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FEEDBACK CANONICAL FORMS OF SINGULAR SYSTEMS

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We consider the action of Proportional-Derivative Feedback and Pure Proportional Feedback groups on the set of all singular systems. Both actions are characterized by complete lists of orbital invariants giving rise to canonical forms. Dynamic interpretations of these invariants are given.

1. INTRODUCTION

We consider the linear, time-invariant singular systems described by:

$$E\dot{x} = Ax + Bu \tag{1.1}$$

where $x \in \mathcal{X} \simeq \mathbb{R}^n \in \mathcal{U} \simeq \mathbb{R}^m$, $(E, A, B) \in \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$. Note that E and A are *non square*. We can always assume without loss of generality that B is monic.

In the sequel, (1.1) will be referred to by the triple (E, A, B). Now, let Σ denote the set of all such (E, A, B)'s. Define $\tau_{PD} = \{(W, V, G, F_P, F_D) \in \mathbb{R}^{q \times q} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{m \times m} \times \mathbb{R}^{m \times n} \times \mathbb{R}^{m \times n}$ with W, V and G non singular$\}$. The action of this group on Σ is defined by:

$$(W, V, G, F_P, F_D) \circ (E, A, B) = (W^{-1}(E + BF_D) V, W^{-1}(A + BF_P) V, W^{-1}BG).$$

Further define a subgroup τ_P of τ_{PD} by the condition $F_D = 0$. τ_P (resp. τ_{PD}) is called the Proportional (resp. Proportional and Derivative) Feedback group.

In Section 2, we consider the action of τ_{PD} on Σ. A complete list of orbital invariants and *P.D. canonical form* for this action are presented algebraically. Then, we consider in Section 3 the action of τ_P on the same set Σ and derive from the P.D. canonical form an *algebraic P. canonical form*. A complete list of orbital invariants for this action is given and is shown to have a precise dynamical meaning (at least

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in the square case ($q = n$)), in the light of which, we further develop, in Section 4, another *dynamical P. canonical form* which exhibits various structural properties of the system in a very transparent way. Section 5 is devoted to concluding remarks.

2. PROPORTIONAL-DERIVATIVE CANONICAL FORM

Associated with (1.1) is the so-called *restricted pencil sNE-NA* [5], where N is a basis matrix for the canonical projection $\mathscr{X} \to \mathscr{X}/\mathrm{Im}\, B$, that is a maximal rank solution of $NB = 0$. Starting from two systems (E, A, B) and (E', A', B') like (1.1) with same dimensions n, q and m, the following result is an obvious consequence of the fact that $(NE, NA) = (N(E + BF_D), N(A + BF_P))$ for all F_P and F_D.

Proposition 2.1. (E, A, B) and (E', A', B') are equivalent under the action of τ_{PD} if and only if their associated restricted pencils are Kronecker's equivalent, that is iff there exist non singular P and Q such that:

$$P[N(sE - A)]\, Q = N'(sE' - A') \tag{2.1}$$

Proof. i) *only if part*: $(E', A', B') = (W^{-1}(E + BF_D)\, V, W^{-1}(A + BF_P)\, V, W^{-1}BG)$ implies $N' = NW$ and thus: $N'(sE' - A') = N[s(E + BF_D)\, V - (A + BF_P)\, V] = N(sE - A)\, V = P[N(sE - A)]\, Q$ with $P = I$ and $Q = V$.

ii) *if part*: $N'(s(E' - A') = P[N(sE - A)]\, Q$ with P and Q invertible. Since both N and N' are epic (of rank $q - m$), there exist invertible T and T' such that $PNT = [\mathbf{1}_{q-m} \mid 0] = N'T'$, that is, with $W =: TT'^{-1}$:

$$N' = PNW. \tag{2.2}$$

This, combined with (2.1), gives $N[W(sE' - A') - (sE - A)\, Q] = 0$, which is equivalent to: $W(sE' - A') - (sE - A)\, Q = BL(s)$, where $L(s)$ is obviously a polynomial matrix of degree one. Write it as: $L(s) = (sF_D - F_P)\, Q$, then $sE' - A' = W^{-1}[s(E + BF_D) - (A + BF_P)]\, V$ with $V = Q$. Finally, from (2.2) directly follows: $NWB' = 0$ which implies $WB' = BG$ for some invertible G, which ends the proof. □

This immediately shows that a complete list of invariants for the action of τ_{PD} on Σ is given by the so-called Kronecker's invariants of the restricted pencil. This corresponds to the following *P.D. canonical form* for (E, A, B) under τ_{PD}:

$$sE_{PD} - A_{PD} = \begin{bmatrix} sE_K - A_K \\ \hdashline \Phi(s) \end{bmatrix} \quad B_{PD} = \begin{bmatrix} \mathbf{0}_{(q-m)\times m} \\ \hdashline \mathbf{1}_{m\times m} \end{bmatrix} \tag{2.3}$$

where $\Phi(s) \equiv \mathbf{0}_{m\times n}$, $\mathbf{1}_{m\times m}$ is the $(m \times m)$ identity and $(sE_K - A_K)$ is the Kronecker normal form [3] of $(sNE - NA)$, that is:

$$(sE_K - A_K) = \text{block diagonal}\,(sE_{K_i} - A_{K_i}) \text{ with:}$$

For the *finite elementary divisors* (f.e.d.), $(sE_{K_1} - A_{K_1})$ = block diag. $(L_{\alpha_{ij}})$:

$$L_{\alpha_{ij}} = \underbrace{\begin{bmatrix} s-\alpha_i & 1 & 0 & . & 0 \\ 0 & & . & . & . \\ . & & . & . & 0 \\ . & & . & . & 1 \\ 0 & & . & 0 & s-\alpha_i \end{bmatrix}}_{\leftarrow\ k_{ij}\ \rightarrow} \begin{matrix}\uparrow \\ k_{ij} \\ \downarrow\end{matrix} \tag{2.4}$$

For the *column minimal indices* (c.m.i.), $(sE_{K_2} - A_{K_2})$ = block diag. (L_{ε_i}):

$$L_{\varepsilon_i} = \underbrace{\begin{bmatrix} s & 1 & 0 & . & . & 0 \\ 0 & s & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0 & s & 1 \end{bmatrix}}_{\leftarrow\ \varepsilon_i + 1\ \rightarrow} \begin{matrix}\uparrow \\ \varepsilon_i \\ \downarrow\end{matrix} \tag{2.5}$$

