Mathematica Bohemica

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*Mathematica Bohemica*, Vol. 116 (1991), No. 3, 319–326

Persistent URL: `http://dml.cz/dmlcz/126171`

116 (1991) MATHEMATICA BOHEMICA No. 3, 319—326

# NATURAL LIFTINGS OF VECTOR FIELDS TO TANGENT BUNDLES OF BUNDLES OF 1-FORMS

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(Received June 13, 1989)

*Summary.* Natural liftings $D: I \to ITT^*$ are classified for $n \geqq 2$. It is proved that they form a 5-parameter family of operators.

*Keywords:* natural bundles, natural liftings, equivariant maps.

*AMS Classification:* 53A55, 58A20.

Kolář classified natural liftings transforming vector fields on manifolds to vector fields on natural bundles which correspond to Weil algebras [3]. A simple example of a natural functor which does not arise from a Weil algebra is the cotangent bundle functor $T^*$ (functors which correspond to Weil algebras are precisely those which are covariant and multiplicative – see [1], [2]). Natural liftings of vector fields to the cotangent bundle of $T^*$ were found by Kolář. In this paper we classify natural liftings of vector fields of $\mathcal{T}TT^*$ – another example of a natural bundle which does not correspond to a Weil algebra.

## 1. PRELIMINARIES

We first recall basic facts concerning natural bundles and introduce the notation we shall need later on. We shall consider natural bundle functors $F: \mathcal{M}_n \to \mathcal{F}\mathcal{M}_n$ where $\mathcal{M}_n$ denotes the category of $n$-dimensional manifolds and local diffeomorphisms whereas $\mathcal{F}\mathcal{M}_n$ is the category of fibred bundles with $n$-dimensional base manifolds and fibre bundle morphisms. It is well known that every natural bundle has finite order and the category of natural bundles of order $\leqq r$ with natural transformations of functors as morphisms is equivalent to the category of $L^r_n$-manifolds and $L^r_n$-equivariant maps. For a natural bundle $F$ we will denote by $F_0$ the corresponding $L^r_n$-manifold, $F_0 = (FR^n)_0$. Similarly, if $p: F \to G$ is a natural transformation of natural bundles, $p_0 = p(R^n)|_{F_0}: F_0 \to G_0$ will denote the $L^r_n$-equivariant map corresponding to $p$. The action of $L^r_n$ on $F_0$ is given by the formula $(j^r_0\varphi)\, z = F(\varphi)(z)$ and this gives formulae which describe the action in the coordinates in particular cases. For example, the canonical coordinates $(x^i)$ on $R^n$ induce the coordinate system $(x^i, p_i)$ on $T^*R^n$ and then the coordinate system $(x^i, p_i, Y^i_1, P^i_1)$ on $TT^*R^n = (TT^*)_0 \times R^n$.

This gives coordinates $(p_i, Y_1^i, P_i^1)$ on $(TT^*)_0$ and by calculating $F(\varphi)$, where $\varphi \in \in \mathrm{Diff}(\mathbf{R}^n, 0)$, we get the action of $L_n^2$ on $(TT^*)_0$ (cf. [5]):

$$(1) \qquad \bar{p}_i = b_i^j p_j, \quad \bar{Y}_1^i = a_j^i Y_1^j, \quad \bar{P}_i^1 = b_i^j P_j^1 - a_{jk}^l b_l^m b_i^j p_m Y_1^k,$$

where $(a_j^i, a_{j_1 j_2}^i) \in L_n^2$ and $(b_j^i)$ is the inverse matrix of $(a_j^i)$. By repeating this procedure again we get coordinates $(p_i, Y_1^i, P_i^1, Y_2^i, P_i^2, Y_3^i, P_i^3)$ on $(TTT^*)_0$ but we will not calculate the action of $L_n^3$. For our purposes it will be sufficient to know the action of the subgroup $B_n^3 = \{j_0^3\varphi \colon \varphi \in \mathrm{Diff}(\mathbf{R}^n, 0),\ j_0^2\varphi = j_0^2 \,\mathrm{id}_{\mathbf{R}^n}\}$. This action is easier to be calculated since if $(a_j^i, a_{j_1 j_2}^i, a_{j_1 j_2 j_3}^i) \in B_n^3$ then $a_j^i = \delta_j^i$ and $a_{j_1 j_2}^i = 0$. Using (1) we get

$$(2) \qquad \bar{P}_i^3 = P_i^3 - a_{ilr}^j p_j Y_1^l Y_2^r$$

and the remaining coordinates do not change.

