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# QUASICONTINUITY AND RELATED PROPERTIES OF FUNCTIONS AND MULTIVALUED MAPS

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*Summary.* The main results presented in this paper concern multivalued maps. We consider the cliquishness, quasicontinuity, almost continuity and almost quasicontinuity; these properties of multivalued maps are characterized by the analogous properties of some real functions. The connections obtained are used to prove decomposition theorems for upper and lower quasicontinuity.

*Keywords*: uniform space, multivalued map, quasicontinuity

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Throughout the paper $(X, T)$ or simply $X$ is a topological space, $\mathrm{Cl}\,A$, $\mathrm{Int}\,A$ are the closure and the interior of a set $A \subset X$, respectively.

A set $A \subset X$ is said to be

– semi-open, if $A \subset \mathrm{Cl}(\mathrm{Int}\,A)$, [10];

– preopen, if $A \subset \mathrm{Int}(\mathrm{Cl}\,A)$;

– semi-preopen, if $A \subset \mathrm{Cl}\left(\mathrm{Int}(\mathrm{Cl}\,A)\right)$, [1].

The union of any family of semi-open (preopen, semi-preopen) sets is a set of the same type, [1, 10]. The intersection of an open set and a semi-open (preopen, semi-preopen) one is again semi-open (preopen, semi-preopen).

Let $X$, $Y$ be topological spaces. A function $f\colon X \to Y$ is called quasicontinuous (almost continuous, almost quasicontinuous) at a point $x \in X$ if for each neighbourhood $V$ of $f(x)$ we have $x \in \mathrm{Cl}\left(\mathrm{Int}\,f^{-1}(V)\right)$, $\left(x \in \mathrm{Int}(\mathrm{Cl}\,f^{-1}(V)), x \in \mathrm{Cl}\left(\mathrm{Int}\left(\mathrm{Cl}\,f^{-1}(V)\right)\right)\right)$, [10, 8, 2]. Equivalently, $f$ is quasicontinuous (almost continuous, almost quasicontinuous) at $x \in X$ if for each neighborhood $V$ of $f(x)$ there is a semi-open (preopen, semi-preopen) set $A$ satisfying $x \in A \subset f^{-1}(V)$. Let us observe

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that the quasicontinuity of $f$ at $x$ is equivalent to the condition $\operatorname{Int}\big(U \cap f^{-1}(V)\big) \neq \emptyset$ for every neighbourhoods $U$, $V$ of $x$ and $f(x)$, respectively, [9].

A function $f$ is called quasicontinuous (almost continuous, almost quasicontinuous) if it has this property at each point. It follows from definitions that continuity implies quasicontinuity and almost continuity; moreover, each of these properties implies almost quasicontinuity but these implications cannot be replaced by equivalences, [14].

Quasicontinuity and almost continuity are independent properties. Furthermore, almost quasicontinuous functions need not have the Baire property.

Example 1. Let $\mathbb{R}$ be the space of real numbers with the natural topology, $B \subset \mathbb{R}$ a Bernstein set and $f$ the characteristic function of $B$. Then $f$ has not the Baire property. Nonetheless, since $B$ and $\mathbb{R} \setminus B$ are dense sets [15], the function $f$ is almost continuous.

Now let $(Y, \varrho)$ be a pseudometric space. A function $f\colon X \to Y$ is said to be cliquish at a point $x_0 \in X$ if

(1) for each $\varepsilon > 0$ and each neighbourhood $U$ of $x_0$ there exists a nonempty open set $W \subset U$ with $\varrho\big(f(x), f(x')\big) < \varepsilon$ for $x, x' \in W$, ([18] for metric spaces).

By $A(f)$ we denote the set of all points at which $f$ is cliquish. A function $f$ is called cliquish if $A(f) = X$.

The symbols $C(f)$ and $E(f)$ are used to denote the sets of all points at which a function $f$ is continuous or quasicontinuous, respectively. Then

(2) $C(f) \subset E(f) \subset A(f) = \operatorname{Cl}\big(A(f)\big)$;

(3) $A(f) \setminus C(f)$ is of the first category;

(4) if $X$ is a Baire space, then $f\colon X \to (Y, \varrho)$ is cliquish iff $X \setminus C(f)$ is of the first category;

(5) a function $f\colon X \to (Y, \varrho)$ is continuous iff it is cliquish and almost continuous.

The statements (2)–(5) are proved in [11, 13, 6, 17] for a metric space $(Y, d)$, but by the same arguments they are valid for pseudometric ones.

