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NATURAL TRANSFORMATIONS TRANSFORMING FUNCTIONS AND VECTOR FIELDS TO FUNCTIONS ON SOME NATURAL BUNDLES

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Summary. A classification of natural transformations transforming functions (or vector fields) to functions on such natural bundles which are restrictions of bundle functors is given.

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1. Bundle functors. All manifolds in this paper are assumed to be paracompact, without boundary, second countable, finite dimensional and of class C^∞. Maps between manifolds will be assumed to be C^∞. Let Mf be the category of all manifolds and all maps, FM the category of all fibered manifolds and their morphisms, and let $B\colon FM \to Mf$ be the base functor. Given a functor $G\colon Mf \to FM$ satisfying $B \circ G = id_{Mf}$ we denote by $p^G_M\colon GM \to M$ its value on a manifold M and by $Gf\colon GM \to GN$ its value in $f\colon M \to N$.

Definition 1.1 ([3]). A bundle functor is a functor $G\colon Mf \to FM$ satisfying $B \circ G = id_{Mf}$ and the following localization condition: if $i\colon U \to M$ is the inclusion of an open subset, then $Gi\colon GU \to \left(p^G_M\right)^{-1}(U)$ is a diffeomorphism.

The Weil functors of A-velocities and the linear functors of higher order tangent bundles are bundle functors ([4], [2]).

Let M, N, P be manifolds. A parametrized family of maps $f_p\colon M \to N$, $p \in P$ is said to be smoothly parametrized if the resulting map $f\colon M \times P \to N$ is of class C^∞.

Proposition 1.1 ([3]). *Every bundle functor $G\colon Mf \to FM$ satisfies the following regularity condition: if $f\colon M \times P \to N$ is a smoothly parametrized family, then the family $\widetilde{Gf}\colon GM \times P \to GN$, $(\widetilde{Gf})_p = G(f_p)$ is also smoothly parametrized.*

Let $G\colon Mf \to FM$ be a bundle functor. If we replace Mf by the category Mf_m of all m-dimensional manifolds and their embeddings, we obtain the concept of natural bundles ([5]). Hence the restriction G_m of G to Mf_m is a natural bundle.

2. Natural transformations transforming functions and vector fields to functions on natural bundles. Let E be a natural bundle on Mf_m. Given a manifold P we denote by $C^\infty(P)$ and $\mathcal{X}(P)$ the vector spaces of all maps $P \to \mathbf{R}$ and all vector fields on P, respectively. Similarly as in [1], we introduce the following definitions.

Definition 2.1. A system $T = \{T(M)\}$ of functions

$$T(M)\colon C^\infty(M) \to C^\infty(EM), \quad M \in \mathrm{obj}(Mf_m)$$

is called a natural transformation transforming functions to functions on E if the following naturality conditon is satisfied: for any $M, N \in \mathrm{obj}(Mf_m)$, any $f \in C^\infty(N)$ and any embedding $\varphi\colon M \to N$ we have $T(M)(f \circ \varphi) = T(N)(f) \circ E\varphi$.

Definition 2.2. A system $T = \{T(M)\}$ of functions

$$T(M)\colon \mathcal{X}(M) \to C^\infty(EM), \quad M \in \mathrm{obj}(Mf_m)$$

is called a natural transformation transforming vector fields to functions on E if the following naturality condition is satisfied: for any $M, N \in \mathrm{obj}(Mf_m)$, any $X \in \mathcal{X}(M)$, any $Y \in \mathcal{X}(N)$ and any embedding $\varphi\colon M \to N$ the assumption $d\varphi(X) = Y \circ \varphi$ implies $T(M)(X) = \big(T(N)(Y)\big) \circ E\varphi$.

Denote by $\mathrm{Trans}_f(E)$ ($\mathrm{Trans}_v(E)$) the set of all natural transformations transforming functions (vector fields) to functions on E. For any $T_1, T_2 \in \mathrm{Trans}_f(E)$ and $\lambda \in \mathbf{R}$ define $T_1 + T_2$, $\lambda T_1 \in \mathrm{Trans}_f(E)$ to be the systems of functions

$$(T_1 + T_2)(M)\colon C^\infty(M) \to C^\infty(EM), \; (T_1 + T_2)(M)(f) = T_1(M)(f) + T_2(M)(f)$$

and

$$(\lambda T_1)(M)\colon C^\infty(M) \to C^\infty(EM), \quad (\lambda T_1)(M)(f) = \lambda\big(T_1(M)(f)\big),$$

$M \in \mathrm{obj}(Mf_m)$. Then $\mathrm{Trans}_f(E)$ is a vector space. Similarly, $\mathrm{Trans}_v(E)$ is also a vector space.

