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EXTENDED TREES OF GRAPHS

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Summary. An extended tree of a graph is a certain analogue of spanning tree. It is defined by means of vertex splitting. The properties of these trees are studied, mainly for complete graphs.

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In this paper we shall introduce the concept of an extended tree (E-tree) of a graph; this concept is a certain analogue of a spanning tree. We consider finite undirected graphs without loops and multiple edges. We use the standard graph-theoretical terminology as eg. in [1].

For defining this concept we will use the concept of vertex splitting. Let G be a graph, let v be a vertex of G of degree at least 2. Let $\mathcal{P} = \{P_1, P_2\}$ be a partition of the set $F(u)$ of edges of G incident with v into two classes. We delete v from G and add two new vertices v^1, v^2. We join the vertex v^1 (or v^2) by edges with exactly those vertices which were joined in G with v by edges from P_1 (or P_2, respectively). Then we say that the resulting graph was obtained from G by splitting the vertex v according to the partition $\mathcal{P}$.

The above mentioned partition $\mathcal{P}$ is called disconnecting, if there is no circuit in G containing edges from both classes of $\mathcal{P}$. Evidently a graph obtained from G by splitting v according to $\mathcal{P}$ has more connected components than G if and only if $\mathcal{P}$ is disconnecting.

Here we see a certain analogy between a disconnecting partition and a bridge; this analogy leads us to the definition of an extended tree of a graph.

Let G be a finite connected graph. If G is a tree, then the unique extended tree of G is G itself. If not, then G contains at least one circuit and thus there exists a vertex v and a partition $\mathcal{P}$ of the set of edges incident with v which is not disconnecting. We choose one of them and perform the splitting according to it. We repeat this procedure, until a tree is obtained. This tree will be called an extended tree (shortly E-tree) of G.

If we have an edge uv of the original graph and the edge uv^1 or uv^2 of the graph obtained by splitting v, we may consider them as identical ones. Therefore we may consider the edge sets of both these graphs to be identical.

Proposition 1. *The above described procedure leads always to a tree. The number of steps is equal to the cyclomatic number $c(G)$ of G.*

Proof. As we perform splittings always according to non-disconnecting partitions, we cannot obtain a disconnected graph. If in some step we obtain a graph which is not a tree, then it contains a circuit and therefore there exists a non-disconnecting partition of $F(v)$ for some vertex v and we may continue. Consider such a graph G' and the graph G'' obtained from it by the above mentioned splitting. If n is the number of vertices and m the number of edges of G', then G'' has $n+1$ vertices and m edges. For the cyclomatic numbers we have

$$c(G'') = m - (n+1) + 1 = m - n = c(G') - 1.$$

Therefore exactly after $c(G)$ steps we obtain a graph whose cyclomatic number is 0 and this is a tree. □

Here again we see an analogy with the spanning tree, which is obtained after $c(G)$ deletions of edges.

Theorem 1. *A finite connected graph G has an E-tree which is a path if and only if it has an Eulerian trail (open or closed).*

Proof. Suppose that G has an open Eulerian trail $(e_1, \dots, e_m)$. For $i = 1, \dots, m-1$ let v_i be the common end vertex of e_i and e_{i+1}; further, let v_0 be the end vertex of e_1 different from v_1 and let v_m be the end vertex of e_m different from v_{m-1}. If the degree of v_0 in G is greater than 1, then we consider the partition $\{\{e_1\}, F(v_0) - \{e_1\}\}$. All edges of G and thus also all edges of $F(v_0) - \{e_1\}$ are in the above mentioned Eulerian trail and therefore there exists $i > 0$ such that $v_i = v_0$. The edges $e_1, \dots, e_i$ form a closed trail, therefore there exists a circuit in G which contains e_1 and e_i and the partition is not disconnecting. We perform the splitting of v_0 according to it; in the resulting graph the vertex v_0^1 has degree 1. The vertex v_1 has degree at least 2 in the resulting graph, because it is incident with e_1 and e_2. If its degree is greater than 2, then consider the partition $\{\{e_1, e_2\}, F(v_1) - \{e_1, e_2\}\}$.

By a similar argument as above we prove that it is not disconnecting. We perform the splitting of v_1 according to it; in the resulting graph the vertex v_1^1 has degree 2. In such a way we proceed further. The final result is an E-tree of G isomorphic to a path.

If G has a closed Eulerian path, then we choose a vertex v and an edge $e \in F(v)$ and perform the splitting of v according to $\{\{e\}, F(v) - \{e\}\}$. The resulting graph has an open Eulerian trail going from v^1 to v^2 and we may continue by the procedure described above. Therefore a graph having an Eulerian trail has an E-tree which is a path.