For a natural bundle $F$ we will denote by $J^rF$ the $r$th jet prolongation of $F$ while $\varrho_k^r \colon J^rF \to J^kF$ $(r \geqq k)$ will denote the canonical projection. Later on we will need a formula for the action of $B_n^{r+1} = \{j_0^{r+1}\varphi \colon \varphi \in \mathrm{Diff}(\mathbf{R}^n, 0),\ j_0^r\varphi = j_0^r \,\mathrm{id}_{\mathbf{R}^n}\}$ on $(J^rT)_0$. It can be obtained by differentiating

$$\bar{X}^i(x) = \left.\frac{\partial \varphi^i}{\partial x^j}\right|_{\varphi^{-1}(x)} X^j(\varphi^{-1}(x)) .$$

We get

$$(3) \qquad \bar{X}_{j_1 \ldots j_r}^i = X_{j_1 \ldots j_r}^i + a_{j_1 \ldots j_{r+1}}^i X^{j_{r+1}}$$

and the other coordinates are not affected.

## 2. VERTICAL BUNDLES AND LIOUVILLE VECTOR FIELDS

Let $F$ be a natural bundle and let $E$ be a natural vector bundle, $E \colon \mathcal{M}_k \to \mathcal{F}\mathcal{M}_k$ where $k = n + \dim F_0$. Then the composition $EF \colon \mathcal{M}_n \to \mathcal{F}\mathcal{M}_n$ is a natural bundle. The projection $EF \to F$ is a natural transformation of functors and will be denoted by $p_F$. The vertical bundle $VEF$ will be defined by the following exact sequence of natural vector bundles over $EF$:

$$(4) \qquad 0 \to VEF \xrightarrow{\ l_{EF}\ } TEF \xrightarrow{\ dp_F\ } p_F^* TF \to 0 ,$$

where $l_{EF}$ denotes the natural inclusion $VEF \to TEF$.

Since $E$ is a natural vector bundle, natural bundles $VEF$ and $EF \times_F EF$ are isomorphic (the natural functor $EF \times_F EF$ is defined so that $EF \times_F EF(M) = EF(M) \times_{F(M)} EF(M)$, and the natural equivalence is given by the formula

$$VEF(M) \ni [\gamma_t] \to \left(\gamma_0, \frac{\mathrm{d}}{\mathrm{d}t}\, \gamma_t|_{t=0}\right) \in EF \times_F EF(M) ,$$

where $\gamma_t$ is a curve in $EF(M)$). It is easy to notice that the natural bundles $p_F^*EF$ and $EF \times_F EF$ are also naturally equivalent. From now on we will identify the functors $VEF$, $EF \times_F EF$ and $p_F^*EF$. Let $L_{EF}$: $EF \to TEF$ denote the composition of the diagonal transformation $EF \to EF \times_F EF \simeq VEF$ with $l_{EF}$: $VEF \to TEF$. Since $p_{EF} \circ L_{EF} = \mathrm{id}_{EF}$, $L_{EF}(M)$ is a vector field on $EF(M)$. This general construction will be used in this paper only in two cases: $E = T^*$, $F = \mathrm{id}_{\mathcal{M}_n}$ and $E = T$, $F = T^*$. In the first case $F(M) = M$ so $F$ is a natural bundle with 0-dimensional fibres, sequence (4) can be rewritten as

$$(5) \qquad 0 \to VT^* \xrightarrow{l_{T^*}} TT^* \xrightarrow{dp} p^*T \to 0\,,$$

$VT^*(M) \simeq T^*(M) \times_M T^*(M)$ is the vertical bundle and $L_{T^*}(M)$ is the Liouville vector field on $T^*(M)$. In the second case we get the sequence

$$(6) \qquad 0 \to VTT^* \xrightarrow{l_{TT^*}} TTT^* \xrightarrow{dp_{T^*}} p_{T^*}^*TT^* \to 0\,,$$

$VTT^*(M) \simeq TT^*(M) \times_{T^*(M)} TT^*(M)$ and $L_{TT^*}(M)$ is a vector field on $TT^*(M)$.