3. Main results.

3. Main results. Let $G\colon \mathcal{M}f \to \mathcal{FM}$ be a bundle functor and m a natural number. Let G_m be the restriction of G to $\mathcal{M}f_m$. We fix a one-point manifold $Q \in \mathrm{obj}(\mathcal{M}f_0)$. Given $h \in C^\infty(G\mathbf{R})$ define $T^h \in \mathrm{Trans}_f(G_m)$ by

$$T^h(M)\colon C^\infty(M) \to C^\infty(GM), \quad T^h(M)(f) = h \circ Gf, \quad M \in \mathrm{obj}(\mathcal{M}f_m).$$

Similarly, for any $h \in C^\infty(GQ)$ define $T_h \in \mathrm{Trans}_v(G_m)$ by

$$T_h(M)\colon \mathcal{X}(M) \to C^\infty(GM), \quad T_h(M)(X) = h \circ G_{q_M}, \ M \in \mathrm{obj}(\mathcal{M}f_m),$$

where $q_M\colon M \to Q$ is a constant map. The main results in this paper are the following theorems.

Theorem 3.1. *The function*

$$G^*\colon C^\infty(G\mathbf{R}) \to \mathrm{Trans}_f(G_m), \quad G^*(h) = T^h$$

is a linear isomorphism. The inverse isomorphism is given by

$$I^*\colon \mathrm{Trans}_f(G_m) \to C^\infty(G\mathbf{R}), \quad I^*(T) = T(\mathbf{R}^m)(p) \circ Gj,$$

where $p\colon \mathbf{R}^m \to \mathbf{R}$ is the projection onto the first factor and $j\colon \mathbf{R} \to \mathbf{R}^m$ is defined by $j(x) = (x, 0, \dots, 0)$.

Theorem 3.2. *If $m \geqslant 2$, then the function*

$$^*G\colon C^\infty(GQ) \to \mathrm{Trans}_v(G_m), \quad {}^*G(h) = T_h$$

is a linear isomorphism. The inverse isomorphism is given by

$$^*I(T) = T(\mathbf{R}^m)(\partial_1) \circ G\tilde{O},$$

where $\partial_1 = \partial/\partial_{x^1}$ is the canonical vector field on $\mathbf{R}^m$ and $\tilde{O}\colon Q \to \{0\} \subset \mathbf{R}^m$.

4. Proof of Theorem 3.1.

4. Proof of Theorem 3.1. It is easy to see that G^* is linear. For any $h \in C^\infty(G\mathbf{R})$ we have $I^* \circ G^*(h) = h \circ Gp \circ Gj = h \circ G(p \circ j) = h$ as $p \circ j = id_{\mathbf{R}}$. Then $I^* \circ G^* = id$. It remains to show the following proposition. □

Proposition 4.1. *If $T_1, T_2 \in \mathrm{Trans}_f(G_m)$ are two natural transformations such that $T_1(\mathbf{R}^m)(p) = T_2(\mathbf{R}^m)(p)$ on $Im(Gj)$, then $T_1 = T_2$.*

Using Proposition 4.1 one can prove that $G^* \circ I^* = id$ as follows. Consider an arbitrary $T \in \mathrm{Trans}_f(G_m)$. Denote $G^* \circ I^*(T) = \overline{T}$. Then

$$\overline{T}(\mathbf{R}^m)(p) \circ Gj = I^*(T) \circ Gp \circ Gj = T(\mathbf{R}^m)(p) \circ Gj.$$

Then by Proposition 4.1. we have $T = \overline{T}$ as well.

Now we shall prove Proposition 4.1. From now on we consider two natural transformations T_1, T_2 such that the assumption of the proposition is satisfied.

Lemma 4.1. $T_1(\mathbf{R}^m)(p) = T_2(\mathbf{R}^m)(p)$.

Proof of the lemma. If $m = 1$, then the assertion is obvious. Let $m \geqslant 2$. Consider an arbitrary $y \in \mathbf{R}^m$. Put $y^* = Gj \circ Gp(y)$. Let $f_t \in \mathbf{R}^m \to \mathbf{R}^m$, $t \in \mathbf{R}$ be the smoothly parametrized family given by $f_t(x) = (x^1, tx^2, \ldots, tx^m)$. Of course, $p \circ f_t = p$, $f_0 = j \circ p$ and f_t is a diffeomorphism provided $t \neq 0$. Hence $T_1(\mathbf{R}^m)(p) = T_1(\mathbf{R}^m)(p) \circ Gf_t$ if $t \neq 0$, and by Proposition 1.1, $Gf_t(y) \to y^*$ as $t \to 0$. Therefore $T_1(\mathbf{R}^m)(p)(y) = T_1(\mathbf{R}^m)(p)(y^*)$. Similarly, $T_2(\mathbf{R}^m)(p)(y) = T_2(\mathbf{R}^m)(p)(y^*)$. Since $y^* \in Im(Gj)$ we have $T_1(\mathbf{R}^m)(p)(y^*) = T_2(\mathbf{R}^m)(p)(y^*)$, and then $T_1(\mathbf{R}^m)(p)(y) = T_2(\mathbf{R}^m)(p)(y)$ as well.

