ABSOLUTE END POINTS OF IRREDUCIBLE CONTINUA

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Summary. A concept of an absolute end point introduced and studied by Ira Rosenholtz for arc-like continua is extended in the paper to be applied arbitrary irreducible continua. Some interrelations are studied between end points, absolute end points and points at which a given irreducible continuum is smooth.

Keywords: arc-like, absolute end point, continuum, decomposition, end point, irreducible, locally connected, smooth

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1. INTRODUCTION

Rosenholtz has proved ([9], Proposition 1.3, p. 1309) an equivalence of certain four conditions related to end points of arc-like continua in the sense of Bing ([2], p. 660) and defined a concept of an absolute end point as a point of an arc-like continuum that satisfies any one of them. It is observed in the paper that each of the conditions under consideration is also equivalent to two other conditions and that the scope of application of the definition can be extended to arbitrary irreducible continua (not necessarily arc-like ones).

2. PRELIMINARIES

All spaces considered in this paper are assumed to be metric and all mappings are continuous. A *continuum* means a compact connected space. If points p and q are in a continuum X and no proper subcontinuum of X contains both p and q, then X is said to be *irreducible between p and q*. If a continuum is irreducible between a pair of its points, then it is said to be *irreducible*. A continuum is called a *triod* provided it contains a subcontinuum whose complement is the union of three nonempty pairwise disjoint open sets. A continuum that contains no subcontinuum which is itself a triod is said to be *atriodic*. A continuum is said to be *decomposable* provided it is the union of two proper subcontinua. Otherwise it is called *indecomposable*. A continuum is said to be (a) *unicoherent*, if it is not the union of two subcontinua whose intersection is not connected, (b) *hereditarily unicoherent*, if each its subcontinuum is unicoherent, (c) *hereditarily unicoherent at a point* p, if the intersection of any two its subcontinua, each of which contains p, is connected. It is easily verified that a continuum X is hereditarily unicoherent if and only if it is hereditarily unicoherent at each of its points, and that it is hereditarily unicoherent at p if and only if, given any point $x \in X$, there exists a unique subcontinuum of X which is irreducible between p and x (see Gordh [5], Theorem 1.3, p. 52).

If a point p of a continuum X is given, then the *composant of* X *belonging to* p is defined to be the union of all proper subcontinua of X which contain p. A set is a *composant of* X provided it is the composant of X belonging to p for some point p of X. A continuum X is said to be (a) *locally connected at a point* p provided each open subset U of X containing p contains an open connected subset also containing p; (b) *connected im kleinen at a point* p of X provided that for each open set U of X containing p, the point p lies in the interior of a connected subset of U; (c) *semi-locally connected at a point* p provided that if U is an open set containing p, then there is an open set V such that $p \in V \subset U$ and finitely many components of the complement of V cover the complement of U. Let p and q be distinct points of a continuum X. We say that X is *aposyndetic at* p *with respect to* q provided there is a subcontinuum of X containing p in its interior and not containing q.

A continuum is said to be *arc-like* (or *chainable*, or *snake-like*) if for each positive number ε there exists an ε-chain covering it. The following well-known theorem summarizes the most important properties of arc-like continua.

Theorem 2.1. *If a continuum X is arc-like, then each subcontinuum of X is arc-like, unicoherent, atriodic, and irreducible. Moreover, it can be homeomorphically embedded into the Euclidean plane.*

Bing in [2], Theorem 13, p. 661 proved the following result.

Theorem 2.2. (Bing). *The following conditions on an arc-like continuum X and a point p of X are equivalent:*

(A) *Each nondegenerate subcontinuum of X containing p is irreducible between p and some other point.*

(B) *If each of two subcontinua of X contains p, then one of the subcontinua contains the other.*

(C) *For each positive number ε there is an ε-chain of X such that only the first link contains p.*

A point p of an arc-like continuum X is called an ***end point*** of X if p and X satisfy any one of conditions (A)–(C).

The following result is proved by Rosenholtz as Theorem 1.0 in [9], p. 1308.