In the sequel we will consider maps with values in uniform spaces. For a uniform space $(Y, \nu)$ we denote by $P_\nu$ a saturated family of pseudometrics inducing $\nu$. For any $y \in Y$, $M$, $M_1 \subset Y$, $\varrho \in P_\nu$ and $r > 0$ we denote

$$B(y, \varrho, r) = \{z \in Y : \varrho(y, z) < r\}, \qquad B(M, \varrho, r) = \bigcup \{B(y, \varrho, r) : y \in M\},$$
$$\varrho(y, M) = \inf_{z \in M} \varrho(y, z), \qquad \varrho(M_1, M) = \sup_{y \in M_1} \varrho(y, M).$$

It is easy to verify

(6) if $M_1$ is compact, then $\varrho(M_1, M) < r$ iff $M_1 \subset B(M, \varrho, r)$.

A function $f\colon X \to (Y,\nu)$ is called cliquish at a point $x \in X$ if for each $\varrho \in P_\nu$ the function $f\colon X \to (Y,\varrho)$ is cliquish at $x$, [3].

We will use the following result:

(7) A function $f\colon X \to (Y,\nu)$ is quasicontinuous if and only if it is cliquish and almost quasicontinuous, [5].

**Theorem 1.** *If $f\colon X \to (Y,\nu)$ is an almost continuous function, then $A(f) = E(f)$.*

Proof. Let $x_0 \in A(f) \setminus E(f)$. Then $\varrho \in P_\nu$, $\varepsilon > 0$ and a neighbourhood $U$ of $x_0$ can be chosen such that each nonempty open set $V \subset U$ contains a point $x_V$ with $\varrho\big(f(x_V), f(x_0)\big) > 2\varepsilon$. Letting $W = B\big(f(x_0),\varrho,\varepsilon\big)$ we have $x_0 \in \operatorname{Int}\big(\operatorname{Cl} f^{-1}(W)\big)$. Since $x_0 \in A(f)$ there is a nonempty open set $U_1 \subset \operatorname{Int}\big(\operatorname{Cl} f^{-1}(W)\big)$ such that $\varrho\big(f(x'), f(x'')\big) < \varepsilon$ for $x'$, $x'' \in U_1$. Furthermore, $\varrho\big(f(x_1), f(x_0)\big) > 2\varepsilon$ for some $x_1 \in U_1$ hence $\varrho\big(f(x), f(x_0)\big) > \varepsilon$ for $x \in U_1$. On the other hand, $U_1 \cap f^{-1}(W) \neq \emptyset$ and for any $x \in U_1 \cap f^{-1}(W)$ we have $\varrho\big(f(x), f(x_0)\big) < \varepsilon$; this is a contradiction completing the proof. $\square$

In a topological space $(X,T)$ the family

$$T_q = \{U \setminus H \colon U \in T,\ H \text{ is of the first category}\}$$

is a topology on $X$. For a set $A \subset X$ the symbols $\operatorname{Cl}_q A$, $\operatorname{Int}_q A$ will be used to denote the $T_q$-closure and the $T_q$-interior of $A$, respectively. Then

(8) $\operatorname{Cl} U = \operatorname{Cl}_q U$, for each set $U \in T_q$, [7];

(9) the spaces $(X,T)$ and $(X,T_q)$ have the same families of the first category sets, [7];

(10) $(X,T)$ is a Baire space iff $(X,T_q)$ is a Baire space, [4].

**Theorem 2.** *Let $(X,T)$ be a Baire space, $(Y,\nu)$ a separable uniform one and let $f\colon X \to Y$ be a function with the Baire property. Then*

(a) *if $f$ if $T_q$-almost continuous, then it is continuous;*

(b) *if $f$ is $T_q$-almost quasicontinuous, then it is quasicontinuous.*

Proof. We fix some $\varrho \in P_\nu$ and an open base $\{V_n \colon n \geqslant 1\}$ of the space $(Y,\varrho)$. Denoting by $C(f,T_q,\varrho)$ the set of all points at which the function $f\colon (X,T_q) \to (Y,\varrho)$ is continuous we obtain

$$X \setminus C(f,T_q,\varrho) = \bigcup_{n=1}^{\infty} f^{-1}(V_n) \setminus \operatorname{Int}_q f^{-1}(V_n).$$

Hence $X \setminus C(f, T_q, \varrho)$ is of the first category; according to (4) it means that $f$ is $T_q$-cliquish. Now, using (5) we have that $f$ is $T_q$-continuous, so—since $Y$ is a regular space—the function $f$ is continuous.