In order to make formulae shorter, we will denote bundle projections as in the following diagram:

$$\begin{array}{ccccccc} TTT^* & \xrightarrow{p_2{}^3} & TT^* & \xrightarrow{p_1{}^2} & T^* & \xrightarrow{p_0{}^1} & \mathrm{id}_{\mathcal{M}_n} \\ \downarrow{\scriptstyle dp_1{}^2} & & \downarrow{\scriptstyle dp_0{}^1} & & & & \\ TT^* & & T & & & & \end{array}$$

We also put $p_1^3 = p_1^2 \circ p_2^3$, $p_0^3 = p_0^1 \circ p_1^3$ and so on.

## 3. NATURAL LIFTINGS

Let $F$ be a natural bundle and let $M \in \mathcal{M}_n$. By $\mathcal{F}(M)$ we will denote the set of sections of the bundle $F(M) \to M$. Similarly, $\mathcal{E}F(M)$ will denote the set of sections of the bundle $EF(M) \to F(M)$. For example, $L_{T^*}(M) \in \mathcal{T}T^*(M)$ and $L_{TT^*}(M) \in \mathcal{T}TT^*(M)$.

A natural lifting $\mathcal{N}$ of vector fields to a natural bundle $F$ is a regular natural differential operator from the tangent bundle functor $T$ to the functor $TF$, $\mathcal{N}$: $\mathcal{T} \to \mathcal{T}F$. It was proved in [3] that the order of such an operator is not greater than the order of $F$. Let $r$ denote the order of $\mathcal{N}$. Then, since $\mathcal{N}$ is regular, there is a corresponding natural transformation $N$: $J^rT \times F \to TF$ such that for $X \in \mathcal{T}(M)$, $z \in F(M)_x$ we have $\mathcal{N}(M)(X)_z = N(M)(j_x^rX, z)$ (regularity is necessary for $N(M)$ to be smooth). Further on we will denote liftings and the corresponding natural transformations by the same letters.

For every natural bundle $F$ there is a fundamental lifting $\mathcal{F}_F$: $\mathcal{T} \to \mathcal{T}F$, which is also called the flow operator: $\mathcal{F}_F(X)$ is a vector field which corresponds to the local 1-parameter group of diffeomorphisms $(F(\varphi_t))$, where $(\varphi_t)$ is a local 1-parameter group of $X$. The order of $\mathcal{F}_F$ is equal to the order of $F$.

We shall begin with examples of natural liftings of vector fields to the bundle $TT^*$. First, we have the flow operator $\mathscr{F}^{1,3} = \mathscr{F}_{TT^*}: \mathscr{T} \to \mathscr{T}TT^*$. We also have the following 'constant' lifting $\mathscr{L}_{TT^*}: \mathscr{T} \to \mathscr{T}TT^*$ defined by the formula

