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AN AXIOM SYSTEM FOR FULL 3-DIMENSIONAL EUCLIDEAN GEOMETRY

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Summary. We present na axiom system for the class of full Euclidean spaces (i.e. of projective closures of Euclidean spaces) and prove the representation theorem for our system, using connections between Euclidean spaces and elliptic planes.

INTRODUCTION

In this paper we investigate a full Euclidean space which results from the projective space (coordinatized by a formally real pythagorean field) by adding a new relation of orthogonality of planes. It will be shown that the full Euclidean space can be obtained as a closure of a Euclidean space by means of one plane; i.e. the full Euclidean space is a natural projective closure of a Euclidean space. A projective point of view helps us to establish connections between Euclidean space geometry and elliptic plane geometry. The same approach to four classical metric plane geometries are to be found in [1].

0. We shall begin with basic definitions, facts and notation

1) Let F be an arbitrary field with char $(F) \neq 2$. A projective space over F is a structure $\langle U_1^F, U_3^F, I^F \rangle$ where U_1^F, U_3^F consist of all equivalence classes in $F^4 \setminus \{(0, 0, 0, 0)\}$ under proportionality relations, and the incidence $I^F \subseteq U_1^F \times U_1^F$ is given by

$$[x_0, x_1, x_2, x_3] \, I^F [a_0, a_1, a_2, a_3] \Leftrightarrow \sum_{i=1}^{3} x_i x_i = 0 \, .$$

We write $|$ instead of I^F.

Elements of U_1^F, points, will be denoted by small letters; planes, elements of U_3^F, will be denoted by small Greek letters. Given $\alpha \neq \beta$ we put $L(\alpha, \beta) := \{a \in U_1^F: a \mid \alpha \wedge a \mid \beta\}$ and call $L(\alpha, \beta)$ a line. Lines will be denoted by capital Roman letters; let U_2^F be the set of all lines. Then a projective space with lines is a structure $\langle U_1^F, U_2^F, U_3^F, | \rangle$ where incidence is extended in a natural way.

Following the known properties of incidence we can introduce and denote

(i) $L(\alpha, b)$, $L(\alpha, \beta)$ – the line determined by two distinct points a, b or planes α, β;

(ii) $P(A, B)$, $P(A, \alpha)$ – the plane containing two intersecting lines A, B or non-incident pair of point and line;

(iii) $p(A, B)$, $p(A, \alpha)$ – the point of intersection of A, B or A and α. Analogously we define a projective plane over $\boldsymbol{F}$.

2) Let $\boldsymbol{F}$ be a formally real pythagorean field. An elliptic plane over $\boldsymbol{F}$ is a structure $\langle P_1^F, P_2^F, |, \perp_e^F \rangle$ such that $\langle P_1^F, P_2^F, | \rangle$ is a projective plane over $\boldsymbol{F}$ and orthogonality $\perp_e^F \subseteq (P_2^F)^2$ is defined by $[a_0, a_1, a_2] \perp_e^F [b_0, b_1, b_2] \Leftrightarrow \sum_{i=0}^{2} a_i b_i = 0$.

A system of axioms for elliptic planes can be obtained by adding the following formulas to the axioms of Fano-Pappian projective planes (cf. [2]) (we write $\perp$ instead of $\perp_e^F$).

EM1: $A \not\perp A$
EM2: $A \perp B \Rightarrow A$
EM3: $(\forall a, A) \Rightarrow (\exists B) [\alpha \mid B \perp A)$
EM4: $(p \mid A, B, C \wedge D \perp A, B) \Rightarrow (A = B \vee D \perp C)$
EM5: $(p \mid A, B \wedge D \perp A, B, C) \Rightarrow (A = B \vee p \mid C)$
EM6: $(A, B \perp C, D) \Rightarrow (A = B \vee C = D)$
EM7: $(A \perp A' \wedge B \perp B' \wedge C \perp C' \wedge \alpha \mid A', B, C \wedge b \mid A, B', C \wedge$
$\wedge\, c \mid A, B, C' \wedge a \nmid A \wedge b \nmid B \wedge c \nmid C \wedge A \not\perp B, C \wedge B \not\perp C) \Rightarrow p \mid C'$
EM8: $(\forall A, B)(\exists C, D) [A \perp B \Rightarrow H(A, B, C, D) \wedge C \perp D]$

where $\boldsymbol{H}$ denotes the relation of harmonic conjugacy.