If $f$ is $T_q$-almost quasicontinuous, then in virtue of (7), $f: (X, T_q) \to (Y, \nu)$ is quasicontinuous. Thus it follows from the regularity of $Y$ that $f$ is quasicontinuous, [4, Cor. 4]. $\square$

Thus Theorem 2 yields decompositions of continuity and quasicontinuity of functions, namely:

**Corollary 1.** *Let $(X, T)$ be a Baire space, $(Y, \nu)$ a separable uniform one and let $f: X \to Y$ be any function. Then*

(a) *$f$ is continuous if and only if it is $T_q$-almost continuous and has the Baire property;*

(b) *$f$ is quasicontinuous if and only if it is $T_q$-almost quasicontinuous and has the Baire property.*

Let us remark that in Theorem 2 (so also in Corollary 1) $T_q$-almost continuity and $T_q$-almost quasicontinuity cannot be replaced by almost continuity or almost quasicontinuity, respectively. For instance, it suffices to take into account the Dirichlet function.

Now we shall formulate some results concerning real functions; in the sequel they will be used in the study of multivalued maps.

Let us put

$$T_1 = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$$

and

$$T_2 = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}.$$

A function $f: X \to \mathbb{R}$ is said to be upper quasicontinuous (almost continuous, almost quasicontinuous) if the function $f: X \to (\mathbb{R}, T_1)$ is quasicontinuous (almost continuous, almost quasicontinuous). Replacing $T_1$ by $T_2$ we obtain definitions of suitable lower forms of generalized continuity. Then for a given $f: X \to \mathbb{R}$ we denote by $E_0^+(f)$ and $E_0^-(f)$ the sets of all points at which $f$ is upper or lower quasicontinuous, respectively.

**Lemma 1.** *If $f: X \to \mathbb{R}$ is an upper (lower) almost continuous function, then*

$$A(f) \subset E_0^+(f) \quad \text{and} \quad \mathrm{Cl}\left(E_0^+(f) \cap E_0^-(f)\right) \subset E_0^+(f)$$
$$\left(A(f) \subset E_0^-(f) \quad \text{and} \quad \mathrm{Cl}\left(E_0^+(f) \cap E_0^-(f)\right) \subset E_0^+(f)\right).$$

Proof. Let $x_0 \in A(f) \setminus E_0^+(f)$; then there exist $\varepsilon > 0$ and a neighbourhood $U_0$ of $x_0$ such that each nonempty open set $U' \subset U_0$ contains a point $x'$ with $f(x_0)+2\varepsilon < f(x')$. Since $f$ is upper almost continuous we have $x_0 \in \operatorname{Int}\big(\operatorname{Cl} f^{-1}\big(-\infty, f(x_0)+\varepsilon\big)\big) = U_1$. The condition $x_0 \in A(f)$ implies a nonempty open set $U \subset U_0 \cap U_1$ can be chosen with $|f(x') - f(x'')| < \frac{1}{2}\varepsilon$ for $x'$, $x'' \in U$. Then for some $x_1 \in U$ we have $f(x_0) + 2\varepsilon < f(x_1)$ and $f(x) > f(x_1) - \frac{1}{2}\varepsilon > f(x_0) + \frac{3}{2}\varepsilon$ for each $x \in U$. But $U \subset \operatorname{Cl} f^{-1}\big(-\infty, f(x_0)+\varepsilon\big)$, so there exist points $x \in U$ satisfying $f(x) < f(x_0)+\varepsilon$, which is a contradiction completing the proof of the first inclusion.

Now let $x_0 \in \operatorname{Cl}\big(E_0^+(f) \cap E_0^-(f)\big)$, $\varepsilon > 0$, $W = \big(-\infty, f(x_0) + \frac{1}{2}\varepsilon\big)$ and let $U$ be a neighbourhood of $x_0$. Since $f$ is upper almost continuous we obtain $x_0 \in U \cap \operatorname{Int}\big(\operatorname{Cl} f^{-1}(W)\big) = U_1$, so $U_1 \cap E_0^+(f) \cap E_0^-(f) \neq \emptyset$. Assume that $x_1 \in U_1 \cap E_0^+(f) \cap E_0^-(f) \setminus f^{-1}(\operatorname{Cl} W)$. Then the conditions $f(x_1) > f(x_0)+\frac{1}{2}\varepsilon$ and $x_1 \in E_0^-(f)$ imply the existence of an open nonempty set $U' \subset U_1$ such that $f(x) > f(x_0) + \frac{1}{2}\varepsilon$ for $x \in U'$. But $U' \subset U_1 \subset \operatorname{Int}\big(\operatorname{Cl} f^{-1}(W)\big)$; this gives $U' \cap f^{-1}(W) \neq \emptyset$ which contradicts the last inequality. Hence $U_1 \cap E_0^+(f) \cap E_0^-(f) \subset f^{-1}(\operatorname{Cl} W)$. Then there exists a nonempty open set $U_2 \subset U_1$ with $f(U_2) \subset \big(-\infty, f(x_0)+\varepsilon\big)$, which means that $x_0 \in E_0^+(f)$ and the proof is complete. □