For the *row minimal indices* (r.m.i.). $(sE_{K_3} - A_{K_3})$ = block diag. (L_{η_i}):

$$L_{\eta_i} = \underbrace{\begin{bmatrix} s & 0 & . & 0 \\ 1 & s & 0 & 0 \\ . & . & . & . \\ 0 & . & 1 & s \\ 0 & . & 0 & 1 \end{bmatrix}}_{\leftarrow\ \eta_i\ \rightarrow} \begin{matrix}\uparrow \\ \eta_i + 1 \\ \downarrow\end{matrix} \tag{2.6}$$

and for the *infinite elementary divisors* (i.e.d.), $(sE_{K_4} - A_{K_4})$ = block diag. (L_{n_i}):

$$L_{n_i} = \underbrace{\begin{bmatrix} 1 & 0 & . & . & 0 \\ s & 1 & 0 & . & 0 \\ . & . & . & . & . \\ . & 0 & s & 1 & 0 \\ . & . & 0 & s & 1 \end{bmatrix}}_{\leftarrow\ n_i\ \rightarrow} \begin{matrix}\uparrow \\ n_i \\ \downarrow\end{matrix} \tag{2.7}$$

This *P.D. canonical form* of (E, A, B) is presented in a more condensed form (Tableau) in Appendix 1.

Recall that $\varepsilon_i = 0$ (resp. $\eta_i = 0$) corresponds to a zero column (resp. row). The polynomials $(s - \alpha_i)^{k_{ij}}$ and the integers ε_i, η_i and n_i are uniquely characterized, for instance in the following geometric way, [7].

Consider the following algorithms (for details about them, see for instance [11]:

$$\mathscr{V}_0 = \mathscr{X}\,, \quad \mathscr{V}_i = A^{-1}(\operatorname{Im} B + E\mathscr{V}_{i-1}) \quad i \geqq 1\,, \quad \text{limit } \mathscr{V}^* \tag{2.8}$$

$$\mathscr{R}_{a0} = 0\,, \quad \mathscr{R}_{ai} = E^{-1}(\operatorname{Im} B + A\mathscr{R}_{ai-1}) \quad i \geqq 1\,, \quad \text{limit } \mathscr{R}_a^* \tag{2.9}$$

and adopt the following notation: let $\{a_i\}$ be a list of positive integers in non decreasing order $(a_1 \geqq a_2 \geqq \ldots)$, then this list is in one-to-one correspondence with the non decreasing list $\{\bar{a}_i\}$ defined by: $\bar{a}_i =:$ number of a_j's which are greater than or equal to i, $i \geqq 1$.

Proposition 2.2. $\{(s - \alpha_i)^{k_{ij}}\}$ is the list of the invariant factors of the map $(E\mathscr{V}^*/E\mathscr{R}^* \| A + BF \| \mathscr{V}^*/\mathscr{R}^*)$, induced by $A + BF$ in the quotient spaces $\mathscr{V}^*/\mathscr{R}^*$ and $E\mathscr{V}^*/E\mathscr{R}^*$, where F is any "friend" of $\mathscr{V}^*$ i.e. such that $(A + BF)\mathscr{V}^* \subset E\mathscr{V}^*$, and $\mathscr{R}^* = \mathscr{V}^* \cap \mathscr{R}_a^*$.

$$\bar{\varepsilon}_{i-1} = \dim\left((\mathscr{V}^* \cap \mathscr{R}_{ai})/(\mathscr{V}^* \cap \mathscr{R}_{ai-1})\right), \quad i \geqq 1$$

$$\bar{\eta}_i = \dim\left((\mathscr{R}_a^* + \mathscr{V}_{i-1})/(\mathscr{R}_a^* + \mathscr{V}_i)\right), \quad i \geqq 1$$

and the total number $(\#)$ of η_i's (including zero ones) is given by:

$$\#\eta_i\text{'s} = n - \dim\left(E\mathscr{V}^* + A\mathscr{R}_a^* + \operatorname{Im} B\right)$$

$$\bar{n}_i = \dim\left((\mathscr{V}^* + \mathscr{R}_{ai})/(\mathscr{V}^* + \mathscr{R}_{ai-1})\right), \quad i \geqq 1$$

We shall now consider the transformation subgroup τ_P and show how to derive from the previous P.D. canonical form an algebraic P. canonical form.

3. ALGEBRAIC P. CANONICAL FORM

An obvious inspection of our previous P.D. canonical form (2.3)–(2.7) shows that the following form can be derived from any (E, A, B), using only (W, V, G, O, O) elements of τ_{PD} (that is *with no feedback at all*):

$$sE_0 - A_0 = \begin{bmatrix} sE_K - A_K \\ \hdashline \Phi(s) \end{bmatrix} \quad B_0 = \begin{bmatrix} \mathbf{0}_{(q-m)\times m} \\ \hdashline \mathbf{1}_{m \times m} \end{bmatrix} \tag{3.1}$$

with $sE_K - A_K$ as described in (2.4)–(2.7) but with now $\Phi(s) \neq 0$:

$$\Phi(s) = s\Phi_D + \Phi_P \tag{3.2}$$

Step 1: Due to the particular canonical form of $(sE_K - A_K)$, a first simple inspection shows that, by using only constant invertible-row-operations on $sE_0 - A_0$, it is possible to eliminate most of the $\Phi_{D_{ij}}$'s entries: all the columns of Φ_D, except those corresponding to the last column of any L_{ε_i} or L_{n_i} block (see (2.5), (2.7)) can be made zero in such a trivial way. Note that this kind of "elementary transformations" do not change the form of $sE_K - A_K$.

Step 2: A further inspection shows that, by using again only (W, V, G, O, O) elements of τ_{PD}, it is also possible, without extra alteration, to eliminate other entries in Φ_D. Indeed, any column ϕ_j of Φ_D can be replaced by $\alpha\phi_j + \beta\phi_i$, $(\alpha \neq 0)$, (and hence be "reduced" by α_i), if and only if i and j respectively correspond to the last column of blocks L_{λ_i} and L_{λ_j} (from (2.5), (2.7)) with either:

i) $\lambda_i = \varepsilon_i \leqq \varepsilon_j = \lambda_j$
ii) $\lambda_i = n_i \geqq n_j = \lambda_j$
iii) $\lambda_i = \varepsilon_i$ and $\lambda_j = n_j$