$$\mathscr{T}(M) \ni X \to L_{TT^*}(M) \in \mathscr{T}TT^*(M) .$$

We can get other natural operators $\mathscr{T} \to \mathscr{T}TT^*$ by composing natural operators $\mathscr{T} \to \mathscr{T}T^*$ and natural operators $\mathscr{T}T^* \to \mathscr{T}TT^*$. We have two natural liftings $\mathscr{F}^{1,2} = \mathscr{F}_{T^*}$ and $\mathscr{L}_{T^*}: \mathscr{T} \to \mathscr{T}T^*$, defined similarly as $\mathscr{F}^{1,3}$ and $\mathscr{L}_{TT^*}$. We also have two natural operators $\mathscr{F}^{2,3}$ and $l: \mathscr{T}T^* \to \mathscr{T}TT^*$, where $\mathscr{F}^{2,3}$ is the flow operator and $l$ is defined in the following way: for $X \in \mathscr{T}T^*(M)$, $Z \in TT^*(M)$ we have $l(X)_Z = l_{TT^*}(Z, X(p_1^2(Z)))$ (we recall that $l_{TT^*}: VTT^* \to TTT^*$ is the natural inclusion from the diagram (6)). Since $\mathscr{F}^{2,3} \circ \mathscr{F}^{1,2} = \mathscr{F}^{1,3}$, we get three more operators: $\mathscr{F}^{2,3} \circ \mathscr{L}_{T^*}$, $l \circ \mathscr{L}_{T^*}$, $l \circ \mathscr{F}^{1,2}$. We will prove that all natural liftings $\mathscr{T} \to \mathscr{T}TT^*$ are generated by the five listed above, provided $n \geqq 2$. But first we shall need some lemmas.

If a group $G$ acts on a set $X$ on the left then $G_x$ will denote the stability group of $x \in X$, $G_x = \{a \in G: ax = x\}$. We have the following obvious lemma:

**Lemma 1.** *If a group $G$ acts on sets $X$, $Y$ on the left and $f: X \to Y$ is $G$-equivariant then $G_x \subset G_{f(x)}$ for all $x \in X$.*

The next lemma comes from the book [4]. Let $V$ denote the vector space $\boldsymbol{R}^n$ with the standard action of the group $L_n^1$ and let $V_{k,l} = \underbrace{V \times \ldots \times V}_{k \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{l \text{ times}}$.

**Lemma 2.** *All smooth $L_n^1$-equivariant maps $V_{k,l} \to V$ are of the form*

$$\sum_{\alpha=1}^{k} g_\alpha(\langle x_\beta, y_\gamma \rangle)\, x_\alpha$$

*where $g_\alpha: \boldsymbol{R}^{kl} \to \boldsymbol{R}$ are smooth functions, $\alpha, \beta = 1 \ldots k$ and $\gamma = 1 \ldots l$.*

Remark 1. A similar statement is true in the case of $L_n^1$-equivariant maps $V_{k,l} \to V^*$ (see [4]).

**Lemma 3.** *Let $c: J^1T \times TT^* \to T$ be a natural transformation and let $n \geqq 2$. Then there exist smooth functions $\alpha, \beta: \boldsymbol{R}^3 \to \boldsymbol{R}$ such that*

$$c(j_x^1 X, Z) = (\alpha \circ \xi(j_x^1 X, Z))\, X + (\beta \circ \xi(j_x^1 X, Z))\, dp_0^1(Z) \tag{7}$$

*where $\xi: J^1T \times TT^* \ni (j_x^1 X, Z) \to (\langle X, p_1^2(Z)\rangle, \langle dp_0^1(Z), p_1^2(Z)\rangle, t) \in \boldsymbol{R}^3$ and $t: J^1T \times TT^* \to \boldsymbol{R}$ is a natural function,*

$$t(j_x^1 X, [\omega_t]) = \frac{\mathrm{d}}{\mathrm{d}t} \langle \omega_t, X \rangle\big|_{t=0} . \tag{8}$$

Proof. Let us consider the $L^2_n$-equivariant map $c_0: (J^1T \times TT^*)_0 \to V$. The injection $L^1_n \ni (a^i_j) \to (a^i_j, 0) \in L^2_n$ allows us to consider $L^1_n$ as a subgroup of $L^2_n$. Let $S^0 \subset (J^1T)_0$ denote the space of 1-jets of constant vector fields on $\boldsymbol{R}^n$. The space $S^0 \times (TT^*)_0$ can be defined in the coordinates $(X^i, X^i_j, p_i, Y^i_1, P^1_i)$ on $(J^1T \times TT^*)_0$ by the conditions $X^i_j = 0$, $i, j = 1 \dots n$. It is easy to notice that $S^0 \times (TT^*)_0$ is $L^1_n$-invariant and, as an $L^1_n$-space, it is equivalent to $V \times V \times V^* \times V^*$. Now we apply lemma 2 to the map $c_0$ restricted to $S^0 \times (TT^*)_0$. We see that