We are now in position to prove Proposition 4.1. Let $M \in \mathrm{obj}(\mathcal{M}f_m)$ be a manifold, let $x \in M$ be a point and $f \in C^\infty(M)$ a map. The proof will be complete after proving that $T_1(M)(f) = T_2(M)(f)$ on the fibre over x.

Assume that $d_x f \neq 0$. There exist an open neighbourhood W of x and a chart $\varphi = (f|W, \varphi^2, \ldots, \varphi^m)$ defined on W. Of course, $f \circ i = p \circ \varphi \circ i$, where $i\colon W \to M$ is the inclusion. It follows from the naturality condition that

$$T_\alpha(M)(f) \circ Gi = T_\alpha(W)(f \circ i) = T_\alpha(\mathbf{R}^m)(p) \circ G(\varphi \circ i)$$

for $\alpha = 1, 2$. Therefore $T_1(M)(f) = T_2(M)(f)$ over x because of Lemma 4.1 and the localization condition.

Now we do not assume that $d_x f \neq 0$. There exist two open subsets $U, V \subset M$ and a map $g \in C^\infty(M)$ such that $x \in \overline{U} \cap \overline{V}$, $g|U = f|U$ and $dg \neq 0$ at each point from V. By the localization condition $T_1(M)(f) = T_1(M)(g)$ over U. By the first case $T_1(M)(g) = T_2(M)(g)$ over V. Thus $T_1(M)(f) = T_1(M)(g) = T_2(M)(g) = T_2(M)(f)$ over x. Proposition 4.1 is proved. □

5. Proof of Theorem 3.2. Using similar arguments to those in the proof of Theorem 3.1 we see that Theorem 3.2 will be proved by proving the following proposition.

Proposition 5.1. *If $T_1, T_2 \in \mathrm{Trans}_v(G_m)$ are two natural transformations such that $T_1(\mathbf{R}^m)(\partial_1) = T_2(\mathbf{R}^m)(\partial_1)$ on $Im(G\tilde{O})$ and $m \geqslant 2$, then $T_1 = T_2$.*

From now on we consider two natural transformations T_1, T_2 such that the assumptions of Proposition 5.1 are satisfied.

Lemma 5.1. $T_1(\mathbf{R}^m)(\partial_1) = T_2(\mathbf{R}^m)(\partial_1)$.

Proof of the lemma. Consider an arbitrary $y \in G\mathbf{R}^m$. Let j, p be as in Theorem 3.1 and let $k\colon \mathbf{R} \to \mathbf{R}^m$ be given by $k(y) = (0, y, 0, \dots, 0)$. Let $f_t, g_t\colon \mathbf{R}^m \to \mathbf{R}^m$, $t \in \mathbf{R}$ be two smoothly parametrized families given by $f_t(x) = (x^1, tx^2, \dots, tx^m)$ and $g_t(x) = (x^1 + x^2, tx^2, \dots, tx^m)$. We see that f_t, g_t are diffeomorphisms preserving ∂_1 provided $t \neq 0$. Using the naturality condition and Proposition 1.1 we derive that

$$T_1(\mathbf{R}^m)(\partial_1)(y) = T_1(\mathbf{R}^m)(\partial_1) \circ Gf_t(y) \to T_1(\mathbf{R}^m)(\partial_1) \circ Gj \circ Gp(y)$$

as $t \to 0$, and then

$$T_1(\mathbf{R}^m)(\partial_1)(y) = T_1(\mathbf{R}^m)(\partial_1) \circ Gj \circ Gp(y).$$

In particular,

$$T_1(\mathbf{R}^m)(\partial_1) \circ Gk \circ Gp(y) = T_1(\mathbf{R}^m)(\partial_1) \circ G\tilde{O} \circ Gq_{\mathbf{R}^m}(y).$$

Using the family g_t instead of f_t we obtain that

$$T_1(\mathbf{R}^m)(\partial_1) \circ Gk \circ Gp(y) = T_1(\mathbf{R}^m)(\partial_1) \circ Gj \circ Gp(y).$$

Hence

$$T_1(\mathbf{R}^m)(\partial_1)(y) = T_1(\mathbf{R}^m)(\partial_1) \circ G\tilde{O} \circ Gq_{\mathbf{R}^m}(y),$$

and similarly for T_2 playing the role of T_1. Therefore from the assumption of the proposition we conclude $T_1(\mathbf{R}^m)(\partial_1)(y) = T_2(\mathbf{R}^m)(\partial_1)(y)$ as well.