**Corollary 2.** *If $f\colon X \to \mathbb{R}$ is an upper and lower almost continuous function, then the set $E_0^+(f) \cap E_0^-(f)$ is closed.*

**Lemma 2.** *Let $f\colon X \to \mathbb{R}$ be a cliquish function. If $f$ is upper (lower) almost quasicontinuous at $x_0 \in X$, then it is upper (lower) quasicontinuous at $x_0$.*

Proof. Suppose $x_0 \notin E_0^+(f)$. Then there exist a neighbourhood $U_0$ of $x_0$ and $\varepsilon > 0$ such that each nonempty open set $U' \subset U_0$ contains a point $x'$ with $f(x_0)+2\varepsilon < f(x')$. Let us denote $W = \big(-\infty, f(x_0)+\varepsilon\big)$. Then $x_0 \in \operatorname{Cl}\big(\operatorname{Int}\big(\operatorname{Cl} f^{-1}(W)\big)\big)$, so $U_1 = U_0 \cap \operatorname{Int}\big(\operatorname{Cl} f^{-1}(W)\big) \neq \emptyset$. Since $f$ is cliquish we can choose a nonempty open set $U_2 \subset U_1$ such that $|f(x') - f(x'')| < \frac{1}{2}\varepsilon$ for $x'$, $x'' \in U_2$. On the other hand, for some $x_1 \in U_2$ we have $f(x_0) + 2\varepsilon < f(x_1)$, so $|f(x) - f(x_1)| < \frac{1}{2}\varepsilon$ for $x \in U_2$. Hence $f(x_0) + 2\varepsilon < f(x_1) < f(x) + \frac{1}{2}\varepsilon$ for $x \in U_2$, i.e. $f(U_2) \subset \big(f(x_0)+\varepsilon, \infty\big)$; but this is impossible because $U_2 \cap f^{-1}(W) \neq \emptyset$. □

Since upper (lower) quasicontinuity of a real function implies cliquishness, Lemma 2 gives a decomposition of upper (lower) quasicontinuity, i.e.:

**Corollary 3.** *A function $f\colon X \to \mathbb{R}$ is upper (lower) quasicontinuous if and only if it is cliquish and upper (lower) almost quasicontinuous.*

**Lemma 3.** *Let $(X,T)$ be a Baire space. A function $f\colon X \to \mathbb{R}$ has the Baire property and is upper and lower $T_q$-almost quasicontinuous if and only if it is upper and lower quasicontinuous.*

Proof. As was shown in the proof of Theorem 2 a function with the Baire property is $T_q$-cliquish. Then by Lemma 2 the function $f$ is upper and lower $T_q$-quasicontinuous. Finally, since $\mathbb{R}$ is a regular space, it is easy to show that $f$ is upper and lower quasicontinuous. The inverse implication is obvious. $\square$

Now let $X$, $Y$ be topological spaces and $F\colon X \to Y$ a multivalued map. For any sets $A \subset X$, $M \subset Y$ we denote $F(A) = \bigcup\{F(x)\colon x \in A\}$, $F^+(M) = \{x \in X\colon F(x) \subset M\}$ and $F^-(M) = \{x \in X\colon F(x) \cap M \neq \emptyset\}$.

A multivalued map $F$ is said to be upper semicontinuous (quasicontinuous, almost continuous, almost quasicontinuous) at a point $x_0 \in X$ if for each open set $V \subset Y$ with $x_0 \in F^+(V)$ we have $x_0 \in \operatorname{Int} F^+(V)$, (resp. $x_0 \in \operatorname{Cl}(\operatorname{Int} F^+(V))$, $x_0 \in \operatorname{Int}(\operatorname{Cl} F^+(V))$ or $x_0 \in \operatorname{Cl}(\operatorname{Int}(\operatorname{Cl} F^+(V))))$, [12, 16].

Equivalently, $F$ is upper semi-continuous (quasicontinuous, almost continuous, almost quasicontinuous) at $x_0$ iff for each open set $V \subset Y$ such that $x_0 \in F^+(V)$ there exists an open (semi-open, preopen, semi-preopen) set $U \subset X$ with $x_0 \in U \subset F^+(V)$. A multivalued map $F$ is called upper semi-continuous (quasicontinuous, etc.) if it has this property at each point.