Theorem 2.3. *(Rosenholtz). The following conditions on an arc-like continuum X and a point p of X are equivalent:*

(1) *If X is irreducible between points x and y, then either x or y is p.*

(2) *$X \setminus \{p\}$ is a composant of X.*

(3) *X is irreducible between p and some other point, and X is connected im kleinen at p.*

(4) *X is irreducible between p and some other point, and X is locally connected at p.*

(5) *X is locally connected at p, and p does not separate X.*

(6) *X is locally connected at p, and whenever $C_1, \ldots, C_n$ is an ε-chain of X with p belonging to a connected link C_k, then either* (a) $k = 1$, (b) $k = n$, (c) $k = 2$ *and $\overline{C_2}$ contains C_1, or* (d) $k = n - 1$ *and $\overline{C_{n-1}}$ contains C_n.*

(7) *For each positive number ε there is a positive number δ such that, if $C_1, \ldots, C_n$ is a δ-chain of X and p belongs to C_k, then either $\bigcup\{C_j : j \in \{1, \ldots, k\}\}$ or $\bigcup\{C_j : j \in \{k, \ldots, n\}\}$ is contained in an ε-neighborhood of p.*

Finally, the validity of a condition (1) *through* (7) *implies the validity of the corresponding statement for each subcontinuum containing p.*

A point p of an arc-like continuum X is called an ***absolute end point*** of X if p and X satisfy any of conditions (1)–(7) above. It is known from Rosenholtz's paper ([9], Remark, p. 1310) that each absolute end point is an end point.

3. Basic structural results

The concept of an end point, in its form introduced by Bing [2] for arc-like continua only, can easily be extended to arbitrary continua if condition (B) is taken as the definition. So, let us accept the following definition. A point p of a continuum X is called an ***end point*** of X if for each two subcontinua of X both containing p, one of the subcontinua contains the other. Note that if X is arc-like, then the new notion coincides with the original one.

In such an extended sense the concept was studied e.g. by Maćkowiak and myself in [4], where it was proved that for an arbitrary continuum X the set of its end points is a G_δ-set in X ([4], Proposition 1, p. 385). Recall that for an arbitrary continuum

X and each point p of X conditions (A) and (B) are equivalent, as was shown by Bing in [2], Theorem 12, p. 661.

Similarly to the previous remark, the concept of an absolute end point, in the form introduced in [9] by Rosenholtz for arc-like continua only, can easily be extended to arbitrary continua, e.g. in such a way that condition (2) is taken for the definition. So, a point p of a continuum X is called an ***absolute end point*** of X if $X \setminus \{p\}$ is a composant of X.

Lemma 3.1. *If a continuum contains an absolute end point p, then it is irreducible between p and some other point.*

Proof. Let a continuum X contain an absolute end point p. Then, by definition, $X \setminus \{p\}$ is a composant of X, i.e. there is a point q of X such that the union of all proper subcontinua of X that contain q is $X \setminus \{p\}$. This implies in particular that no proper subcontinuum of X contains both p and q, and therefore X is irreducible between these two points. □

By the above lemma, if absolute end points are under consideration, we can restrict our study of arbitrary continua to irreducible continua only.

Proposition 3.2. *If a continuum X is irreducible, then for any point p of X conditions (1)–(4) are equivalent. Moreover, each of them is also equivalent to the following ones:*

(8) *X is irreducible between p and some other point, and X is aposyndetic at p with respect to any other point of X.*

(9) *X is irreducible between p and some other point, and X is semi-locally connected at p.*

Proof. The equivalence of conditions (1)–(4) has been stated in Rosenholtz's paper [9] as Proposition 1.3, p. 1309 for arc-like continua. However, the same proof is in fact valid for our more general setting, with the only change that the unique irreducible subcontinua between the points considered (the uniqueness is a consequence of the condition that each subcontinuum of an arc-like continuum is unicoherent, see [9], Theorem 0.0, p. 1306 and Lemma 1.2, p. 1308) are replaced by arbitrary ones which do exist by Theorem 1 of §48, I, p. 192 of the Kuratowski monograph [7]. The rest of the conclusion is a consequence of Theorem 20 of Thomas' paper [10], p. 28, where it is shown that local connectedness at a point, semi-local connectedness at a point, and aposyndeticity at a point with respect to any other point of a continuum coincide provided the continuum is irreducible. □

Remark 3.3. It can be easily observed that irreducibility of the continuum X is an indispensable assumption in Proposition 3.2.

As the reader can see, of the seven conditions (1)–(7) only the last two are expressed in terms of chains; all the others are formulated in a manner which can be applied to arbitrary continua. So, we shall consider not only conditions (1)–(4) but also (5) as special properties of points of any continuum, not necessarily an arc-like one. To study some relations between conditions (1)–(4) and (5) for arbitrary (irreducible) continua, we need a lemma, which generalizes Lemma 1.4 of Rosenholtz's paper [9], p. 1310.