Every "elementary transformation" of this kind can be easily formalized (details are given in Appendix 3) and we just illustrate here these transformations through simple but sufficiently explicit examples:

i) $\boldsymbol{\varepsilon}_1 = \mathbf{0}\,, \quad \boldsymbol{\varepsilon}_2 = \mathbf{1}$ i.e.:

$$\begin{bmatrix} 0 & L_{\varepsilon_2} \\ \hline \multicolumn{2}{c}{\Phi_D(s)} \end{bmatrix} = \begin{bmatrix} \boxed{0} & \boxed{s \;\; 1} \\ \hline s & 0 \;\; s \end{bmatrix}$$

$$\begin{bmatrix} \boxed{0} & \boxed{s \;\; 1} \\ \hline s & 0 \;\; s \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boxed{0} & \boxed{s \;\; 1} \\ \hline s & 0 \;\; 0 \end{bmatrix}$$

ii) $\mathbf{n}_1 = \mathbf{2}, \; \mathbf{n}_2 = \mathbf{1}$ i.e.:

$$\begin{bmatrix} L_{n_1} & 0 \\ 0 & L_{n_2} \\ \hline \multicolumn{2}{c}{\Phi_D(s)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ s & 1 & \\ & & 1 \\ \hline 0 & s & s \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \\ s & 1 & \\ & & 1 \\ \hline 0 & s & s \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \\ s & 1 & \\ & & 1 \\ \hline 0 & s & 0 \end{bmatrix}$$

iii) $\boldsymbol{\varepsilon}_1 = \mathbf{1}, \; \mathbf{n}_1 = \mathbf{1}$ i.e.:

$$\begin{bmatrix} L_{\varepsilon_1} & 0 \\ 0 & L_{n_1} \\ \hline \multicolumn{2}{c}{\Phi_D(s)} \end{bmatrix} = \begin{bmatrix} s & 1 & \\ & & 1 \\ \hline 0 & s & s \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s & 1 & \\ & & 1 \\ \hline 0 & s & s \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s & 1 & \\ & & 1 \\ \hline 0 & s & 0 \end{bmatrix}$$

Step 3: The $\mathbf{1}_{m \times m}$ block of B_0 in (3.1) finally allows, by using Proportional Feedback F_P, to eliminate Φ_P in (3.2).

In view of Step 2, let us order lists $\{\varepsilon_i\}$ and $\{n_i\}$ as:

$$\varepsilon_1 \leqq \varepsilon_2 \leqq \ldots \quad \text{and} \quad n_1 \geqq n_2 \geqq \ldots$$

All the above-mentioned elementary transformations obviously bring $\Phi(s)$ in (3.2) towards a special form where all the columns are zero except some (usually not all) associated with the last columns of blocks L_{ε_i} or L_{n_i}.

After reordering blocks ε_i and n_i, $\Phi(s)$ can be written as:

$$\Phi(s) = \left[\begin{array}{ccc:cccc} 0 & . & 0 & \boxed{0 \;.\; 0 \; s} & . & . & . \\ 0 & . & 0 & . & \boxed{0 \;.\; 0 \; s} & . & . \\ 0 & . & 0 & . & . & . & \boxed{0 \;.\; 0 \; s} \\ 0 & . & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This reduction (of $\Phi(s)$) obviously amounts to splitting each list $\{\varepsilon_i\}$ and $\{n_i\}$ into two separate sublists, depending on the associated *zero or non zero column in* $\Phi(s)$.

We are now able to give our first main Theorem describing the *algebraic P. canonical form.* For that, let use the following notation: block diag. (i) denotes the matrix where all blocks are zero except those (i) situated along the main diagonal and row block (i) denotes the matrix formed by each row block (i)

Theorem 3.1. Any (E, A, B) system like (1.1) can be given, through the action of τ_P, the following equivalent form:

$$[sE_P - A_P] = \text{block diag.}\,[sE_{P_i} - A_{P_i}] \quad B_P = \text{row block}\,[B_{P_i}]\,, \quad i = 1 \text{ to } 6, \tag{3.2}$$

with *for the f.e.d.*:

$$[sE_{P_1} - A_{P_1}] = \text{block diag.}\,[L_{\alpha_{ij}}] \qquad B_{P_1} = 0 \tag{3.3}$$

and $L_{\alpha_{ij}}$ as described in (2.4),

for the r.m.i.:

$$[sE_{P_2} - A_{P_2}] = \text{block diag.}\,[L_{\eta_i}] \qquad B_{P_2} = 0 \tag{3.4}$$

and L_{η_i} as described in (2.6),

for the "two types" of c.m..i.:

$$* \; [sE_{P_3} - A_{P_3}] = \text{block diag.}\,[L_{\gamma_i}] \qquad B_{P_3} = 0 \tag{3.5}$$

$$\text{with } L_{\gamma_i} = \underbrace{\begin{bmatrix} s & 1 & 0 & . & . & 0 \\ 0 & s & 1 & . & . & . \\ . & . & . & . & . & . \\ . & . & . & 0 & s & 1 \end{bmatrix}}_{\gamma_i} \Bigg\updownarrow \gamma_i - 1$$

$$* \; [sE_{P_4} - A_{P_4}] = \text{block diag.}\,[L_{\sigma_i}] \quad B_{P_4} = [\text{block diag.}\,[b_{\sigma_i}] \;\; 0] \tag{3.6}$$

$$\text{with } L_{\sigma_i} = \underbrace{\begin{bmatrix} s & 1 & 0 & . & 0 \\ 0 & s & 1 & . & . \\ . & . & . & . & . \\ . & . & . & s & 1 \\ . & . & . & 0 & s \end{bmatrix}}_{\sigma_i} \Bigg\updownarrow \sigma_i \text{ and } b_{\sigma_i} = \begin{bmatrix} 0 \\ . \\ 0 \\ 0 \\ 1 \end{bmatrix} \Bigg\updownarrow \sigma_i$$

and for the "two types" of i.e.d.:

$$* \left[sE_{P_5} - A_{P_5}\right] = \text{block diag.}\left[L_{q_i}\right] \quad B_{P_5} = 0 \tag{3.7}$$

$$\text{with } L_{q_i} = \begin{bmatrix} 1 & 0 & . & . & 0 \\ s & 1 & 0 & . & 0 \\ . & . & . & . & . \\ . & 0 & s & 1 & 0 \\ . & . & 0 & s & 1 \end{bmatrix} \begin{matrix} \uparrow \\ \\ q_i \\ \\ \downarrow \end{matrix} \qquad \leftarrow q_i \rightarrow$$

$$* \left[sE_{P_6} - A_{P_6}\right] = \text{block diag.}\left[L_{p_i}\right] \quad B_{P_6} = \left[0 \ \text{block diag.}\left[b_{pi}\right]\right] \tag{3.8}$$

$$\text{with } L_{p_i} = \begin{bmatrix} 1 & 0 & . & 0 \\ s & 1 & 0 & 0 \\ . & . & . & . \\ . & 0 & s & 1 \\ . & . & 0 & s \end{bmatrix} \begin{matrix} \uparrow \\ \\ p_i + 1 \\ \\ \downarrow \end{matrix} \text{ and } b_{pi} = \begin{bmatrix} 0 \\ . \\ . \\ 0 \\ 1 \end{bmatrix} \begin{matrix} \uparrow \\ \\ p_i + 1 \\ \\ \downarrow \end{matrix} \qquad \leftarrow p_i \rightarrow$$

Further remark that:

$\gamma_i = 1$ corresponds to a zero column in $[sE_P - A_P]$,

$\eta_i = 0$ corresponds to a zero row both in $[sE_P - A_P]$ and in B_P, while

$p_i = 0$ corresponds to a zero row in $[sE_P - A_P]$ and a non zero row in B_P.

