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ON FORMAL THEORY OF DIFFERENTIAL EQUATIONS III

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Summary. Elements of the general theory of Lie-Cartan pseudogroups (including the intransitive case) are developed within the framework of infinitely prolonged systems of partial differential equations (diffieties) which makes it independent of any particular realizations by transformations of a geometric object. Three axiomatic approaches, the concepts of essential invariant, subgroup, normal subgroup and factorgroups are discussed. The existence of a very special canonical composition series based on Cauchy characteristics is proved and relations to the equivalence problem, theory of geometrical objects and connection theory are briefly mentioned.

Keywords: Lie-Cartan pseudogroups, diffieties, Cauchy characteristics, composition series, equivalence problem, geometrical object.

MS classification: 58H05, 22E65.

With regard to the experiences of the preceding Part II, we still postpone the general theory of diffieties in order to settle the equivalence problem, that is, the Lie-Cartan pseudogroups. In principle two approaches are available: either the pseudogroups are introduced as families of local diffeomorphisms which are solutions of a (rather special) system of partial differential equations with the property that the compositions of solutions are again solutions (cf., e.g., [11, 14, 7, 6]) or, alternatively, they may be identified with the families of automorphisms of appropriate canonical structures, in reality with these structures themselves ([1, 12, 15, 9]). We follow the latter possibility but, in contrast to the common conception, we try to study just the pseudogroups and not their accidental realizations by transformations of a geometrical object. Then the use of infinite prolongations is necessary.

In outline, let $x^1, \ldots, x^m$ be coordinates on a manifold $\mathbf{M}$, let $\tilde{\mathbf{M}}$ be a duplicate of $\mathbf{M}$ with the relevant coordinates $\tilde{x}^1, \ldots, \tilde{x}^m$. Transformations $\mathbf{f}$ of a pseudogroup acting on $\mathbf{M}$ may be represented by the graphs $\tilde{x}^i \equiv f^i(x^1, \ldots, x^m)$, submanifolds of the direct product $\mathbf{M} \times \tilde{\mathbf{M}}$, where f^i satisfy some differential equations specifying the pseudogroup. After the prolongation (we pass to the induced jet-to-jet mappings of infinite order), the manifolds $\mathbf{M}, \tilde{\mathbf{M}}$ are replaced by certain spaces $\mathbf{X}, \tilde{\mathbf{X}}$ with coordinates $x^1, \ldots, x^m, x^{m+1}, \ldots$ and $\tilde{x}^1, \ldots, \tilde{x}^m, \tilde{x}^{m+1}, \ldots$, and the mappings $\mathbf{f}$ turn into some (generalized) diffeomorphisms $\mathbf{g}$ of $\mathbf{X}$ represented by certain graphs

$\tilde{x}^j \equiv g^j(x^1, \ldots, x^{m(j)})$ (here $g^1 = f^1, \ldots, g^m = f^m$) where g^j are solutions of a peculiar system Ω of differential equations with infinite number of independent (!) and dependent variables, coordinates in $\mathbf{X}$, $\tilde{\mathbf{X}}$, respectively. The system Ω can be expressed as the requirement of invariance of certain functions k (*invariants*) and differential forms ξ (*Maurer-Cartan forms*): $k = \mathbf{g}^* k$, $\xi = \mathbf{g}^* \xi$. Alternatively, in terms of graphs, Ω is identified with the module generated by the differences $\xi - \tilde{\xi}$ on the subspace $\mathbf{J} \subset \mathbf{X} \times \tilde{\mathbf{X}}$ defined by the equations $k - \tilde{k} = 0$. Here k, ξ and $\tilde{k}$, $\tilde{\xi}$ (the duplicates) are regarded as functions on the direct product $\mathbf{X} \times \tilde{\mathbf{X}}$.We shall start just at this level in the second chapter and refer to [1] for more elementary comments and a lot of examples.

It follows that pseudogroups are nothing else than a special kind (existence of the symmetry $\mathbf{X} \Leftrightarrow \tilde{\mathbf{X}}$) of "generalized diffitiees" for which the $\mathscr{D}im$ axiom is omitted. This topic will be discussed in the first chapter. On this occasion, our previous methods will be substantially revised and a little attempt will be performed to improve the notation and terminology.

Among the main achievements of the present paper, we mention the existence of a canonical composition series for any pseudogroup, an indispensable tool for any reasonable intrinsical structure theory. Our approach is extremely simple from the conceptual point of view but (unfortunately) it is of a little use for practice. We intend to deal with this important problem in the next Part IV.