Lemma 3.4. *Let a point p of an irreducible continuum X be given such that X is connected im kleinen at p and hereditarily unicoherent at p. Then either p is an absolute end point of X, or X is the union of two subcontinua having only p in common and each having p as an absolute end point. In the latter case, p separates X.*

Proof. The proof of Lemma 1.4 in [9], p. 1310, presented by Rosenholtz for arc-like continua is valid here, because the only properties of the continuum X used in that proof are its irreducibility and hereditary unicoherence at the point p. □

Proposition 3.5. *If a continuum X is irreducible, then for any point p of X any of conditions (1)–(4) implies condition (5). And if it is additionally assumed that*

(3.6) $\qquad X$ *is hereditarily unicoherent at* p,

then (5) implies any of conditions (1)–(4).

Proof. By Proposition 3.2 conditions (1)–(4) are equivalent. Assume any of them. Then X is locally connected at p by (4), and since $X \setminus \{p\}$ is a composant of X by (2), it is connected. Thus (5) follows. Conversely, if (3.6) is assumed, then the conclusion follows from Lemma 3.4. □

Corollary 3.7. *If a continuum X is irreducible and hereditarily unicoherent at a point p, then conditions (1)–(5), (8) and (9) are equivalent.*

Question 3.8. Is hereditary unicoherence of X at p an essential assumption in Proposition 3.5 and Corollary 3.7?

Remark 3.9. Note that none of conditions (1)–(5) implies hereditary unicoherence of the continuum X at the point p. To see this consider the disjoint union U of two copies X_1 and X_2 of a topologist's sine curve. Let a_i and b_i be end points of the limit segment of X_i, and let c_i denote the only absolute end point of X_i for $i \in \{1, 2\}$. Next, identify a_1 with a_2 and b_1 with b_2, and let $f : U \to X$ be the identification mapping. The resulting space X is a continuum which is irreducible between two absolute end points $f(c_1)$ and $f(c_2)$, conditions (1)–(5) are satisfied with either $f(c_1)$ or $f(c_2)$ as p, while X is not hereditarily unicoherent at any point. This continuum is homeomorphic to the one described by Maćkowiak in Example 5.4 of [8], p. 89.

Proposition 3.10. *If a continuum contains an absolute end point, then it is decomposable.*

Proof. Really, every composant of an indecomposable continuum is of the first category (see [7], §48, VI, Theorem 6, p. 212), so it is a boundary set, and hence it equals the complement of $\{p\}$ for no point p of the continuum. □

Examples are known of continua for which the set of end points is dense (Maćkowiąk and the author, [4]), or even equals the whole continuum (the pseudo-arc). Nothing similar can happen if absolute end points are under consideration.

Proposition 3.11. *Each continuum has at most two absolute end points.*

Proof. If a continuum is not irreducible, then the set of its absolute end points is empty, according to Proposition 3.1. If it is irreducible, then Proposition 3.2 can be applied, and the conclusion follows from condition (1). □

Remark 3:12. Even for arc-like continua all three possibilities may occur: a pseudo-arc has no absolute end point, a topologist's sine curve has one, and an arc has two of them.

4. Decompositions

We shall observe some relations between absolute end points of an irreducible continuum X and elements of the minimal upper semi-continuous monotone linear decomposition of X (as defined and studied by Kuratowski in §48 of [7], Sections III and IV, p. 195–204, especially Theorem 3 on p. 200, and by Thomas in [10]). In terms of these decompositions two more conditions can be added to those considered above in Proposition 3.2, provided the irreducible continuum X in question has a special structure. To describe it, we recall some definitions.

A decomposition $\mathscr{D}$ of a continuum X irreducible between points x and y is said to be ***admissible*** provided it has more than one element (i.e. it is nondegenerate), is upper semi-continuous, every element of $\mathscr{D}$ is a subcontinuum of X (i.e. $\mathscr{D}$ is monotone), and every element of $\mathscr{D}$ not containing x or y separates X. An irreducible continuum which has an admissible decomposition is said to be *of type A*. It is known that for each continuum of type A there exists a minimal admissible decomposition (i.e. such that it refines any other admissible decomposition), and that this minimal decomposition is unique (see Thomas' paper [10], Theorem 3 on p. 8 and compare Theorem 3 of Kuratowski's monograph [7], §48, IV, p. 200). This unique minimal admissible decomposition of X is called the ***canonical decomposition of*** X. Elements of the canonical decomposition of an irreducible continuum X are called *layers* of X. A continuum is said to be *of type A'* if it is of type A and has an admissible decomposition each of whose elements has void interior (see [10], p. 13 and compare [7], the footnote on p. 197, where the name "of type λ" is used in the same sense). It is known that a continuum X is of type A' if and only if every subcontinuum of X with nonvoid interior is decomposable (Theorem 10 of [10], p. 15 and Theorem 3 of [7], §48, VII, p. 216). Further, if $\mathscr{D}$ is the canonical decomposition of an irreducible continuum X of type A', then the quotient space $X/\mathscr{D}$ is an arc, and we may assume that $X/\mathscr{D} = [0, 1]$.