Note that if B is not monic (general case), we only have to add zero columns in B_p.

This *Algebraic P. canonical form* is described in a more condensed form (Tableau) in Appendix 2. Blocks of type 1 and 2 have already been geometrically described in Proposition 2.2. The following geometric characterization of blocks of type 3 to 6 establish invariance and thus *canonicity*. It directly derives from the inspection, on the previous "canonical" form, of algorithms (2.8) and (2.9) and also of the following variant of (2.9):

$$\mathcal{M}_{a0} = \operatorname{Ker} E\,, \quad \mathcal{M}_{ai} = E^{-1}(\operatorname{Im} B + A\mathcal{M}_{ai-1}) \quad i \geqq 1\,, \tag{3.9}$$

the limits of which satisfies $\mathcal{M}_a^* = \mathcal{R}_a^*$ (since $\mathcal{R}_{a0} \subset \mathcal{M}_{a0} \subset \mathcal{R}_{a1} \subset \ldots$),

Remembering the notation of Proposition 2.2, we have:

Proposition 3.2.

$$\bar{\gamma}_i = \dim\left((\mathcal{V}^* \cap \mathcal{M}_{ai-1})/(\mathcal{V}^* \cap \mathcal{R}_{ai-1})\right), \quad i \geqq 1 \tag{3.10}$$

$$\bar{\sigma}_i = \dim\left((\mathcal{V}^* \cap \mathcal{R}_{ai})/(\mathcal{V}^* \cap \mathcal{M}_{ai-1})\right), \quad i \geqq 1 \tag{3.11}$$

$$\bar{q}_i = \dim\left(\mathcal{M}_{ai-1}/\mathcal{R}_{ai-1}\right) - \bar{\gamma}_i\,, \quad i \geqq 1 \tag{3.12}$$

$$\bar{p}_i = \dim\left(\mathcal{R}_{ai}/\mathcal{M}_{ai-1}\right) - \bar{\sigma}_i\,, \quad i \geqq 1 \tag{3.13}$$

and total number of p_i's (with zero ones) $= m - \bar{\sigma}_1$.

We are now able to derive the *dynamical P. canonical form*.

4. DYNAMICAL P. CANONICAL FORM

The limits $\mathscr{V}^*$ and $\mathscr{R}_a^*$ of algorithms (2.7) and (2.9)–(3.9) are respectively known as the supremal $(A, E, \operatorname{Im} B)$-invariant subspace and the almost reachability subspace in $\mathscr{X}$. $\mathscr{R}^* = \mathscr{V}^* \cap \mathscr{R}_a^*$ is the famous supremal reachability subspace of $\mathscr{X}$ (for details and references, see for instance [11]).

Concepts like reachability and controllability are now well understood. Precise geometric characterizations have first been given in [9] for regular (square) systems and after for regularizable (square) systems [10]. This geometric result has also been extended to *non square* systems [2] (using Differential Inclusions technics). However, this non square case reveals some sharp specificities (the solution of the underlying Differential& Algebraic Equation is not unique and the associated degrees of freedom contribute to this reachability concept) which would make the justification of the following indices rather long. That is the reason why *we shall from now on restrict our attention to square system* (1.1) (i.e.: $q = n$). Note however that no other assumption will be needed (as regularity or regularizability or controllability...).

Since we shall in fact be able to check each of such properties directly from our invariants, some minimal definitions are certainly needed.

(E, A, B) is *regular* when $\det(sE - A)$ is not identically zero i.e.: $(sE - A)$ only has f.e.d. and i.e.d.,

(E, A, B) is *proper* when $(sE - A)^{-1}$ not only exists but also is proper i.e. $(sE - A)$, only has f.e.d. and i.e.d. of order ≤ 1,

(E, A, B) is *strictly proper* when $(sE - A)^{-1}$ not only exists but also is strictly proper i.e.: $(sE - A)$ only has f.e.d.

Regular, proper and strictly proper is changed into P. (or (P.D.)) *regularizable*, P. (or (P.D.)) *properizable* and P. (or (P. D.)) *strictly properizable* when the respective property holds for $s(E + BF_D) - (A + BF_P)$ (or $sE - (A + BF_P)$) for some F_P (or (F_D, F_P)), in place of $(sE - A)$.

For a given (E, A, B), the properties of *reachability* and *controllability* are defined in accordance with the ability the system has to move from any initial point to any other one (reachability) or to the origin (controllability) along *non impulsive trajectories* (see for instance [9]). Then:

(E, A, B) is called P. (or (P. D.)) *controllable* (or *reachable*) when this property holds for some F_P (or (F_D, F_P)).

The fine reachability structure of $\mathscr{R}^*$ has been described in [11] where *proper and non proper reachability indices* have been introduced (see also [6] for their algebraic definition). The distinction between proper and non proper ones can be summarized as follows. $\mathscr{R}^*$ can always be decomposed as the direct sum of singly generated reachability subspaces of dimension r_i, corresponding to reachability chains like:

$$0 = Ax_k + Bu_k\,, \quad Ex_k = Ax_{k-1} + Bu_{k-1}, \ldots, Ex_2 = Ax_1 + Bu_1\,,$$
$$Ex_1 = Bu_0\,.$$

Such a chain is called a *proper* (resp. *non proper*) *reachability chain* when $Ex_1 \neq 0$ (resp. $Ex_1 = 0$). Their lengths define the *proper* (resp. *non proper*) *reachability indices* noted as r_{pi} (resp. r_{npi}). The combined use of (2.8), (2.9) and (3.9) gives a nice description for r_{pi}'s and r_{npi}'s (see [11]):

$$\bar{r}_{npi} = \dim\left((\mathscr{V}^* \cap \mathscr{M}_{ai-1})/(\mathscr{V}^* \cap \mathscr{R}_{ai-1})\right), \quad i \geqq 1 \tag{4.1}$$

$$\bar{r}_{pi} = \dim\left((\mathscr{V}^* \cap \mathscr{R}_{ai})/(\mathscr{V}^* \cap \mathscr{M}_{ai-1})\right), \quad i \geqq 1 \tag{4.2}$$