Now let G have an E-tree T which is a path. If two edges have a common end vertex in T, they have it also in G. If we run over the E-tree from one end vertex to the other, then the corresponding edges in G form an Eulerian trail. □

Proposition 2. *A finite connected graph G has an E-tree which is a star if and only if G is a star.*

Proof. If G is a star, then evidently its unique E-tree is G itself, i.e. a star. Now suppose that G is not a star. If G is a tree, then its unique E-tree is G itself, therefore it is not a star. If G is not a tree, it contains a circuit C; the length of C is at least 3. Then we may take a vertex v of C and a non-disconnecting partition of $F(v)$ and perform the splitting according to it. The vertices v^1, v^2 in the resulting graph are obviously not adjacent and have no common neighbour, because it would have to be joined with v in the original graph by two edges. Therefore the distance between v_1^1 and v_2^1 is at least 3. By further splittings it cannot be decreased, therefore any E-tree must contain a path of length at least 3 and cannot be a star. □

Theorem 2. *Let G be a finite connected graph, let q be the number of vertices of odd degrees in G. Let T be an E-tree of G, let t be the number of its terminal vertices. Then*

$$\tfrac{1}{2}q + 1 \leqslant t.$$

Proof. If $q = 0$, the assertion is evident; thus suppose $q \geqslant 2$. If we perform a splitting of a vertex v, then the sum of the degrees of v^1 and v^2 in the resulting graph is equal to the degree of v in the original graph. If the degree of v is odd, then exactly one of the vertices v^1, v^2 has an odd degree and thus the number of vertices of odd degrees cannot decrease at splittings. Therefore if T is an E-tree of G and q_0 is the number of its vertices of odd degrees, then $q_0 \geqslant q$. Let t be the number of terminal vertices of T. We have $2 \leqslant t \leqslant q_0$, because each terminal vertex has the odd degree 1. We shall prove the inequality $\frac{1}{2}q_0 + 1 \leqslant t$ by induction according to t. If $t = 2$, then T is a path and $q_0 = 2$; then $\frac{1}{2}q_0 + 1 = t$. Now let $t = k \geqslant 3$ and suppose that the assertion holds for $t \leqslant k-1$. Let u be a terminal vertex of T, let v be the vertex of degree at least 3 (at least one must exist) whose distance from u is minimal. Let P be the path in T connecting u and v, let T' be the tree obtained from T by deleting all edges and vertices of P except v. The tree T' has $t' = k-1$ terminal vertices. If v has an even (or odd) degree in T', then it has an odd (or even) degree in T' and $q_0' = q_0$ (or $q_0' = q_0 - 2$, respectively) for the number q_0' of vertices of odd degrees in T'. As $t' = k-1$, we have $\frac{1}{2}q_0' + 1 \leqslant t'$ and therefore $\frac{1}{2}q_0 \leqslant \frac{1}{2}q_0' + 1 \leqslant t' = t-1$ and hence $\frac{1}{2}q_0 + 1 \leqslant t$. As $q_0 \geqslant q$, we have also $\frac{1}{2}q + 1 \leqslant t$. □

Theorem 3. *Let n be an even positive integer. Then there exists an E-tree T of the complete graph K_n having exactly $\frac{1}{2}n + 1$ terminal vertices.*

Remark. The graph K_n for n even contains n vertices of the odd degree $n-1$.

Proof. For $n = 2$ the assertion is evident. Suppose $n \geqslant 4$. Choose a linear factor L in K_n. Let G be the graph obtained from K_n by deleting all edges of L; it is a connected regular graph of the even degree $n-2$. For each edge $e \in E(L)$ choose one of its end vertices and denote it by $u(e)$; let M be the set of all vertices $u(e)$ for e from L. We split each vertex $u(e)$ according to the partition $\{\{e\}, F(u(e)) - \{e\}\}$ and denote the graph thus obtained by H. The vertices $u^1(e)$ for $u(e) \in M$ are of degree 1 in H and the subgraph of H induced by the set of all vertices $u^2(e)$ is G. By Theorem 1 we can construct an E-tree of G which is a path; the construction described in the proof of the theorem is such that both the terminal vertices of this path come from one (arbitrarily chosen) vertex of the original graph. If we choose this vertex to be a vertex $u^2(e_0)$ for some $e_0 \in E(L)$, then by this construction we obtain from H a tree T whose terminal vertices are all $u(e)$ for $e \in E(L)$ and one of the terminal vertices of this path (the other is joined with $u^1(e_0)$ and has the degree 2). This is the required tree. □