Replacing in the above definitions $F^+$ by $F^-$ we obtain suitable lower forms of generalized continuity.

In a uniform space $(Y,\nu)$ we denote by $\mathcal{Z}(Y)$ the family of all nonempty compact subsets of $Y$. Then the sets

$$\{(M_1,M_2) \in \mathcal{Z}(Y) \times \mathcal{Z}(Y)\colon M_1 \subset B(M_2,\varrho,r) \text{ and } M_2 \subset B(M_1,\varrho,r)\},$$
$$\varrho \in P_\nu,\ r > 0$$

form a base of the uniformity $\tilde{\nu}$ on $\mathcal{Z}(Y)$.

For any pseudometric $\varrho \in P_\nu$ the Hausdorff pseudometric $\tilde{\varrho}$ is given by

$$\tilde{\varrho}(M_1,M_2) = \max\{\varrho(M_1,M_2), \varrho(M_2,M_1)\},$$

and then

$$P_{\tilde{\nu}} = \{\tilde{\varrho}\colon \varrho \in P_\nu\}.$$

A multivalued map $F\colon X \to Y$ with compact values is said to be cliquish at a point $x \in X$ if the function $F\colon X \to (\mathcal{Z}(Y),\tilde{\nu})$ is cliquish at $x$. For a multivalued map $F$ the set of all cliquishness points will be denoted as $A(F)$.

Let us take $y \in Y$, a finite set $L \subset Y$ and $\varrho \in P_\nu$. If $F\colon X \to Y$ is a multivalued map, then by $\psi_{F,y,\varrho}$, $\varphi_{F,L,\varrho}$ (or simply $\psi_{y,\varrho}$ and $\varphi_{L,\varrho}$) we denote real functions given by

$$\psi_{y,\varrho}(x) = \varrho\big(y, F(x)\big) \quad \text{and} \quad \varphi_{L,\varrho}(x) = \varrho\big(F(x), L\big).$$

**Theorem 3.** *A multivalued map $F\colon X \to (Y,\nu)$ is lower semicontinuous (quasicontinuous, almost continuous, almost quasicontinuous) at a point $x_0 \in X$ if and only if there is a dense set $D \subset Y$ such that $\psi_{y,\varrho}$ is upper semicontinuous (quasicontinuous, almost continuous, almost quasicontinuous) at $x_0$ for each $\varrho \in P_\nu$ and $y \in D$.*

Proof. Let us take $y \in Y$, $\varrho \in P_\nu$, $\varepsilon > 0$ and $r = \psi_{y,\varrho}(x_0)$. Then $F(x_0) \cap B(y,\varrho,r+\varepsilon) \neq \emptyset$ and by assumptions on $F$ there exists an open (semi-open, preopen, semi-preopen) set $U \subset X$ with $x_0 \in U$ and $F(x) \cap B(y,\varrho,r+\varepsilon) \neq \emptyset$ for $x \in U$. This yields $\psi_{y,\varrho}(x) < r + \varepsilon$ for $x \in U$, so $\psi_{y,\varrho}$ is upper semicontinuous (quasicontinuous, almost continuous, almost quasicontinuous) at $x_0$.

Conversely, let $W \subset Y$ be an open set with $F(x_0) \cap W \neq \emptyset$. Then $F(x_0) \cap B(y,\varrho,\varepsilon) \neq \emptyset$ for some $y \in D$, $\varrho \in P_\nu$ and $\varepsilon > 0$, so $\psi_{y,\varrho}(x_0) < \varepsilon$. The assumptions on $\psi_{y,\varrho}$ imply the existence of an open (semi-open, preopen, semi-preopen) set $U$ with $x_0 \in U$ and $\psi_{y,\varrho}(x) < \varepsilon$ for $x \in U$. Hence $F(x) \cap B(y,\varrho,\varepsilon) \neq \emptyset$ for $x \in U$, which completes the proof. □

In the sequel, by $\mathcal{L}(D)$ we denote the family of all finite subsets of a set $D \subset Y$ and write $\mathcal{L}$ instead of $\mathcal{L}(Y)$.