3) There are several (logically equivalent) ways in which a Euclidean space can by defined. It will be convenient for us to understand it as a structure consisting of points and planes together with incidence (membership) and orthogonality. Planes are sets described by linear equations, two planes with equations $Ax + By + Cz = 0$ and $A'x + B'y + C'z = 0$ are orthogonal if $AA' + BB' + CC' = 0$. Consequently, an affine space will be a structure consisting of points, planes and an incidence. Then we obtain

Theorem. *A projective space with one plane point-wise deleted is an affine space.*

Theorem. *A Euclidean space with ortgonality dropped is an affine space.*
Moreover, we shall use the following

Theorem. *Every projection bijection preserves harmonic quadruples.*

1. Analytical model. Let $\langle U_1^F, U_3^F, | \rangle$ be a projective space over a formally real pythagorean field $\boldsymbol{F}$. Then a full Euclidean space over $\boldsymbol{F}$ is a structure $\langle U_1^F, U_3^F, \perp \rangle$, where the relation $\perp \subseteq (U_3^F)^2$ is given by

$$[\alpha_0, \alpha_1, \alpha_2, \alpha_3] \perp [b_0, b_1, b_2, b_3] \Leftrightarrow \sum_{i=1}^{3} a_i b_i = 0\,.$$

Clearly, the above formula defines a relation in U_3^F.

Consequently, if $\langle U_1^F, U_2^F, U_3^F, |\rangle$ is a projective space with lines then $\langle U_1^F, U_2^F, U_3^F, |, \perp\rangle$ is called a full Euclidean space with lines. In every full Euclidean space there is exactly one plane distinguished, the plane $\omega = [1, 0, 0, 0]$. This implies

Theorem 1. $(\exists!\, \omega)(\forall\alpha)\,[\omega \perp \alpha]$

Theorem 2. $\alpha \perp \alpha \Rightarrow (\forall\beta)\,[\alpha \perp \beta]$

The plane ω will be called the special plane. The points and lines incident with ω will be called special too; the remaining points, lines and planes will be called regular.

Let $(U_1^F)^*$ be the set of special points, $(U_2^F)^*$ – the set of special lines.

Theorem 3. *Let* $V_1^F = U_1^F \setminus (U_1^F)^*$, $V_3^F = U_3^F \setminus \{\omega\}$, $|' = | \cap (V_1 \times V_3)$, $\perp' = \perp \cap (V_1 \times V_3)$. *Then* $\langle V_1, V_3, |', \perp'\rangle$ *is a Euclidean space.*

Proof. Let us consider a standard isomorphism φ of $\langle V_1, V_3, |'\rangle$ onto an affine space over F, defined by $[x_0, x_1, x_2, x_3] \longrightarrow (x_1/x_0, x_2/x_0, x_3/x_0)$. It is known that under this isomorphism the affine planes correspond to the coefficients in their equations. It is also seen then that φ preserves orthogonality. Thus $\langle V_1, V_3, |', \perp'\rangle$ is (up to an isomorphism) a Euclidean space. ■

In the sequel we shall be concerned with orthogonality of lines, especially with orthogonality of special lines. Let us define

W1 $L(\alpha, \omega) \perp_0 L(\beta, \omega) \Leftrightarrow \alpha \perp \beta$.

The planes determining the corresponding lines are not uniquely defined. Yet, it is easy to verify that the following holds:

$$L(\alpha, \omega) = L(\alpha', \omega) \wedge L(\beta, \omega) = L(\beta', \omega) \wedge \alpha \perp \beta \Rightarrow \alpha' \perp \beta',$$

and W1 is a correct definition of orthogonality in $(U_2^F)^*$.

Theorem 4. $\langle (U_1^F)^*, (U_2^F)^*, |, \perp_0\rangle$ is an elliptic plane.

Proof. It is known that $\langle (U_1^F)^*, (U_2^F)^*, |\rangle$ is a projective plane. Let $\alpha = [a_0, a_1, a_2, a_3]$, $A = L(\alpha, \omega) \in (U_2^F)^*$. Then A is represented by $[a_1, a_2, a_3]$. W1 implies $[a_1, a_2, a_3] \perp_0 [b_1, b_2, b_3] \Leftrightarrow \sum_{i=1}^{3} a_i b_i = 0$ which completes the proof. ■

The formula W1 can be also used to define orthogonality in the space, for a given orthogonality on a fixed plane.

Let $U_{1|\omega} = \{a \in U_1^F \colon a \mid \omega\}$; $U_{2|\omega} = \{L \in U_2^F \colon L \mid \omega\}$.