$$c_0(X^i, 0, p_i, Y^i_1, P^1_i) = g(X^i p_i, X^i P^1_i, Y^i_1 p_i, Y^i_1 P^1_i)\, X + \\ + h(X^i p_i, X^i P^1_i, Y^i_1 p_i, Y^i_1 P^1_i)\, Y_1$$

where $g, h: \boldsymbol{R}^4 \to \boldsymbol{R}$ are smooth functions. We will prove that $g$ and $h$ do not depend on the fourth variable. Let $B_1 \subset B^2_n$ be the stability group of $j^1_0 X$, where $X = \partial/\partial x_1$ is a constant vector field on $\boldsymbol{R}^n$. If $(a^i_{jk}) \in B_1$ then formula (3) implies that $a^i_{j1} = 0$ and, since $c_0$ is $B_1$-equivariant,

$$g(X^i p_i, X^i P^1_i, Y^i_1 p_i, Y^i_1 P^1_i) = g(X^i p_i, X^i P^1_i, Y^i_1 p_i, Y^i_1 P^1_i - a^j_{ik} p_j Y^i_1 Y^k_1)\,.$$

This formula implies that $g$ does not depend on the fourth variable. The same argument can be applied to $h$. It follows from (8) that in the coordinates we have

$$(9) \qquad t_0(X^i, X^i_j, p_i, Y^i_1, P^1_i) = P^1 X^k + X^m_l Y^l_1 p_m\,.$$

Consequently, there exist smooth functions $\alpha, \beta: \boldsymbol{R}^3 \to \boldsymbol{R}$ such that $c_0$ satisfies (7) on $S^0 \times (TT^*)_0$. Since $c_0$ is $L^2_n$-equivariant, (7) is satisfied on $L^2_n(S^0 \times (TT^*)_0)$, which is a dense subset of $(J^1T \times TT^*)_0$. This completes the proof because $c_0$ is continuous. QED.

For the fibre product of two fibre bundles we will denote by $pr_1$ and $pr_2$ the projections to the first and to the second factor of the product, respectively.

**Lemma 4.** *Let $f: J^1T \times TT^* \to TT^*$ be a natural transformation such that $p^2_1 \circ f = p^2_1 \circ pr_2$ and let $n \geqq 2$. Then there exist smooth functions $\alpha, \beta, \gamma: \boldsymbol{R}^3 \to \boldsymbol{R}$ such that*

$$(10) \qquad f = (\alpha \circ \xi)\, \mathscr{F}^{1,2} + (\beta \circ \xi)\, pr_2 + (\gamma \circ \xi)\, \mathscr{L}_{T^*}$$

*where $\xi$ is defined as in lemma 3.*

Proof. Let us consider the natural transformation $c = dp^1_0 \circ f: J^1T \times TT^* \to T$. Then $c$ is as in (7) for suitable functions $\alpha, \beta$. Let us define

$$(11) \qquad f_1 = f - (\alpha \circ \xi)\, \mathscr{F}^{1,2} - (\beta \circ \xi)\, pr_2\,,$$