**Theorem 4.** *Let $F\colon X \to (Y,\nu)$ be a multivalued map with compact values. The map $F$ is upper semicontinuous (quasicontinuous, almost continuous, almost quasicontinuous) at a point $x_0 \in X$ if and only if there exists a dense set $D \subset Y$ such that $\varphi_{L,\varrho}$ is upper semicontinuous (quasicontinuous, almost continuous, almost quasicontinuous) at $x_0$ for each $\varrho \in P_\nu$ and $L \in \mathcal{L}(D)$.*

Proof. Let $L \in \mathcal{L}$, $\varrho \in P_\nu$, $\varepsilon > 0$ and $r = \varphi_{L,\varrho}(x_0)$; then we have $F(x_0) \subset B(L,\varrho,r+\varepsilon)$. It follows from the properties of $F$ that there exists an open (semi-open, preopen, semi-preopen) set $U$ such that $x_0 \in U$ and $F(U) \subset B(L,\varrho,r+\varepsilon)$. This implies $\varphi_{L,\varrho}(x) < \varphi_{L,\varrho}(x_0) + \varepsilon$ for $x \in U$, so $\varphi_{L,\varrho}$ is upper semicontinuous (quasicontinuous, almost continuous, almost quasicontinuous) at $x_0$.

Conversely, let $W \subset Y$ be an open set with $F(x_0) \subset W$. Since $F(x_0)$ is compact we can choose $\varrho \in P_\nu$, $L \in \mathcal{L}(D)$ and $\varepsilon > 0$ such that $F(x_0) \subset B(L,\varrho,\varepsilon) \subset W$; hence $\varphi_{L,\varrho}(x_0) < \varepsilon$. By the assumptions on $\varphi_{L,\varrho}$ there exists an open (semi-open, preopen, semi-preopen) set $U$ such that $x_0 \in U$ and $\varphi_{L,\varrho}(x) < \varepsilon$ for $x \in U$. Thus $\varrho\big(F(x),L\big) < \varepsilon$ for $x \in U$, so $F(U) \subset B(L,\varrho,\varepsilon) \subset W$ and the proof is complete. □

In a similar way the cliquishness of a multivalued map can be characterized:

**Theorem 5.** *Assume that a multivalued map $F\colon X \to (Y,\nu)$ is cliquish at a point $x_0 \in X$. Then*

(a) *for each $y \in Y$, $\varrho \in P_\nu$ the function $\psi_{y,\varrho}$ is cliquish at $x_0$;*

(b) *if $F$ has compact values, then for each $L \in \mathcal{L}$, $\varrho \in P_\nu$ the function $\varphi_{L,\varrho}$ is cliquish at $x_0$.*

Proof. Let $y \in Y$, $\varrho \in P_\nu$, $\varepsilon > 0$ be fixed and let $U$ be a neighbourhood of $x_0$. Then there exists a nonempty open set $U_1 \subset U$ such that $F(x') \subset B\big(F(x''), \varrho, \frac{1}{2}\varepsilon\big)$ for $x'$, $x'' \in U$. For $x' \in U_1$ let us put $r = \psi_{y,\varrho}(x')$; hence $F(x') \cap B\big(y, \varrho, r + \frac{1}{2}\varepsilon\big) \neq \emptyset$. Consequently, we have $F(x'') \cap B(y, \varrho, r + \varepsilon) \neq \emptyset$ for each $x'' \in U_1$, so $\psi_{y,\varrho}(x'') < r + \varepsilon = \psi_{y,\varrho}(x') + \varepsilon$ for each $x'' \in U_1$. This leads to the inequality $|\psi_{y,\varrho}(x') - \psi_{y,\varrho}(x'')| < \varepsilon$ for $x'$, $x'' \in U_1$ and (a) is proved.

Now, let $F$ be a compact valued map; we fix $L \in \mathcal{L}$, $\varrho \in P_\nu$, $\varepsilon > 0$ and a neighbourhood $U$ of $x_0$. Then we choose a nonempty open set $U_1 \subset U$ such that $\varrho\big(F(x'), F(x'')\big) < \frac{1}{2}\varepsilon$ for $x'$, $x'' \in U_1$. For $x' \in U_1$ we denote $r = \varphi_{L,\varrho}(x')$; then $F(x') \subset B\big(L, \varrho, r + \frac{1}{2}\varepsilon\big)$. Thus $F(x'') \subset B\big(F(x'), \varrho, \frac{1}{2}\varepsilon\big) \subset B(L, \varrho, r+\varepsilon)$ for each $x'' \in U_1$. This implies $\varphi_{L,\varrho}(x'') < r + \varepsilon = \varphi_{L,\varrho}(x') + \varepsilon$ for $x'$, $x'' \in U_1$ and consequently $|\varphi_{L,\varrho}(x') - \varphi_{L,\varrho}(x'')| < \varepsilon$ for $x'$, $x'' \in U_1$, which completes the proof. □