Proposition 4.1. *Let an irreducible continuum X be of type A', with $\mathscr{D}$ as its canonical decomposition, and with $\varphi\colon X \to X/\mathscr{D}$ as the quotient mapping. Then each of conditions* (1)–(4), (8) *and* (9) *is equivalent to any of the following two:*

(10) *X is irreducible between p and some other point, and $\{p\} \in \mathscr{D}$.*

(11) *The singleton $\{p\}$ is equal either to $\varphi^{-1}(0)$ or to $\varphi^{-1}(1)$.*

P r o o f. It follows from Theorem 20 of Thomas' paper [10], p. 28, that if a continuum X is of type A', then X is locally connected at a point $x \in X$ if and only if the singleton $\{x\}$ is a member of the canonical decomposition of X. Thus the conclusion follows from Proposition 3.2. □

R e m a r k 4.2. The assumption that X is of type A' is essential in Proposition 4.1 because if the continuum X consists of a chain of indecomposable continua K_1, K_2, ... converging to a point p and of an arc A joined together in such a way that for each natural index n the intersections $K_n \cap K_{n+1}$ as well as $A \cap K_1$ are singletons (as is pictured in Fig. 7 of [10], p. 29), then X is of type A (while not of type A'), the point p is an absolute end point of X, and the singleton $\{p\}$ is an element of no admissible decomposition of X.

The following result is due to Rosenholtz, Remark on p. 1310 of [9].

Proposition 4.3. (Rosenholtz). *If a continuum X is arc-like, then each absolute end point of X is an end point of X.*

This result cannot be extended to arbitrary (irreducible) continua (i.e. the assumption that X is arc-like is indispensable in 4.3) because of an example presented below.

Example 4.4. *There exists a hereditarily unicoherent irreducible continuum X of type A' which contains two absolute end points p and q such that*

(a) *each subcontinuum of X containing q is irreducible between q and some other point,*

(b) *there is a simple triod Y in X that contains the point p, and thereby the point p does not have the property considered in* (a)*;*

(c) *the point p is not, while q is, an end point of X.*

Proof. In the Euclidean plane equipped with the Cartesian rectangular coordinate system take a topologist's sine curve S defined by

$$S = L \cup \left\{(x, y) \colon y = \sin \frac{1}{x} \text{ and } 0 < x \leqslant 1\right\},$$

where $L = \{(0, y) \colon -1 \leqslant y \leqslant 1\}$ stands for the limit arc of S. Let $m = (0,0)$ be the middle point of L, and $q = (1, \sin 1)$ the only absolute end point of S. Put $p = (-1, 0)$ and denote by A the straight line segment joining p with m. Then $X = A \cup S$ has all the properties needed. □

Remark 4.5. The same Example 4.4 shows also that Lemma 3.1 cannot be extended to subcontinua of the continuum considered which contain the absolute end point.

Note that the continuum X of Example 4.4, being not arc-like, is not atriodic, according to Theorem 2.1. However, we do not know if "atriodic" can be substituted for "arc-like" in Proposition 4.3. So we have the following question.

Question 4.6. Does there exist an atriodic irreducible continuum containig an absolute end point which is not an end point?

5. Irreducible smooth continua

Recall that a continuum X is said to be *smooth at a point* $p \in X$ (see Maćkowiak's paper [8], p. 81) if for each convergent sequence $x_1, x_2, \ldots$ of points of X and for each subcontinuum K of X such that $p, x \in K$, where $x = \lim x_n$, there exists a sequence $K_1, K_2, \ldots$ of subcontinua of X with $p, x_n \in K_n$ for each $n \in \{1, 2, \ldots\}$ and $\operatorname{Lim} K_n = K$ (equivalently, if for each subcontinuum K of X which contains p and for each open set V which contains K there exists an open connected set U such that $K \subset \dot{U} \subset V$, see [8], Theorem 3.1, p. 83). A continuum is said to be *smooth* if it is smooth at some of its points. It is shown by Maćkowiak in [8], Theorems 4.3 and 5.3, p. 85 and 88, that for continua which are either hereditarily unicoherent at a point or irreducible this concept coincides with that introduced and studied earlier by Gordh in his papers [5] and [6]. Irreducible smooth continua were investigated e.g. by the author in [3] and by Maćkowiak in §5 of [8].