$f_1: J^1T \times TT^* \to TT^*$. Since $dp^1_0 \circ \mathscr{F}^{1,2} = \varrho^1_0 \circ pr_1$, we have $dp^1_0 \circ f_1 = 0$, and from the exact sequence (5) we get that $f_1$ takes values in $VT^*$. But $VT^*$ is naturally equivalent to $(p_0^1)^* T^* = T^* \times T^*$ and, since the diagram

$$\begin{array}{ccc} J^1T \times TT^* & \xrightarrow{f_1} & T^* \times T^* \\ & {\scriptstyle p_1^2 \circ pr_2} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & T^* & \end{array}$$

commutes, it is enough to find $f_2 = pr_2 \circ f_1 \colon J^1T \times TT^* \to T^*$. Similarly as in the proof of lemma 3 one can show that $f_2 = (\gamma \circ \xi)\, p_1^2 \circ pr_2$, where $\gamma \colon \mathbf{R}^3 \to \mathbf{R}$ is a smooth function (see Remark 1). Since $f_1 = l_{T^*} \circ (p_1^2 \circ pr_2, f_2)$ we see that $f_1 = (\gamma \circ \xi)\, \mathscr{L}_{T^*}$ and from (11) we get (10). QED.

**Theorem.** *Let $D \colon I \to ITT^*$ be a natural lifting and let $n \geqq 2$. Then there exist smooth functions $\alpha, \gamma, \alpha', \beta', \gamma' \colon \mathbf{R}^3 \to \mathbf{R}$ such that*

$$D = (\alpha \circ \xi)\, \mathscr{F}^{1,3} + (\gamma \circ \xi)\, \mathscr{F}^{2,3} \circ \mathscr{L}_{T^*} + (\alpha' \circ \xi)\, l \circ \mathscr{F}^{1,2} + + (\beta' \circ \xi)\, \mathscr{L}_{TT^*} + (\gamma' \circ \xi)\, l \circ \mathscr{L}_{T^*}\,. \tag{12}$$

Proof. Let us consider the corresponding natural transformation $D \colon J^2T \times TT^* \to TTT^*$. Then the diagram

$$\begin{array}{ccc} J^2T \times TT^* & \xrightarrow{D} & TTT^* \\ & {\scriptstyle pr_2} \searrow \quad \swarrow {\scriptstyle p_2^3} & \\ & TT^* & \end{array}$$

commutes. We will consider the natural transformation $dp_1^2 \circ D \colon J^2T \times TT^* \to TT^*$. Let $A^1 = (\varrho_0^1)^{-1} (V \setminus \{0\}) \subset (J^1T)_0$ and $A^2 = (\varrho_0^2)^{-1} (V \setminus \{0\}) \subset (J^2T)_0$. Formula (3) implies that $B_n^3$ acts transitively on fibres of the bundle $\varrho_1^2 \colon A^2 \to A^1$. Since $B_n^3$ acts trivially on $(TT^*)_0$, $dp_1^2 \circ D$ is constant on fibres of the bundle $A^2 \to A^1$. But $A^1$ is dense in $(J^1T)_0$ and consequently $dp_1^2 \circ D$ is constant on fibres of $(J^2T)_0 \to \to (J^1T)_0$. Therefore there exists $f \colon J^1T \times TT^* \to TT^*$ such that the diagram

$$\begin{array}{ccc} J^2T \times TT^* & \xrightarrow{dp_1^2 \circ D} & TT^* \\ & {\scriptstyle \varrho_1^2 \times \mathrm{id}_{TT^*}} \searrow \quad \swarrow {\scriptstyle f} & \\ & J^1T \times TT^* & \end{array}$$

commutes. We apply lemma 4 to $f$ and find that $f = (\alpha \circ \xi)\, \mathscr{F}^{1,2} + (\beta \circ \xi)\, pr_2 + + (\gamma \circ \xi)\, \mathscr{L}_{T^*}$. Let

$$D_1 = D - (\alpha \circ \xi)\, \mathscr{F}^{1,3} - (\gamma \circ \xi)\, \mathscr{F}^{2,3} \circ \mathscr{L}_{T^*}\,. \tag{13}$$

This implies that

$$dp_1^2 \circ D_1(j_x^2X, Z) = (\beta \circ \xi)\, Z\,. \tag{14}$$

We will prove later that $\beta \circ \xi \equiv 0$. Then it follows from the exact sequence (6) that $D_1$ takes values in $VTT^*$ and we have the following commutative diagram:

$$\begin{array}{ccc} J^2T \times TT^* & \xrightarrow{D_1} & TT^* \times_{T^*} TT^* \\ & {\scriptstyle pr_2} \searrow \quad \swarrow {\scriptstyle pr_1} & \\ & TT^* & \end{array}$$