**Theorem 6.** *Let $X$ be a Baire space, $(Y,\nu)$ a separable uniform one and let $F\colon X \to Y$ be a multivalued map with compact values. If for each $\varrho \in P_\nu$, $y \in D$, $L \in \mathcal{L}(D)$ the functions $\psi_{y,\varrho}$, $\varphi_{L,\varrho}$ are cliquish, where $D$ is a countable dense subset of $Y$, then $F$ is cliquish.*

Proof. For a fixed $\varrho \in P_\nu$ we denote by $C_\varrho(F)$ the set of all points at which the function $F\colon X \to \big(\mathcal{Z}(Y), \tilde{\varrho}\big)$ is continuous. Further, let $C_0^+(\varphi_{L,\varrho})$ and $C_0^+(\psi_{y,\varrho})$ be the sets of upper semicontinuity points of $\varphi_{L,\varrho}$ and $\psi_{y,\varrho}$, respectively. Then according to Theorem 3 and 4 we obtain

$$C(F) = \bigcap_{L \in \mathcal{L}(D)} C_0^+(\varphi_{L,\varrho}) \cap \bigcap_{y \in D} C_0^+(\psi_{y,\varrho}) \supset \bigcap_{L \in \mathcal{L}(D)} C(\varphi_{L,\varrho}) \cap \bigcap_{y \in D} C(\psi_{y,\varrho}).$$

Since $\varphi_{L,\varrho}$ and $\psi_{y,\varrho}$ are cliquish functions, in virtue of (4) the set

$$\bigcap_{L \in \mathcal{L}(D)} C(\varphi_{L,\varrho}) \cap \bigcap_{y \in D} C(\psi_{y,\varrho})$$

is dense $G_\delta$ in $X$, so $X \setminus C_\varrho(F)$ is of the first category. Now, using (4) once more we obtain that $F\colon X \to \big(\mathcal{Z}(Y), \tilde{\varrho}\big)$ is a cliquish function, which completes the proof. □

The results on decomposition of the quasicontinuity of functions obtained earlier permit to formulate a theorem on decomposition of the upper and lower quasicontinuity of multivalued maps.

**Theorem 7.** *Let $F\colon X \to (Y,\nu)$ be a cliquish multivalued map with compact values. Then $F$ is lower (upper) almost quasicontinuous at $x_0$ if and only if it is lower (upper) quasicontinuous at $x_0$.*

Proof. Since $F$ is cliquish, following Theorem 5 the functions $\varphi_{L,\varrho}$, $\psi_{y,\varrho}$ are cliquish for each $y \in Y$, $L \in \mathcal{L}$ and $\varrho \in P_\nu$. If $F$ is lower (upper) almost quasicontinuous at $x_0$, then it follows from Theorem 3 (Theorem 4) that all functions $\psi_{y,\varrho}$ ($\varphi_{L,\varrho}$) are upper almost quasicontinuous at $x_0$. Now, applying Lemma 2 we obtain that all $\psi_{y,\varrho}$ ($\varphi_{L,\varrho}$) are upper quasicontinuous at $x_0$. Finally, Theorem 3 (Theorem 4) gives the lower (upper) quasicontinuity of $F$ at $x_0$, which completes the proof. □

**Corollary 4.** *Let $X$ be a Baire space, $(Y,\nu)$ a separable uniform one and let $F\colon X \to Y$ be a multivalued map with compact values. Then $F$ is lower (upper) quasicontinuous if and only if it is cliquish and lower (upper) almost quasicontinuous.*

Proof. Let $\varrho \in P_\nu$ be fixed. By $C^+_\varrho(F)$ and $C^-_\varrho(F)$ we denote the sets of all points at which the map $F\colon X \to (Y,\varrho)$ is upper or lower semicontinuous, respectively. If $F$ is lower (upper) quasicontinuous, then according to [3, Cor. 3] both $X \setminus C^+_\varrho(F)$ and $X \setminus C^-_\varrho(F)$ are of the first category. This means that $F\colon X \to (\mathcal{Z}(Y), \tilde{\varrho})$ is a cliquish function; consequently $F\colon X \to Y$ is cliquish.

The converse is a consequence of Theorem 7. □

A multivalued map $F\colon X \to Y$ is said to have the Baire property if for each open set $V \subset Y$ the set $F^+(V)$ has the Baire property.

**Theorem 8.** *Let $X$ be a Baire space, $(Y,\nu)$ a separable uniform one and let $F\colon X \to Y$ be a multivalued map with compact values. Then*

(a) *the map $F$ is upper and lower semicontinuous if and only if $F$ is upper and lower $T_q$-almost continuous and has the Baire property;*

(b) *$F$ is upper and lower quasicontinuous if and only if $F$ is upper and lower $T_q$-almost quasicontinuous and has the Baire property.*

Proof. If $F$ is upper and lower semicontinuous (quasicontinuous), then the conclusion is evident.