The same kind of decomposition can be performed on $(\mathscr{R}_a^*/\mathscr{R}^*)$, leading to *proper and non proper almost reachability indices* noted $\{r_{api}\}$ and $\{r_{anpi}\}$ (note that we could call them *purely almost reachability indices* since purely reachability ones are inside $\mathscr{R}^*$). These integers correspond to the dimensions of the (proper and non proper) *singly generated almost reachability subspaces* described by chains like:

$$Ex_k = Ax_{k-1} + Bu_{k-1}, \ldots, Ex_2 = Ax_1 + Bu_1, \quad Ex_1 = Bu_0,$$

and satisfying either $Ex_1 \neq 0$ or $Ex_1 = 0$.

The following geometric characterization of r_{api} and r_{anpi} makes the pair with (4.1)–(4.2):

$$\bar{r}_{anpi} = \dim(\mathscr{M}_{ai-1}/\mathscr{R}_{ai-1}) - \bar{\gamma}_i \quad i \geqq 1 \tag{4.3}$$

$$\bar{r}_{api} = \dim(\mathscr{R}_{ai}/\mathscr{M}_{ai-1}) - \bar{\sigma}_i, \quad i \geqq 1 \tag{4.4}$$

Note that $\dim(\mathscr{M}_{ai-1}/\mathscr{R}_{ai-1})$ is naturally associated with *all* non proper almost reachability chains, that is including also those which are *purely* reachability ones, hence the term $(-\bar{\gamma}_i)$ in (4.3) to retain only those which are *purely* (non proper) almost reachability chains. Idem for (4.4).

Direct comparison between (3.10)–(3.13) and (4.1)–(4.4) leads to:

$$r_{npi} = \gamma_i; \quad r_{pi} = \sigma_i; \quad r_{anpi} = q_i \quad \text{and} \quad r_{api} = p_i, \quad i \geqq 1 \tag{4.5}$$

Due to this dynamical interpretation of the previous algebraic lists $\{\gamma_i\}, \ldots, \{p_i\}$, it is now possible to describe the dynamical P. canonical form. For that, first note that the inspection of the rectangular blocks in $[sE_P - A_P]$ ((3.2)) and the fact that E and A are square obviously lead to the following relationship:

$$\#\{\gamma_i\} = \#\{p_i = 0\} + \#\{p_i \geqq 1\} + \#\{\eta_i\} \tag{4.6}$$

where # means "total number of elements in the set".

We can thus, by associating γ_i-terms with p_i or η_i ones, define *independent subsystems* which dynamic properties are precise and different.

The three possibilities are:

i) P-feedback association of γ_i with $p_i = 0$: this gives rise to a *regular and reachable subsystem*,

ii) P-feedback association of γ_i with $p_i \geqq 1$: this gives rise to a *regular subsystem*,

iii) association of γ_i with η_i: this gives rise to a *non regular subsystem*.

Examples are given below, the first matrix on the left being $[sE_P - A_P \vdots B_P]$:

i) $\gamma_1 = 2, p_1 = 0$:

$$\left[\begin{array}{cc|c} s & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \overset{F_P}{\Leftrightarrow} \left[\begin{array}{cc|c} s & 1 & 0 \\ 1 & 0 & 1 \end{array}\right] \overset{(W,V)}{\Leftrightarrow} \left[\begin{array}{cc|c} 1 & s & 0 \\ 0 & 1 & 1 \end{array}\right]$$

ii) $\gamma_1 = 2, p_1 = 1$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ s & 0 & 0 & 1 \\ 0 & s & 1 & 0 \end{array}\right] \overset{F_P}{\Leftrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ s & 1 & 0 & 1 \\ 0 & s & 1 & 0 \end{array}\right] \overset{(W,V)}{\Leftrightarrow} \left[\begin{array}{ccc|c} 1 & s & 0 & 0 \\ 0 & 1 & s & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

iii) $\gamma_1 = 2, \eta_1 = 1$:

$$\left[\begin{array}{ccc|c} s & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & s & 1 & 0 \end{array}\right] \overset{(W,V)}{\Leftrightarrow} \left[\begin{array}{ccc|c} 1 & s & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Associations of type i), ii), and iii) leading to systems with more or less "pathologies" (non reachable, non regular), we shall adopt the following ***association rule*** which "minimizes pathologies": assume that γ_i's are in non decreasing order and decompose this list in three ordered parts. Then association iii) is performed with "smallest" γ_i's, ii) with "medium" γ_j's and i) with "highest" γ_i's. This can be summarized in the following *dynamical P. canonical form* expressed in terms of $r_{npi}, r_{pi}, \ldots$ (see (4.5)):

For any μ, let us denote $\mathbf{1}_\mu$, J_μ and b_μ respectively the $(\mu \times \mu)$ identity matrix and the following $(\mu \times \mu)$ nilpotent matrix and $(\mu \times 1)$ vector:

$$J_\mu = \begin{bmatrix} 0 & 1 & 0 & . \\ 0 & . & 1 & 0 \\ . & . & 0 & 1 \\ 0 & . & 0 & 0 \end{bmatrix} \quad \text{and} \quad b_\mu = \begin{bmatrix} 0 \\ . \\ 0 \\ 1 \end{bmatrix}.$$