Let $D_2 = pr_2 \circ D_1 \colon J^2T \times TT^* \to TT^*$. Then $D_1 = l_{TT^*} \circ (pr_2, D_2)$. Similarly as above we find that there exist $\alpha', \beta', \gamma' \colon \boldsymbol{R}^3 \to \boldsymbol{R}$ such that $D_2 = (\alpha' \circ \xi)\, \mathscr{F}^{1,2} + (\beta' \circ \xi)\, pr_2 + (\gamma' \circ \xi)\, \mathscr{L}_{T^*}$. This formula and (13) imply (12).

It remains to prove that $\beta \circ \xi \equiv 0$. We shall use the coordinate systems on $(J^2T \times TT^*)_0$ and $(TTT^*)_0$ introduced in the first section. Let

$$(p, Y_1, P^1, \tilde{Y}_2, \tilde{P}^2, \tilde{Y}_3, \tilde{P}^3) = D_1(j_0^2X, p, Y_1, P^1)\,.$$

Since $D_1 \colon (J^2T \times TT^*)_0 \to (TTT^*)_0$ is $B_n^3$-equivariant, lemma 1 and formulae (2), (3) imply

$$(15) \qquad a^i_{jkl} X^l = 0 \Rightarrow a^i_{jkl} p_i Y_1^k \tilde{Y}_2^l = 0$$

for all $(a^i_{jkl}) \in B_n^3$. Since $dp_0^2 \circ D_1(j_0^2X, p, Y_1, P^1) = \tilde{Y}_2$, it follows from (14) that $\tilde{Y}_2 = (\beta \circ \xi)\, Y_1$. This and (15) imply that if $X$ and $Y$ are linearly independent, $p \neq 0$, then $\beta \circ \xi = 0$. Consequently, $\beta \circ \xi$ vanishes on a dense subset of $(J^2T \times TT^*)_0$ and since it is smooth, $\beta \circ \xi \equiv 0$. QED.

Remark 2. In a similar way one can get classification of natural differential operators $\mathscr{T} \to \mathscr{T}T^*$. We consider a natural transformation $D \colon J^1T \times T^* \to TT^*$. Since $B_n^2$ acts transitively on fibres of $A_1 \to V \setminus \{0\}$ and trivially on $V$, the map $dp_0^1 \circ D \colon J^1T \times T^* \to T$ factorizes through $D_1 \colon T \times T^* \to T$. Then we use Lemma 2 and find that $dp_0^1 \circ D = \alpha \varrho_0^1 \circ pr_1$ and the natural transformation $D - \alpha \mathscr{F}_{T^*}$ takes values in $VT^* = T^* \times T^*$. Therefore it is enough to find all natural operators $\tilde{D} \colon J^1T \times T^* \to T^*$. As before we find smooth $\beta \colon \boldsymbol{R} \to \boldsymbol{R}$ such that $\tilde{D} = \beta pr_2$ and conclude that if $D \colon \mathscr{T} \to \mathscr{T}T^*$ is a natural differential operator, then

$$D = \alpha \mathscr{F}_{T^*} + \beta \mathscr{L}_{T^*}\,.$$

Note that we do not need $n \geqq 2$ in this case. This result and the idea of proof is due to Kolář (not published).

Remark 3. In the case $n = 1$ the bundle $TTT^* \to TT^*$ is four-dimensional and, since there are five different liftings, new invariants might appear, at least locally.

I would like to thank professor Kolář for suggestions and corrections.

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## Souhrn

## PŘIROZENÉ LIFTY VEKTOROVÝCH POLÍ DO TANGENCIÁLNÍCH BANDLŮ BANDLŮ 1-FOREM


PIOTR KOBAK



Autor klasifikuje přirozené lifty $D: I \to ITT^*$. Dokazuje, že tvoří 5-parametrickou soustavu operátorů.



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