Theorem 4. *Let T be an E-tree of a complete graph K_n, let t be the number of its terminal vertices. Then*

$$t \leqslant \tfrac{1}{2}n^2 - \tfrac{3}{2}n + 2.$$

Proof. The assertion is clear for $n = 2$; thus suppose $n \geqslant 3$. The cyclomatic number of K_n is $c(K_n) = \frac{1}{2}n(n-1) - n + 1 = \frac{1}{2}n^2 - \frac{3}{2}n + 1$. The E-tree T is obtained from K_n by $c(K_n)$ vertex splittings. For each vertex v of K_n we denote by $S(v)$ the set of vertices of T which were obtained by successive splittings of the vertex v; obviously $S(v)$ has at most $n-1$ elements and the sum of degrees of vertices from $S(v)$ in T is equal to the degree of v in G, i.e. to $n-1$. If $M(v)$ contains $n-1$

vertices of degree 1, then these vertices were obtained by $n-2$ splittings. But such a case may occur for at most one vertex; otherwise there would exist an edge joining two vertices of degree 1 in T, which is impossible. For any other vertex v the set $S(v)$ consists of $s(v) \leqslant n-3$ vertices of degree 1 and at least one vertex of degree at least 2. These $s(v)$ vertices are obtained by at least $s(v)$ splittings. In total, t vertices of degree 1 can be obtained by at least $t-1$ splittings. The number of splittings is $c(K_n)$, hence $t-1 \leqslant c(K_n)$ and

$$t \leqslant c(K_n) + 1 = \tfrac{1}{2}n^2 - \tfrac{3}{2}n + 2.$$

□

Theorem 5. *For every integer $n \geqslant 2$ there exists an E-tree of K_n having exactly $\frac{1}{2}n^2 - \frac{3}{2}n + 2$ terminal vertices.*

Proof. Choose a spanning tree S of K_n and choose one terminal vertex v_0 of S. Further, assign orientations to all edges of $E(K_n) - E(S)$ in such a way that all edges of $E(K_n) - E(S)$ which are incident with v_0 will be oriented so that v_0 will be their terminal vertex, other edges of $E(K_n) - E(S)$ will be oriented arbitrarily. Now for each edge $e \in E(K_n) - E(S)$ we perform a splitting of its terminal vertex (in the orientation) according to the partition in which one class is $\{e\}$. In this way the required E-tree is obtained. Its terminal vertices are all terminal (in the orientation) vertices of edges of $E(K_n) - E(S)$, whose number is $c(K_n)$, and moreover one vertex which was obtained from v_0 by splittings (a vertex incident with an edge of $E(S)$).

□

In the end we will mention the diameters of E-trees of complete graphs.

Theorem 6. *Let T be an E-tree of a complete graph K_n for $n \geqslant 4$, let δ be its diameter. Then*

$$\delta \geqslant 4.$$

Proof. The diameter of T cannot be 2, because then T would be a star, which is impossible by Proposition 2. Suppose $\delta = 3$. Then T exactly two vertices of degrees greater than 1 and therefore it has $\frac{1}{2}n(n-1) - 1 = \frac{1}{2}n^2 - \frac{1}{2}n - 1$ terminal vertices. According to Theorem 4 this must be less than or equal to $\frac{1}{2}n^2 - \frac{3}{2}n + 2$. But this implies $n \leqslant 3$. Therefore for $n \geqslant 4$ we have $\delta \geqslant 4$. □

Remark. For the graph K_2 there exists a unique E-tree, namely K_2 itself; its diameter is 1. For the graph K_3 all E-trees are isomorphic to a path of length 3 and therefore they have diameter 3.

Theorem 7. *For each integer $n \geqslant 4$ there exists an E-tree T of K_n whose diameter is* 4.

Proof. The required tree T is constructed in the way described in the proof of Theorem 5, choosing the spanning tree S to be a star. □

Theorem 8. *Let T be an E-tree of a complete graph K_n, let δ be its diameter. Then*

$$\delta \leqslant n(n-1) \quad \text{for } n \text{ odd},$$
$$\delta \leqslant n^2 - n + 1 \quad \text{for } n \text{ even}.$$

Proof. If n is odd, then K_n is an Eulerian graph and has an E-tree which is a path; this path has $\frac{1}{2}n(n-1)$ edges. If n is even, then by Theorem 2 the tree T has at least $\frac{1}{2}n + 1$ terminal vertices. A diametral path of T can contain only two of them and therefore its length is at most $\frac{1}{2}n(n-1) - \frac{1}{2}n + 1 = \frac{1}{2}n^2 - n + 1$. □

Theorem 9. *For each even positive integer there exists an E-tree T of K_n whose diameter is $\frac{1}{2}n^2 - n + 1$.*

Proof. This is the tree described in the proof of Theorem 3. □

References

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