Theorem 4.1. Any (E, A, B) like (1.1) is equivalent, through the action of τ_P, to the following form:

$$[\text{block diag. } E'_{P_i}]\, \dot{x} = [\text{block diag. } A'_{P_i}]\, x + \text{row block } [B'_{P_i}]\, u \tag{4.7}$$

with:

$$[E'_{P_1} \vdots A'_{P_1}] = \text{block diag. } [\mathbf{1}_{k_{ij}} \vdots J_{k_{ij}} - \alpha_i \mathbf{1}_{k_{ij}}]$$
$$B'_{P_1} = 0 \tag{4.8}$$

$$[E'_{P_2} \vdots A'_{P_2}] = \text{block diag. } [\mathbf{1}_{r_{pi}} \vdots J_{r_{pi}}]$$
$$B'_{P_2} = [\text{block diag. } [b_{r_{pi}}] \vdots 0 \vdots 0] \tag{4.9}$$

$$[E'_{P_3} \vdots A'_{P_3}] = \text{block diag. } [J_{r_{anpi}} \vdots \mathbf{1}_{r_{anpi}}]$$
$$B'_{P_3} = 0 \tag{4.10}$$

$$\begin{aligned}[E'_{P_4} \vdots A'_{P_4}] &= \text{block diag.} [J_{r_{npi}} \vdots \mathbf{1}_{r_{npi}}] \\ B'_{P_4} &= [0 \vdots \text{block diag.} [b_{r_{npi}}] \vdots 0]\end{aligned} \tag{4.11}$$

$$\begin{aligned}[E'_{P_5} \vdots A'_{P_5}] &= \text{block diag.} [J_{r_{npj}+r_{apk}} \vdots \mathbf{1}_{r_{npj}+r_{apk}}] \\ B'_{P_5} &= [0 \vdots 0 \vdots \text{block diag.} [\tilde{b}_{r_{npj}+r_{apk}}])]]\end{aligned} \tag{4.12}$$

with:

$$\tilde{b}_{r_{npj}+r_{apk}} = \begin{bmatrix} 0 \\ . \\ 0 \\ 1 \\ \cdots \\ 0 \\ . \\ 0 \end{bmatrix} \begin{matrix} \uparrow \\ r_{npj} \\ \downarrow \\ \uparrow \\ r_{apk} \\ \downarrow \end{matrix}$$

$$[E'_{P_6} \vdots A'_{P_6}] = \text{block diag.} [J_{r_{npl}+\eta_s} \vdots \tilde{\mathbf{1}}_{r_{npl}+\eta_s}] \quad B'_{P_6} = 0 \tag{4.13}$$

with:

$$\tilde{\mathbf{1}}_{r_{npl}+\eta_s} = \mathbf{1}_{r_{npl}+\eta_s} - \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ & & \cdots & & \\ & \cdots & 0 & \cdots & \\ & \cdots & 0\ 1\ 0 & \cdots & \\ & \cdots & 0 & \cdots & \\ & & \cdots & & \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \uparrow \\ r_{npl} \\ \downarrow \\ \uparrow \\ \eta_s \\ \downarrow \end{matrix}$$
$$\leftarrow \ r_{npl} \ \rightarrow\leftarrow \ \eta_s \ \rightarrow$$

For each subsystem, we can describe the Kronecker structure of each block of $[\lambda E'_{P_i} - A'_{P_i}]$ and $[\lambda E'_{P_i} - A'_{P_i} \vdots B'_{P_i}]$:

i	block of $[\lambda E'_{P_i} - A'_{P_i}]$	block of $[\lambda E'_{P_i} - A'_{P_i} \vdots B'_{P_i}]$
1	f.e.d. of order k_{ij} (at α_i)	f.e.d. of order k_{ij} & zero c.m.i.
2	f.e.d. of order r_{pi} at $\lambda = 0$	c.m.i of order r_{pi}
3	i.e.d. of order r_{anpi}	i.e.d. of order r_{anpi} & zero c.m.i.
4	i.e.d. of order r_{npi}	i.e.d. of order 1 & c.m.i. of order $r_{npi} - 1$
5	i.e.d. of order $r_{npj} + r_{apk}$	i.e.d. of order $r_{apk} + 1$ & c.m.i. of order $r_{npj} - 1$
6	c.m.i. of order $r_{npl} - 1$ & r.m.i. of order η_s	c.m.i. of order $r_{npl} - 1$ & r.m.i. of order η_s & zero c.m.i.

Dynamic properties of (E, A, B) can also be inspected through the existence $(\times)$ or absence $(\emptyset)$ of certain invariants:

Property of (E, A, B)	$(s - \alpha_i)^{k_{ij}}$	η_i	r_{npi}	r_{pi}	r_{anpi}	r_{api}
General (E, A, B)	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
P. D. "regularizability"	$\times$	$\emptyset$	$\times$	$\times$	$\times$	$\times$
P. "regularizability"	$\times$	$\emptyset$	$\times$	$\times$	$\times$	$\times$
P. D. "properizability"	$\times$	$\emptyset$	$\times$	$\times$	$=1$	$=1$
P. "properizability"	$\times$	$\emptyset$	$\times$	$\times$	$=1$	$\emptyset$
P. D. "strictly properizability"	$\times$	$\emptyset$	$\times$	$\times$	$\emptyset$	$\emptyset$
P. "strictly properizability"	$\times$	$\emptyset$	$\emptyset$	$\times$	$\emptyset$	$\emptyset$
P. D. controllability	$\emptyset$	$\emptyset$	$\times$	$\times$	$=1$	$=1$
P. controllability	$\emptyset$	$\emptyset$	$\times$	$\times$	$=1$	$\emptyset$
P. D. reachability	$\emptyset$	$\emptyset$	$\times$	$\times$	$\emptyset$	$\emptyset$
P. reachability	$\emptyset$	$\emptyset$	$\times$	$\times$	$\emptyset$	$\emptyset$
regularity for all F_P	$\times$	$\emptyset$	$\emptyset$	$\times$	$\times$	$\emptyset$
regularity for all F_P & all F_D	$\times$	$\emptyset$	$\emptyset$	$\emptyset$	$\times$	$\emptyset$