PRELIMINARIES ON DIFFETIES

1. Underlying spaces. Let $\mathbb{R}^\infty$ be the space of all sequences $t = (t^1, t^2, \ldots)$ of real numbers with the usual direct product topology. An infinite product of open intervals $a^i < t^i < b^i$ is called a *box*. A topological subspace $\mathbf{U} \subset \mathbb{R}^\infty$ which is the union of a family of such boxes is called a *model space*. Let $\mathbf{J}$ be a topological space, $\mathscr{F} = \mathscr{F}(\mathbf{J})$ (an abbreviation) a family of real functions on $\mathbf{J}$ which satisfies the following condition: there exists an open covering $\mathbf{J} = \bigcup \mathbf{V}_\alpha$ (α varies in an index set) and homeomorphisms $\mathbf{f}_\alpha = (f_\alpha^1, f_\alpha^2, \ldots)\colon \mathbf{V}_\alpha \to \mathbf{U}_\alpha$ (here $f_\alpha \in \mathscr{F}$) onto some model spaces $\mathbf{U}_\alpha$ such that $f \in \mathscr{F}$ if and only if $f = F_\alpha(f^1, \ldots, f^{n(\alpha)})$ on each $\mathbf{V}_\alpha$, where F_α are appropriate smooth in the common sense functions on $\mathbb{R}^{n(\alpha)}$. Then $\mathbf{J}$ is called an *underlying space*, $\mathscr{F}$ the relevant family of *structural functions*, and the auxiliary functions f_α^i appearing in the definition are *coordinates* (of $\mathbf{J}$, on $\mathbf{V}_\alpha$).

The above covering need not be unique, of course, but one can then easily find the transformation rules (based on the existence of the functions F_α) between various families of coordinates, resembling the common theory of finite-dimensional manifolds (cf. [5]). One can also see that the underlying spaces are objects rather near to the inverse limits of such manifolds (used in the previous papers [3, 4]), but we are not interested in these questions. In fact, the common finite-dimensional manifolds appear if $\mathbb{R}^\infty$ is replaced by $\mathbb{R}^m$ in the above definition and in order not to exclude some easy (but rather important) realizations of our theory, the term "underlying space" may also denote such manifolds.

All concepts defined in terms of the structural functions are intrinsical by nature. For instance, *admissible mappings* $\mathbf{g}: \mathbf{J}' \to \mathbf{J}$ between underlying spaces are introduced as continuous transformations satisfying $\mathbf{g}^*\mathscr{F}' \subset \mathscr{F}$ (here $\mathscr{F}' = \mathscr{F}(\mathbf{J}')$). However, some other notions can be most easily introduced by the use of coordinates. For instance, if liftings by the pull-back $\mathbf{g}^*$ of coordinates on every subset $\mathbf{V}_\alpha \subset \mathbf{J}$ can be completed to coordinates on $\mathbf{V}'_\alpha = \mathbf{g}^{-1}(\mathbf{V}_\alpha) \subset \mathbf{J}'$ (at leat locally), then $\mathbf{J}$ is called a *factorspace* of $\mathbf{J}'$ and $\mathbf{g}$ a *fibering* of $\mathbf{J}'$. (In greater detail: for every $p \in \mathbf{V}'_\alpha$ we postulate the coordinates $g^1, g^2, \ldots \in \mathscr{F}'$ on a neighbourhood of p such that $\mathbf{g}^*f^i_\alpha \equiv g^{i(\alpha)}$ for appropriate indices $i(\alpha)$.) Analogously, if the coordinates of $\mathbf{J}'$ can be derived from a part of coordinates of $\mathbf{J}$, then $\mathbf{J}'$ is called a *subspace* of $\mathbf{J}$ with the *inclusion* $\mathbf{g}$. (More precisely: for every α, a certain subfamily of the family $\mathbf{g}^*f^1, \mathbf{g}^*f^2, \ldots$ may serve for the coordinates on $\mathbf{V}'$.)

The fibre mappings are usually assumed surjective (which can be easily achieved). Analogously, the inclusion mappings will be assumed injective so that we may even identify $\mathbf{J}' = \mathbf{g}(\mathbf{J}') \subset \mathbf{J}$. One can then see that the subset $\mathbf{J}' \subset \mathbf{J}$ is determined by certain equations between the coordinates (schematically $f^i_\alpha \equiv F^i_\alpha(\ldots, g^j_\beta, \ldots)$ where $g^j_\beta \equiv \mathbf{g}^*f^j_\beta$ are the coordinates of $\mathbf{J}'$ and F^i_α are smooth functions of a finite number of arguments). It is to be noted that even in the case that these equations are absent, $\mathbf{J}'$ need not be an open subset of the surrounding pace $\mathbf{J}$. We shall speak of a *box subspace* $\mathbf{J}' \subset \mathbf{J}$ in this case. (For instance, the model spaces are box subspaces of $\mathbb{R}^\infty$.) The box subspaces currently appear in practice if some "irregular points" must be left out of the original underlying space to ensure various needed properties (e.g., invertibility of mappings, regularity, existence of bases of various modules, and so on). Such measures will be often taken without explicit warning.