Theorem 5. *Let* $\langle U_1^F, U_2^F, U_3^F, |\rangle$ *be a projective space with lines,* $\omega \in U_3^F$. *Assume that* $\langle U_{1|\omega}, U_{2|\omega}, |, \perp_0\rangle$ *is an elliptic plane. Let* $\perp \subseteq (U_3^F)^2$ *satisfy* W1 *and* $(\forall\alpha)\,.\,[\alpha \perp \omega]$. *Then* $\langle U_1^F, U_2^F, U_3^F, |, \perp\rangle$ *is a full Euclidean space with lines.*

Proof. It suffices to choose a coordinate system in the given projective space such that $\omega = [1, 0, 0, 0]$. Then the formula for orthogonality on ω is known. Moreover, the coordinate field must be formally real and pythagorean (since $\perp_0$ is elliptic). The rest is clear. ■

2. System of axioms. Theorems 3, 4, 5 show a close connection between Euclidean and elliptic geometries. A system of axioms for full Euclidean spaces will be constructed such that this connection will come into prominence.

We shall show that a system of axioms for full Euclidean spaces can be obtained by adding the formulas BM1, ..., BM9 to a system of axioms for projective spaces (the formula B1, ..., B4 (cf. [1]).

B1 $(a, b, c \mid \alpha, \beta \wedge \alpha, b \mid \alpha, \beta, \gamma) \Rightarrow (a = b \vee \alpha = \beta \vee c \mid \gamma)$

B2 $(\forall a, b, c)(\exists \delta)[a, b, c \mid \delta]$

B3 $(\forall \alpha, \beta, \gamma)(\exists d)[\alpha, \beta, \gamma \mid d]$

B4 $(\exists a_1, a_2, a_3, a_4, a_5, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)[a_i \mid \alpha_i \Leftrightarrow i - j = \pm 2 (\mathrm{mod}\ 5)]$

BM1 $\alpha \perp \beta \Rightarrow \beta \perp \alpha$

BM2 $(\exists! \alpha)(\forall \beta)[\beta \perp \alpha]$ – this unique plane will be denoted by ω in the axioms below.

BM3 $\omega \perp \omega \Rightarrow (\forall \alpha)\{\omega \perp \alpha]$

BM4 $(\forall a, b, \alpha)(\exists \beta)[a, b \mid \beta \perp \alpha]$

BM5 $(p, q \mid \alpha, \beta, \gamma \wedge \delta \perp \alpha, \beta) \Rightarrow (\alpha = \beta \vee p = q \vee \delta \perp \gamma)$

BM6 $(\omega \perp \omega \wedge \gamma \neq \omega \wedge \alpha, \beta \mid p \nmid \omega \wedge q \mid \alpha, \beta, \omega \wedge \gamma \perp \alpha, \beta, \delta) \Rightarrow$
$\Rightarrow (\alpha = \beta \vee q \mid \delta)$

BM7 $(\omega \perp \omega \wedge \gamma \neq \omega \wedge \alpha, \beta \mid p \nmid \omega \wedge \alpha, \beta \perp \gamma, \delta \wedge q \mid \omega, \gamma) \Rightarrow (\alpha = \beta \vee q \mid \delta)$

BM8 $(\omega \perp \omega \wedge \alpha \not\perp \beta, \gamma \wedge \beta \not\perp \gamma \wedge \alpha \perp \alpha' \wedge \beta \perp \beta' \wedge \gamma \perp \gamma' \wedge$
$\wedge\ \omega \nmid p \mid \alpha, \beta, \gamma, \alpha', \beta', \gamma' \wedge a \mid \alpha', \beta, \gamma \wedge b \mid \alpha, \beta', \gamma \wedge c \mid \alpha, \beta, \gamma' \wedge$
$\wedge\ a \nmid \alpha \wedge b \nmid \beta \wedge c \nmid \gamma \wedge s \mid \alpha', \beta') \Rightarrow s \mid \gamma'.$

BM9 $(\omega \perp \omega \wedge \alpha, \beta \neq \omega \wedge \alpha \perp \beta \wedge p \neq q \wedge p, q \mid \alpha, \beta) \Rightarrow$
$\Rightarrow (\exists \gamma, \delta, a, b, c, d)[p, q \mid \gamma, \delta \wedge a \mid \alpha \wedge b \mid \beta \wedge c \mid \gamma \wedge d \mid \delta \wedge a \nmid \beta \wedge$
$\wedge\ \gamma \perp \delta \wedge H(a, b; c, d)]$

Theorem 6. Every full Euclidean space satisfies axioms BM1 – BM9.