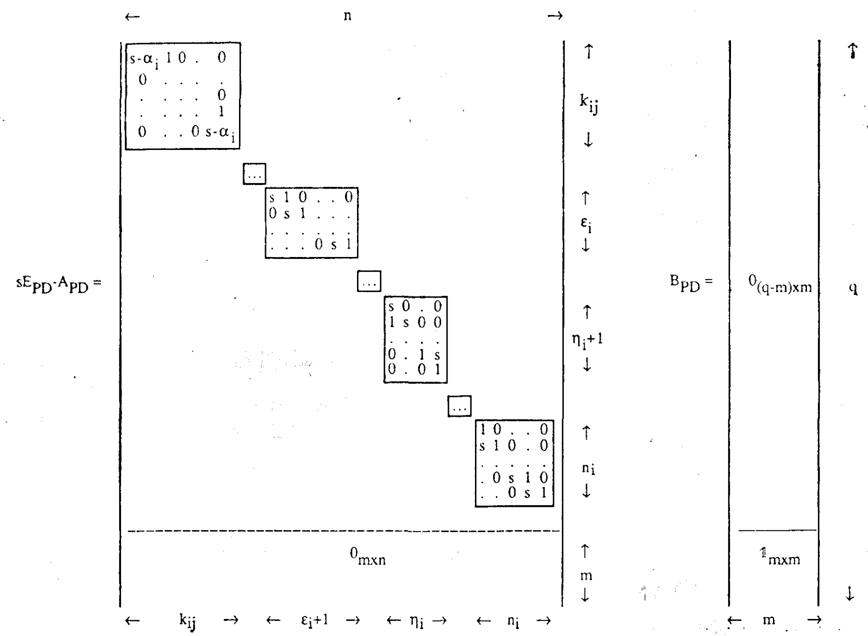
5. CONCLUSION

We have characterized the orbits of the set Σ of all singular systems (E, A, B) with E and A non square, under the actions of both Proportional & Derivative and Pure Proportional Feedback groups. Non square (E, A, B) models can be very usefull, for instance (with $q < n$) for describing systems with extra degrees of freedom (we could say with extra "unspecified inputs") which the designer does not want to choose once and for ever in his model. As a direct application of this flexibility offered by non square descriptions, we can mention the description of some variable structure systems (systems with variable order, variable gain, and to some extend time-variable parameters) (see [1]).

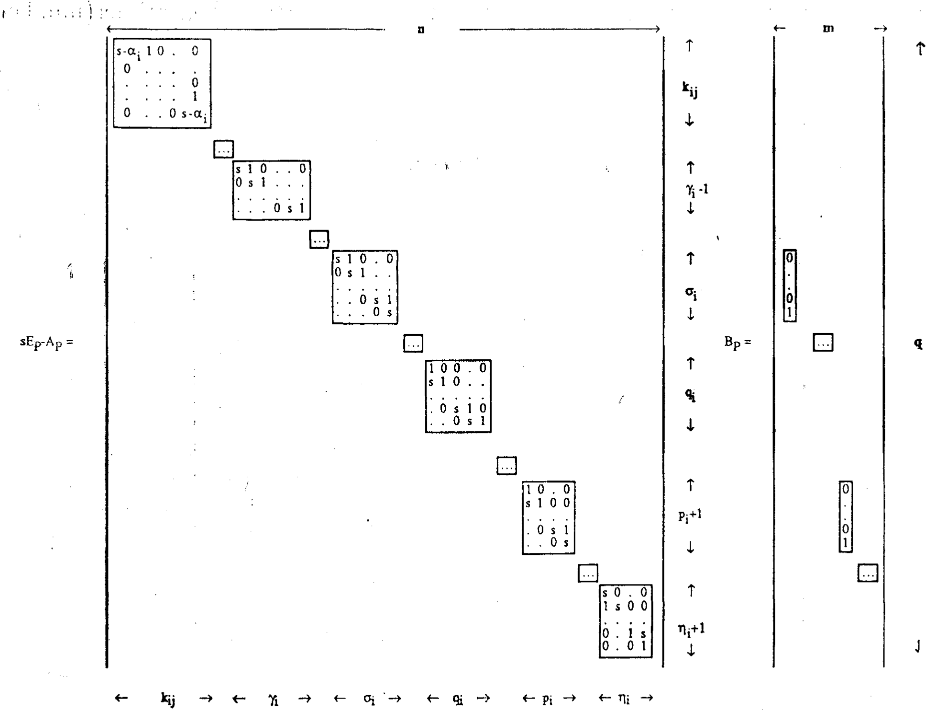
No restrictive assumption (like regularity, regularizability, controllability, ...) has been made. Our results are very similar to those of [4] obtained independently but restricted only to the case of regular & controllable systems. Moreover, it has to be noticed that this kind of algebraic P. canonical form has been extended to systems with an output equation [8] (the transformation group also includes P or P. D. output injections and changes of bases in the output space.).

To conclude we must note that further geometric and dynamical interpretations of the indices introduced in the paper and justifications/modifications of the terminology need to be developed in the general case of non square state descriptions (number of equations different from the number of state variables.).

APPENDIX 1: P. D. CANONICAL FORM



APPENDIX 2: ALGEBRAIC P. CANONICAL FORM



APPENDIX 3: TRANSFORMATIONS LEADING TO THE P. ALGEBRAIC FORM

This Appendix is devoted to the development of the elementary $(W, V, G, 0, 0)$ transformations used in Step 1 and Step 2 of our procedure for the obtention of our Algebraic P. Canonical Form from the P.D. Canonical one (recall that Step 3 is just the application of some Proportional Feedback which will cancell the remaining Φ_P term). More precisely we can describe some Algorithm giving rise to the canonical form of Theorem 3.1:

We can always assume that $sE - A$ and B have the following special form (if not, an abvious $(W_0, V_0, G_0, 0, 0)$ transformation does the job):

$$sE - A = \begin{bmatrix} sE_K - A_K \\ \hline \Phi(s) \end{bmatrix} \quad B = \begin{bmatrix} \mathbf{0}_{(q-m)\times m} \\ \hline \mathbf{1}_{m \times m} \end{bmatrix}$$

with $sE_K - A_K$ the Kronecker Normal form of $N(sE - A)$ (recall (2.4)–(2.7)) and $\Phi(s) = s\Phi_D + \Phi_P$.

i) With the help of invertible rows operations, it is possible to cancel most of the Φ_{Dij}'s entries except those corresponding to the last column of $L_{\varepsilon i}$ or L_{ni} blocks (recall (2.3)–(2.5)).

ii) With the help of invertible column operations, it is possible to reorder the columns of $(sE - A)$ in such a way that $\Phi(s)$ has the following form with $\varepsilon_1 \leqq \varepsilon_2 \leqq \ldots$ $\ldots$ and $n_1 \geqq n_2 \geqq \ldots$:

← ε_1	→	← ε_2	→	...	← n_1	→	← n_2	→	
	×		×			×		×	
	·		·			·		·	
(0)	·	(0)	·	...	(0)	·	(0)	·	...
	·		·			·		·	
	×		×			×		×	

where × stands for unspecified entries. We shall note $l_1 = \varepsilon_1$, $l_2 = \varepsilon_1 + \varepsilon_2, \ldots$, the corresponding unspecified columns of $\Phi(s)$.

iii) If the first column of this reordered $\Phi(s)$ is not zero, then perform the following reductions. Else go directly to iv).

iii) a) Normalize column l_1 in order that $\Phi(s)$ has the following form:

← ε_1	→	← ε_2	→	...	← n_1	→	← n_2	→	
	1		×			×		×	
	0		·			·		·	
(0)	·	(0)	·	...	(0)	·	(0)	·	...
	·		·			·		·	
	0		×			×		×	

iii b) Reduce the first row, with the help of elementary transformations of type T_1 and T_2 (described below):

← ε_1	→	← ε_2	→	...	← n_1	→	← n_2	→	
	1		0			0		0	
	0		×			×		×	
(0)	·	(0)	·	...	(0)	·	(0)	·	...
	·		·			·		·	
	0		×			×		×	

iv) consider then the following column l_2, and perform a similar treatment as in iii). Do this for each column l_i, and use elementary transformations of type T_3 when columns of blocks L_{n_i}'s are only concerned.

v) When all the columns l_i have been normalized, then reorder the different blocks and just obtain the desired result.