Let a continuum X be irreducible between a point p and some other point of X. Consider the following three conditions which the point p may satisfy:

(α) p is an absolute end point of X;

(β) p is an end point of X;

(γ) X is smooth at p.

Then neither (α) nor (β) implies the other two, even for hereditarily unicoherent continua X; moreover, both (α) and (β) together do not imply (γ), even if the continuum X is arc-like, while (γ) implies both (α) and (β) for all irreducible continua X. Indeed, Example 4.4 shows that (α) does not imply (β), and an end point of the limit segment of a topologist's sine curve X is an end point, while not an absolute one, of X, so (β) does not imply (α) for arc-like continua. Further, take the accordionlike continuum M (see e.g. Thomas' paper [10], Fig. 1, p. 12), note that M is arc-like, consider the canonical decomposition $\mathscr{D}$ of M into its layers $L_t = \varphi^{-1}(t)$ for $t \in [0,1]$, where $\varphi\colon M \to M/\mathscr{D} = [0,1]$ means the quotient mapping, and shrink its end layers L_0 and L_1 to points, say p and q. The resulting continuum X is then a monotone image of M, and hence it is arc-like by Bing's Theorem 3 of [1], p. 47. The points p and q of X are absolute end points and hence end points of X, while X is smooth at none of its points, which can be seen by a characterization of points of smoothness in irreducible continua given in my paper [3], Theorem, p. 48. Finally, an argument for the only true implication among the discussed interrelations between conditions (α), (β) and (γ) is presented in the proof of the theorem below.

Theorem 5.1. *Let a continuum X be irreducible between a point p and some other point of X. If X is smooth at p, then p is both an absolute end point and an end point of X.*

Proof. If a continuum is smooth at p, then it is locally connected at this point by Maćkowiak's Corollary 3.2 in [8], p. 84. Since X is irreducible from p to some other point of X, condition (4) holds, and so p is an absolute end point of X by Proposition 3.2.

Further, Theorem 5.3 of Maćkowiak [8], p. 88, states that if an irreducible continuum is smooth at a point, then it is hereditarily unicoherent at this point. So X is hereditarily unicoherent at p, and thereby it is of type A' according to my Proposition 1 of [3], p. 46. Let $\varphi\colon X \to [0,1]$ be the quotient mapping for the canonical decomposition of X into its layers. Then, since we have already shown that p is an absolute end point of X, by (11) of Proposition 4.1 we may assume that $\{p\} = \varphi^{-1}(0)$.

Suppose on the contrary that p is not an end point of X, i.e. there are two subcontinua M and N of X, both containing p, such that

$$M \setminus N \neq \emptyset \neq N \setminus M. \tag{5.2}$$

Put $t_1 = \sup \varphi(M)$ and $t_2 = \sup \varphi(N)$, and observe that

$$M \subset \varphi^{-1}([0,t_1]) \quad \text{and} \quad N \subset \varphi^{-1}([0,t_2]). \tag{5.3}$$

We claim that $t_1 = t_2$. By the symmetry of assumptions, suppose that $t_1 < t_2$. Since the family

$$\mathscr{F} = \{\varphi^{-1}([0,t])\colon t \in [0,1]\}$$

is strictly monotone (see Theorem 2 of §48, III of Kuratowski's monograph [7], p. 195), we conclude that

$$M \subset \varphi^{-1}([0,t_1]) \subset \varphi^{-1}([0,(t_1+t_2)/2]) \subset \text{ int } N \subset N,$$

contrary to (5.2). Thus the claim is proved.

Since each member $\varphi^{-1}([0,t])$ of the family $\mathscr{F}$ considered is an irreducible continuum between the point p and any point of its boundary ([7], §48, III, Theorem 1, p. 195), and since its boundary is contained in $\varphi^{-1}(t)$ (see Theorems 6 and 7 of §48, III, p. 197 of [7]), we conclude from the claim that

$$\varphi^{-1}([0,t_1)) \subset M \cap N. \tag{5.4}$$

Since the continuum X is smooth at the point p, it follows from my characterization of points of smoothness in irreducible continua (see [3], Proposition 3, p. 47) that the layer $\varphi^{-1}(t_1)$ is of left cohesion, which means that it is contained in the closure of the set $\varphi^{-1}([0,t_1))$. Therefore (5.4) implies that $\varphi^{-1}([0,t_1]) = \varphi^{-1}(t_1) \cup \varphi^{-1}([0,t_1)) \subset M \cap N$, whence we conclude by (5.3) that $M = N$ contrary to (5.2). The proof is complete. □

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