Proof. BM1 follows from the definition, BM2, BM3 correspond to Theorems 1 and 2. BM4 states solvability of some system of linear equations; it follows from the linear algebra.

ad BM5: Let $\alpha \neq \beta$, $p \neq q$. If p, q are special, then α, β, γ intersect in the same special line. By W1 we obtain $\alpha \perp \delta \Leftrightarrow \gamma \perp \delta$. If at least one of p, q is regular then we consider lines $A = L(\alpha, \omega)$, $B = L(\beta, \omega)$, $C = L(\gamma, \omega)$, $D = L(\delta, \omega)$ and the point $s = p(L(p, q), \omega)$. We have $s \mid A, B, C \wedge A, B \perp_0 D \wedge A \neq B$. EM4 yields $D \perp_0 C$, thus $\delta \perp \gamma$.

ad BM6: If $\delta = \omega$ then $q \mid \delta$. Assume $\delta \neq \omega$, $\alpha \neq \beta$. Let $A = L(\alpha, \omega)$, $B = L(\beta, \omega)$, $C = L(\delta, \omega)$, $D = L(\gamma, \omega)$. Then we have $q \mid A, B, \wedge D \perp_0 A, B, C \wedge \wedge A \neq B$. By Theorem 4, EM5 we obtain $q \mid C$, thus $q \mid \delta$.

ad BM7: Assume $\alpha \neq \beta$. Let us consider lines $A = L(\alpha, \omega)$, $B = L(\beta, \omega)$, $C = L(\gamma, \omega)$, $D = L(\delta, \omega)$. We have: $A \neq B \wedge A, B \perp_0 C, D$. By BM6 we obtain $C = D$. The assumption $q \mid \omega, \gamma$ implies $q \mid C$, $q \mid D$, and finally $q \mid \delta$.

ad BM8: Consider again lines $A = L(\alpha, \omega)$, $B = L(\beta, \omega)$, $C = L(\gamma, \omega)$, $D = L(\delta, \omega)$ and analogously A', B', C', D'. Let $a' = p(\omega, L(p, a))$, $b' = p(\omega, L(p, b))$, $c' = p(\omega, L(p, c))$. Notice first that $L(p, a) \mid \alpha', \beta, \gamma$; $L(p, b) \mid \alpha, \beta', \gamma$; $L(p, c) \mid \alpha, \beta, \gamma'$. Moreover, $L(p, a) \nmid \alpha$, $p \nmid \alpha$; thus $a' \nmid \alpha$, $a' \nmid A$. Analogously $b' \nmid B$, $c' \nmid C$. Then W1 yields $a' \mid A', B, C \wedge b' \mid A, B', C \wedge c' \mid A, B, C' \wedge a' \nmid A \wedge b' \nmid B \wedge \wedge c' \nmid C \wedge A \perp_0 A' \wedge B \perp_0 B' \wedge C \perp_0 C' \wedge A \not\perp_0 B, C \wedge B \not\perp_0 C$. In such a case we infer by EM7 (and Theorem 4) that $r \mid A', B' \Rightarrow r \mid C'$. Let $s \mid \alpha', \beta'$. If $s = p$ then $s \mid C'$ by the assumptions. Let $s \neq p$ and consider $s' = p(\omega, L(s, p))$. Then $s' \mid \alpha', \beta'$, thus $s' \mid A', B'$. Therefore $s' \mid C'$, $s' \mid \gamma'$, $L(s, p)) \mid \gamma'$ and finally $s \mid \gamma'$.

ad BM9: Let $\alpha \perp \beta$; p, q as assumed. If $p, q \mid \omega$ then by BM5 we obtain $\alpha = \omega \vee \vee p = q \vee \beta \perp \beta$ – inconsistency. Let us assume p is regular, let $A = L(\alpha, \omega)$, $B = L(\beta, \omega)$. We have $A \perp_0 B$, thus by EM8 there exist lines C, D with $C \perp_0 D$, $H(A, B; C, D)$. Consider the planes $\gamma = P(C, p)$, $\delta = P(D, p)$ and the point $p' = p(A, B)$. We have $p' \mid \alpha, \beta$, thus $L(p', p) = L(p, q) = L(\alpha, \beta)$. Moreover, $p' \mid A, B$ and $H(A, B; C, D)$, thus $p' \mid C, D$. This implies $L(p, p') \mid \gamma$ and $p, q \mid \gamma$. Analogously we prove $p, q \mid \delta$. Let E be any line incident with ω, nonincident with p' and consider $a = p(A, E)$, $b = p(B, E)$, $c = p(C, E)$, $d = p(D, E)$. Then $H(a, b; c, d)$ and the appropriate incidencies hold. ∎