We are now in position to prove Proposition 5.1. Let $M \in \mathrm{obj}(\mathcal{M}f_m)$ be a manifold, $x \in M$ a point and $X \in \mathcal{X}(M)$ a vector field. The proof will be complete after proving that $T_1(M)(X) = T_2(M)(X)$ on the fibre over x.

Suppose that $X_x \neq 0$. There exists an open neighbourhood W of x and a chart φ defined on W such that $d\varphi^{-1}(\tilde{\partial}_1) = X \circ \varphi^{-1}$, where $\tilde{\partial}_1 \in \mathcal{X}(Im\varphi)$ is the restriction of ∂_1. Using the localization and naturality conditions and Lemma 5.1 we obtain that $T_1(M)(X) = T_2(M)(X)$ over x.

Now we do not assume that $X_x \neq 0$. There exist two open subsets $U, V \subset M$ and a vector field $Y \in \mathcal{X}(M)$ such that $x \in \overline{U} \cap \overline{V}$, $X|U = Y|U$ and $Y \neq 0$ at each point from V. By the localizatin condition $T_1(M)(Y) = T_1(M)(X)$ over U. By the first case $T_1(M)(Y) = T_2(M)(Y)$ over V. Hence $T_1(M)(X) = T_1(M)(Y) = T_2(M)(Y) = T_2(M)(X)$ over x. Proposition 5.1 is proved. $\square$

6. Applications. Let $G\colon Mf \to FM$ be a bundle functor such that $G\mathbf{R}$ is diffeomorphic to $\mathbf{R}^N$, $N = \dim G\mathbf{R}$. (For example, if G has a point property, i.e. $G(\text{point}) = \text{point}$, then $G\mathbf{R}$ is diffeomorphic to $\mathbf{R}^N$, see [3].) Let $H = (H^1, \dots, H^N)\colon G\mathbf{R} \to \mathbf{R}^N$ be a diffeomorphism. By virtue of Theorem 3.1 one can define N natural transformations $T^1, \dots, T^N \in \mathrm{Trans}_f(G_m)$ by $T^i = G^*(H^i)$. We have the following corollary of Theorem 3.1.

Corollary 6.1. *The function*

$$C^\infty(\mathbf{R}^N) \ni \Phi \to \Phi \circ (T^1, \dots, T^N) \in \mathrm{Trans}_f(G_m)$$

is a linear isomorphism.

Proof. The function is equal to $G^* \circ H^*$, where G^* is the isomorphism defined in Theorem 3.1 and $H^*\colon C^\infty(\mathbf{R}^N) \to C^\infty(G\mathbf{R})$ is the isomorphism given by $H^*(f) = f \circ H$. $\square$

In particular, when $G = T^{p,r}$ is the bundle of p^r-velocities ([1]) one can consider $H = (H^\alpha)\colon T^{p,r}\mathbf{R} \to \mathbf{R}^N$, the diffeomorphism given by $H^\alpha(j_0^r\gamma) = \frac{1}{\alpha!}D_\alpha\gamma(0)$, $\alpha \in (\mathbf{N} \cup \{0\})^p$, $|\alpha| \leqslant r$. Of course, $G^*(H^\alpha) = \mathcal{L}^\alpha$ are the α-lifts of functions to $T^{p,r}$, [1]. Therefore we have the following subcorollary.

Corollary 6.2. *The function*

$$C^\infty(\mathbf{R}^{\dim T^{p,r}\mathbf{R}}) \ni \Phi \to \Phi \circ \left(\mathcal{L}^\alpha : \alpha \in (\mathbf{N} \cup \{0\})^p, |\alpha| \leqslant r\right) \in \mathrm{Trans}_f(T^{p,r}|Mf_m)$$

is a linear isomorphism.

Now, we give an application of Theorem 3.2. Let $G\colon Mf \to FM$ be a bundle functor and $m \geqslant 2$ a natural number. Let $\mathcal{L} = \{\mathcal{L}_M\}$, $\mathcal{L}_M\colon \mathcal{X}(M) \to \mathcal{X}(GM)$, $M \in \mathrm{obj}(Mf_m)$ be a quasi-lifting of vector fields to G_m, [1]. Owing to the decomposition theorem, [1], $\mathcal{L} = \mathcal{L}^v + c(\ )G(\ )$, where $\mathcal{L}^v$ is the lifting of vertical type, $G(\ )$ is the complete lifting of vector fields to G_m and $c(\ )$ is the element of $\mathrm{Trans}_v(G_m)$. $\mathcal{L}$ is a lifting of vector fields to G_m, [1], if and only if $c(\ ) = c \in \mathbf{R}$. Therefore we have the following corollary of Theorem 3.2. (This corollary generalizes the result of [1, p. 41].)

Corollary 6.3. *Any quasi-lifting of vector fields to G_m is a lifting if and only if G has a point property.*

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