathsf{P}(F_m). \qquad \square$$

Theorem. Suppose that Y is bounded, the random vector v satisfies condition (20), function $S(x, y)$ satisfies

$$\forall \xi > 0 \quad \exists \delta > 0 \quad \forall x \in X: \|y_1 - y_2\| \leqq \delta \Rightarrow |S(x, y_1) - S(x, y_2)| \leqq \xi \quad (21)$$

and the parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ satisfy

$$\varepsilon_1 + \varepsilon_2 < \varepsilon_3. \quad (22)$$

Then the procedure of the semistochastic decomposition scheme is finite with probability 1.

Proof. Set

$$\varepsilon_4 = \tfrac{1}{2}(\varepsilon_3 - \varepsilon_1 - \varepsilon_2). \quad (23)$$

By (21) there exists $\delta > 0$ such that

$$\forall x \in X: \|y_1 - y_2\| \leqq \delta \Rightarrow |S(x, y_1) - S(x, y_2)| \leqq \varepsilon_4. \quad (24)$$

Let $\bar{E}$ denote the event that there exists $n \in N$ such that $\{v_1, \ldots, v_n\}$ is a δ-net of Y. By Lemma 4 we have $\mathsf{P}(\bar{E}) = 1$. Supposing that $\bar{E}$ occurs, we shall show that the procedure terminates not later than after n iterations and the theorem is thus proved for $\mathsf{P}(\bar{E}) = 1$.

Therefore suppose that $\{v_1, \ldots, v_n\}$ is a δ-net for some $n \in \mathbb{N}$. If the procedure does not terminate before the nth iteration, at the nth iteration we have

$$|s' - S(x^*, \bar{y})| \leqq \varepsilon_1 \quad (11)$$

$$|\inf_{y \in \bar{Y}} S(x^*, y) - S(x^*, \bar{y})| \leqq \varepsilon_1 \quad (25)$$

and

$$|s'' - S(x^*, y^*)| \leqq \varepsilon_2. \quad (12)$$

Since $\{v_1, \ldots, v_n\} \subset \bar{Y}$, there exists $y' \in \bar{Y}$ such that

$$|S(x^*, y^*) - S(x^*, y')| \leqq \varepsilon_4. \quad (26)$$

From (12) and (26) it follows

$$|S(x^*, y') - s''| \leqq \varepsilon_2 + \varepsilon_4$$

whence

$$\inf_{y \in \bar{Y}} S(x^*, y) \leqq S(x^*, y') \leqq \varepsilon_2 + \varepsilon_4 + s'' \leqq \varepsilon_2 + \varepsilon_4 + \inf_{y \in \bar{Y}} S(x^*, y)$$

which entails

$$|S(x^*, y') - \inf_{y \in \bar{Y}} S(x^*, y)| \leqq \varepsilon_2 + \varepsilon_4. \quad (27)$$

Combining (26), (27), (25) and (23) we obtain

$$|S(x^*, y^*) - S(x^*, \bar{y})| \leqq \varepsilon_1 + \varepsilon_2 + 2 . \varepsilon_4 = \varepsilon_3 .$$

So criterion (8) is satisfied and the procedure terminates. □

Remark 4. Condition (21) is satisfied if $S(x, y)$ is uniformly continuous on $X \times Y$ or $S(x, y)$ is continuous on $X \times Y$ and X, Y are compact sets.

Remark 5. If Y satisfies

$$\forall \delta > 0 \quad \forall y \in Y: \mu(B_y^\delta) > 0 ,$$

where μ is the Lebesgue measure, then each random vector v with positive density function on Y, e.g. the uniformly distributed vector, satisfies condition (20).

Remark 6. Analogously we can establish the semistochastic decomposition for the following problem $\inf_{x \in X} \sup_{y \in Y} S(x, y)$.

(Received January 22, 1990.)

REFERENCES

[1] G. B. Dantzig and P. Wolfe: The decomposition algorithm for linear programs. Econometrica *29* (1961), 4, 767—778.

[2] J. F. Benders: Partitioning procedures for solving mixed variable programming problems. Numer. Math. *4* (1962), 238—260.

[3] A. Geoffrion: Generalized Benders decomposition. J. Optim. Theory Appl. *10* (1972), 237—260.

[4] T. J. Van Roy: Cross decomposition for mixed integer programming. Mathematical Programming *25* (1983), 46—63.

[5] I. Kornai and T. Liptak: Planning on two levels. In: Application of Mathematics in Economical Research 4, Moscow 1965. In Russian.

[6] R. E. Burkard, H. W. Hamacher and J. Tind: On general decomposition schemes in mathematical programming. Mathematical Programming Study *24* (1985), 238—252.

[7] V. Z. Belinskij, V. A. Volkonskii, S. A. Ivankov, A. B. Pomanskii and A. D. Shapiro: Iterative Methods in Game Theory and Mathematical Programming. Nauka, Moscow 1974. In Russian.

[8] V. V. Fedorov: Maximin Numerical Methods. Nauka, Moscow 1979.

[9] I. Ekeland and R. Temam: Analyse Convexe et Problemes Variationnels. Dunod, Paris 1974.

[10] G. D. Maistrovskii: Gradient methods for finding saddle points. Econom. i Mat. Metody *12* (1976), 917—929. In Russian.

[11] E. G. Golstein: Gradient methods for determination of saddle points and modified Lagrangians. In: Proc. of the Workshop "Math. Optimierung — Theorie und Anwendungen" Wartburg/Eisenach 1983.

[12] R. T. Rockafellar: A dual approach to solving nonlinear programming problems by unconstrained optimization. Math. Programming *5* (1973), 354—373.

[13] Tran Quoc Chien: Nondifferentiable and quasidifferentiable duality in vector optimization theory. Kybernetika *21* (1985), 4, 298—321.

[14] Tran Quoc Chien: Fenchel-Lagrange duality in vector fractional programming via abstract duality scheme. Kybernetika *22* (1986), 4, 299—319.

[15] Tran Quoc Chien: Perturbation theory of duality in vector optimization via the abstract duality scheme. Kybernetika *23* (1987), 1, 67–81.
[16] Tran Quoc Chien: Vector nonconvex perturbational duality theory via the abstract duality scheme. Essays on Nonlinear Analysis and Optimization Problems. Ha-Noi 1987.
[17] B. Gnedenko: The Theory of Probability. Nauka, Moscow 1973.
[18] N. N. Vorobiev: Elements of Game Theory: Noncooperative Games. Nauka, Moscow 1984.

RNDr. Tran Quoc Chien, DrSc., matematicko-fyzikální fakulta UK (Faculty of Mathematics and Physics — Charles University), Malostranské nám. 25, 118 00 Praha 1. Czechoslovakia. Permanent address: *Department of Mathematics — Polytechnical School of Da-nang. Vietnam.*