Transformations $T_1 = (W_1, V_1, G_1, 0, 0)$: cancellation of a cross entry $(X = \alpha s)$ in the case $\varepsilon_i \leqq \varepsilon_j$.

$$W_1 = \mathbf{1}_q + W(\varepsilon_i, \varepsilon_j), \quad V_1 = \mathbf{1}_n + V(\varepsilon_i, \varepsilon_j), \quad G_1 = \mathbf{1}_m$$

with all blocks in $W(\varepsilon_i, \varepsilon_j)$ equal to 0 except the block $W\varepsilon_i\varepsilon_j$ in row stripe ε_i and in column stripe ε_j, with all blocks in $V(\varepsilon_i, \varepsilon_j)$ equal to 0 except the block $V\varepsilon_i\varepsilon_j$ in row stripe $\varepsilon_i + 1$ and in column stripe $\varepsilon_j + 1$, and:

$$W\varepsilon_i\varepsilon_j = \underset{\longleftarrow \quad \varepsilon_j \quad \longrightarrow}{\begin{bmatrix} 0 & . & 0 & \alpha & 0 & . & 0 \\ 0 & . & 0 & 0 & \alpha & 0 & . \\ . & . & . & . & . & . & . \\ 0 & . & 0 & 0 & . & 0 & \alpha \end{bmatrix}} \begin{matrix} \uparrow \\ \varepsilon_i \\ \downarrow \end{matrix} \qquad V\varepsilon_i\varepsilon_j = \underset{\longleftarrow \quad \varepsilon_j + 1 \quad \longrightarrow}{\begin{bmatrix} 0 & . & 0 & -\alpha & 0 & . & 0 \\ 0 & . & 0 & 0 & -\alpha & 0 & . \\ . & . & . & . & . & 0 & . \\ 0 & . & 0 & 0 & . & 0 & -\alpha \end{bmatrix}} \begin{matrix} \uparrow \\ \varepsilon_i + 1 \\ \downarrow \end{matrix}$$

Transformations $T_2 = (W_2, V_2, G_2, 0, 0)$: cancellation of a cross entry $(X = \alpha s)$ in the case ε_i, n_j.

$$W_2 = \mathbf{1}_q + W(\varepsilon_i, n_j), \quad V_2 = \mathbf{1}_n + V(\varepsilon_i, n_j), \quad G_2 = \mathbf{1}_m$$

with all blocks in $W(\varepsilon_i, n_j)$ equal to 0 except the block $W\varepsilon_i n_j$ in row stripe ε_i and in column stripe n_j, with all blocks in $V(\varepsilon_i, n_j)$ equal to 0 except the block $V\varepsilon_i n_j$ in row stripe $\varepsilon_i + 1$ and in column stripe n_j, and:

$$W\varepsilon_i n_j = \underset{\longleftarrow \quad n_j \quad \longrightarrow}{\begin{bmatrix} 0 & . & 0 & \alpha & 0 & . & 0 \\ 0 & . & 0 & 0 & \alpha & 0 & \\ . & . & . & . & . & . & . \\ 0 & . & 0 & 0 & . & 0 & \alpha \end{bmatrix}} \begin{matrix} \uparrow \\ \varepsilon_i \\ \downarrow \end{matrix} \text{ if } n_j \geqq \varepsilon_i \text{ and } W\varepsilon_i n_j = \underset{\longleftarrow \ n_j \ \longrightarrow}{\begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ \alpha & 0 & . & 0 \\ . & . & . & . \\ 0 & . & 0 & \alpha \end{bmatrix}} \begin{matrix} \uparrow \\ \varepsilon_i \\ \downarrow \end{matrix} \text{ if } n_j \leqq \varepsilon_i$$

$$V\varepsilon_i n_j = \underset{\longleftarrow \quad n_j \quad \longrightarrow}{\begin{bmatrix} 0 & . & 0 & -\alpha & 0 & . & 0 \\ 0 & . & 0 & 0 & -\alpha & 0 & . \\ . & . & . & . & . & . & . \\ 0 & . & 0 & 0 & . & 0 & -\alpha \end{bmatrix}} \begin{matrix} \uparrow \\ \varepsilon_i + 1 \\ \downarrow \end{matrix} \text{ if } n_j \geqq \varepsilon_i + 1 \text{ and}$$

$$V\varepsilon_i n_j = \underset{\longleftarrow \ n_j \ \longrightarrow}{\begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ -\alpha & 0 & . & 0 \\ . & 0 & . & . \\ 0 & . & 0 & -\alpha \end{bmatrix}} \begin{matrix} \uparrow \\ \varepsilon_i + 1 \\ \downarrow \end{matrix} \text{ if } n_j < \varepsilon_i + 1$$

Transformations $T_3 = (W_3, V_3, G_3, 0, 0)$: cancellation of a cross entry ($X = \alpha s$) in the case $n_i \geqq n_j$.

$$W_2 = \mathbf{1}_q + W(n_i, n_j), \quad V_2 = \mathbf{1}_n + V(n_i, n_j), \quad G_2 = \mathbf{1}_m$$

with all blocks in $W(n_i, n_j)$ equal to 0 except the block Wn_in_j in row stripe n_i and in column stripe n_j, with all blocks in $V(n_i, n_j)$ equal to 0 except the block Vn_in_j in row stripe n_i and in column stripe n_j, and:

$$Wn_in_j = \underbrace{\begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ \alpha & 0 & . & 0 \\ . & . & . & . \\ 0 & . & 0 & \alpha \end{bmatrix}}_{\leftarrow\ n_j\ \rightarrow} \updownarrow n_i \quad \text{and} \quad Vn_in_j = \underbrace{\begin{bmatrix} 0 & . & . & 0 \\ . & . & . & . \\ 0 & . & . & 0 \\ -\alpha & 0 & . & 0 \\ . & . & . & . \\ 0 & . & 0 & -\alpha \end{bmatrix}}_{\leftarrow\ n_j\ \rightarrow} \updownarrow n_i$$

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