Theorem 4 simplifies the proof of Theorem 6. Analogously, Theorem 5 considerably simplifies the proof of the representation theorem. Namely, it suffices to prove that the geometry induced by W1 on special plane is an elliptic geometry, i.e. it satisfies axioms EM1, ..., EM8.

Theorem 7. *Formulas* B1, ..., BM9 *constitute a system of axioms for the class of full Euclidean spaces.*

Proof. When the prove that formulas EM1, ..., EM8 are valid on a special plane we use W1 in the form: if $A, B \mid \omega$ then $A \perp_0 B \Leftrightarrow (\exists \alpha, \beta)[\alpha \perp \beta \wedge A = L(\alpha, \omega) \wedge B = L(\beta, \omega)]$. However, then W1 easily follows from BM5 with p, q being special points. To prove EM1 we consider arbitrary regular points α, a special line A and the plane $\alpha = P(a, A)$. If $\alpha \perp \alpha$ then a cannot be regular. Therefore $\alpha \not\perp \alpha$.Thus $A \not\perp_0 A$.

EM2 states that $\perp_0$ is symmetric. To prove it we consider an arbitrary regular point a. Then $A \perp_0 B$ implies $P(a, A) \perp P(a, B)$, $P(a, B) \perp P(a, A)$, and $B \perp_0 A$. To prove EM3 let a be a special point, A – a special line. We consider a regular point b and a plane $\alpha = P(b, A)$. By BM4 there exists β such that $\beta \perp \alpha$, $\beta \mid a, b$.

Let $B = P(\beta, \omega)$, then $a \mid B \perp_0 A$. Analogously EM4 follows from BM5, EM6 follows from BM6, EM6 from BM7, EM7 from BM8.

Finally, to prove EM8 let A, B be special lines, with $A \perp_0 B$. Let r be a regular point, let $p = p(A, B)$, $\alpha = P(r, A)$, $\beta = P(r, B)$. From BM9 we get planes γ, δ and points a, b, c, d such that $\gamma \perp \delta$; $p, r \mid \gamma, \delta$; $a \mid \alpha$, $b \mid \beta$, $c \mid \gamma$, $d \mid \delta$, $a \not\mid \beta$, $H(a, b; c, d)$ It is easy to see that $\mp(a, b, c, d)$ and $a, b, c, d \not\mid L(p, r)$. Consider a plane ε such that $a, b, c, d \mid \varepsilon$. Let ξ be a perspectivity with centre r mapping ε^* onto ω^*. Then $a' = \xi(a)$, $b' = \xi(b)$, $c' = \xi(c)$, $d' = \xi(d)$ form a harmonic quadruple. Let $C = L(p, c')$, $D = L(p, d')$. We have $p \mid A, B, C, D$, $a' \mid A$, $b' \mid B$, $c' \mid C$, $d' \mid D$. Therefore $H(A, B; C, D)$. Since $\gamma \perp \delta$ we obtain $C \perp_0 D$, which ends the proof. ■

As we can see, a system of axioms for full Euclidean spaces is relatively short and elegant. It is worthwhile to notice that the axioms we have obtained can be easily translated into the language of the traditional Euclidean geometry. For example BM4 can be read as:

„Through any line we can drop a plane orthogonal to the given plane."

BM5 gives two variants:

„A plane orthogonal to a given line is orthogonal to every plane containing this line."

Or, if p, q are special:

„A plane orthogonal to a given plane is orthogonal to every plane parallel to the given one."

BM6: „Planes and lines orthogonal to a given plane are parallel."

BM7: „Planes posessing a common orthogonal line are parallel."

References

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Souhrn

SYSTEM AXIÓMŮ PRO PLNOU 3-DIMENSIONÁLNÍ EUKLIDOVSKOU GEOMETRII

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Je předložen systém axiómů pro třídu plných euklidovských prostorů (tj. projektivních uzávěrů euklidovských prostorů). S použitím vztahů mezi euklidovskými prostory a eliptickými rovinami je pro tento systém dokázána věta o representaci.

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