Let $D$ be a countable dense subset of $Y$, $L \in \mathcal{L}(D)$, $y \in D$ and $\varrho \in P_\nu$. For any $r > 0$ we will denote $B_c(y,\varrho,r) = \{z \in Y : \varrho(z,y) \leqslant r\}$. Then, because $F(x)$ is compact, for every numbers $a$, $b$ with $0 \leqslant a < b$ the conditions $\varphi_{L,\varrho}(x) \geqslant a$ and $\psi_{y,\varrho}(x) \leqslant b$ imply $F(x) \cap [Y \setminus B(L,\varrho,a)] \neq \emptyset$ and $F(x) \cap B_c(y,\varrho,b) \neq \emptyset$, respectively.

Thus by simple calculus we obtain

$$\varphi_{L,\varrho}^{-1}([a,b)) = F^{-}\big(Y \setminus B(L,\varrho,a)\big) \cap F^{+}\big(B(L,\varrho,b)\big),$$
$$\psi_{y,\varrho}^{-1}((a,b]) = F^{-}\big(B_c(y,\varrho,b)\big) \cap F^{+}\big(Y \setminus B_c(y,\varrho,a)\big),$$

so all functions $\varphi_{L,\varrho}$ and $\psi_{y,\varrho}$ have the Baire property. Now, using arguments analogous to those in the proof of Theorem 2, we have that functions $\varphi_{L,\varrho}$, $\psi_{y,\varrho}\colon (X,T_q) \to \mathbb{R}$ are cliquish; thus Theorem 6 implies the cliquishness of the map $F\colon (X,T_q) \to Y$.

Assume that $F$ is not upper $T_q$-semicontinuous at $x_0 \in X$. We can choose $\varrho \in P_\nu$, $\varepsilon > 0$ such that each $T_q$-neighbourhood of $x_0$ contains a point $x$ with $F(x) \not\subset B\big(F(x_0),\varrho,3\varepsilon\big)$. Let us put $W_1 = B\big(F(x_0),\varrho,\varepsilon\big)$ and $W_2 = \big\{y \in Y\colon \varrho\big(y,F(x_0)\big) > 3\varepsilon\big\}$. Then from the upper $T_q$-almost continuity we have $x_0 \in \operatorname{Int}_q\big(\operatorname{Cl}_q F^{+}(W_1)\big)$; furthermore $F(x_1) \cap W_2 \neq \emptyset$ for some $x_1 \in \operatorname{Int}_q\big(\operatorname{Cl}_q F^{+}(W_1)\big)$. The lower $T_q$-almost continuity gives $x_1 \in \operatorname{Int}_q\big(\operatorname{Cl}_q F^{-}(W_2)\big)$. Since the multivalued map $F$ is $T_q$-cliquish there is a nonempty set $U \in T_q$ with $U \subset \operatorname{Int}_q\big(\operatorname{Cl}_q F^{+}(W_1)\big) \cap \operatorname{Int}_q\big(\operatorname{Cl}_q F^{-}(W_2)\big)$ and $F(x') \subset B\big(F(x''),\varrho,\varepsilon\big)$ for $x'$, $x'' \in U$, so $U \cap F^{+}(W_1) \neq \emptyset$ and $U \cap F^{-}(W_2) \neq \emptyset$. Hence we can choose points $x_2$, $x_3 \in U$ with $F(x_2) \subset W_1$ and $F(x_3) \cap W_2 \neq \emptyset$. It means that $F(x_3) \not\subset B\big(F(x_2),\varrho,\varepsilon\big)$ which is a contradiction. Thus we have shown that $F$ is upper $T_q$-semicontinuous. In the similar way it can be proved that $F$ is lower $T_q$-semicontinuous. Then, since $Y$ is a regular space, the map $F$ is upper and lower semicontinuous. Finally, we suppose that $F$ is upper and lower $T_q$-almost quasicontinuous. In virtue of Theorem 7 it is upper and lower $T_q$-quasicontinuous. Then, according to [4, Th. 2], $F$ is upper and lower semicontinuous, which completes the proof. $\square$

We remark that Theorem 8 is not a consequence of Theorem 2 applied to the function $F\colon X \to \big(\mathcal{Z}(Y),\tilde{\nu}\big)$ since the upper and lower almost continuity (almost quasicontinuity) of a multivalued map $F$ do not imply the almost continuity (almost quasicontinuity) of the function $F\colon X \to \big(\mathcal{Z}(Y),\tilde{\nu}\big)$.

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