Most of our considerations will be of local nature so that the model spaces with the global systems of coordinates are in reality quite sufficient for the underlying spaces.

2. Several results. For the convenience of the reader, we recall some basic terminology and simple results from [4, 5] omitting (elementary) proofs.

(i) A sequence $\zeta^1, \zeta^2, \ldots \in \mathscr{V}$ is a *basis* of an $\mathscr{F}$-module $\mathscr{V}$ if every $\zeta \in \mathscr{V}$ admits a unique expression $\zeta = \sum z_i\zeta^i$ ($z_i \in \mathscr{F}$) by a finite sum. Let $\mathscr{V}^\wedge$ be the $\mathscr{F}$-module of all $\mathscr{F}$-linear mappings $Z: \mathscr{V} \to \mathscr{F}$ (the values are denoted $\zeta(Z) \in \mathscr{F}$, $\zeta \in \mathscr{V}$). A sequence $Z_1, Z_2, \ldots \in \mathscr{V}$ is a *weak basis* of $\mathbf{V}^\wedge$ if $\zeta(Z_i) = 0$ for every $\zeta \in \mathscr{V}$ and all $i \geqq i(\zeta)$ large enough, and every $Z \in \mathscr{V}^\wedge$ can be uniquely expressed by a weakly convergent series $Z = \sum^\infty z^iZ_i$ ($z^i \in \mathscr{F}$). The bases and the dual bases are coupled in *dual pairs* satisfying $\zeta^i(Z_j) \equiv \delta^i_j$. The existence of a basis will be tacitly supposed for every countably generated module under consideration, then the weak basis for the dual module can be introduced by the above mentioned duality.

(ii) Let $\mathfrak{m}_p \subset \mathscr{F}$ be the maximal ideal of functions $f \in \mathscr{F}$ vanishing at a point $p \in \mathbf{J}$. The $\mathbb{R}$-linear space $\mathscr{V}_p = \mathscr{V}/\mathfrak{m}_p\mathscr{V}$ is called the *localization* (of $\mathscr{V}$ at p). We shall speak of a regular $\mathscr{F}$-module $\mathscr{V}$ if for every (equivalently: for an appropriate) basis $\zeta^1, \zeta^2, \ldots$ of $\mathscr{V}$, the localizations $\zeta_p^1, \zeta_p^2, \ldots$ provide a basis of the $\mathbb{R}$-linear space $\mathscr{V}_p$. In this case, the localizations $(Z_1)_p, (Z_2)_p, \ldots \in \mathscr{V}_p^\wedge$ of a weak basis $Z_1, Z_2, \ldots$ of $\mathscr{V}^\wedge$ provide a weak basis of the $\mathbb{R}$-linear space $\mathscr{V}_p^\wedge$ (in the obvious sense).

(iii) If $\mathscr{V}$ is a regular module, then every $\mathscr{F}$-homomorphism $h: \mathscr{V} \to \mathscr{W}$ into an $\mathscr{F}$-module $\mathscr{W}$ admits the (obviously defined) localization $h_p: \mathscr{V}_p \to \mathscr{W}_p$ (easy verification, compare with [3, page 383]).

(iv) The natural inclusion $\mathscr{U} \subset \mathscr{V}$ of a submodule $\mathscr{U}$ into a module $\mathscr{V}$ is called *regular* if every (equivalently: appropriate) basis of $\mathscr{U}$ can be completed into a basis of $\mathscr{V}$. In this case, if $\mathscr{V}$ is regular then both $\mathscr{U}$ and $\mathscr{V}/\mathscr{U}$ are so. Moreover $(\mathscr{V}/\mathscr{U})_p = \mathscr{V}_p/\mathscr{U}_p$ and $\mathscr{U}^{\perp\perp} = \mathscr{U}$. (Recall that for a subset $A \subset \mathscr{V}$, the submodule $A^\perp \subset \mathscr{V}^\wedge$ consists of all $Z \in \mathscr{V}^\wedge$ satisfying $\zeta(Z) = 0$ for every $\zeta \in A$. Quite analogously the submodule $B^\perp \subset \mathscr{V}$ is defined for a subset $B \subset \mathscr{V}^\wedge$.)

(v) The letter ℓ will denote the *dimension of modules* (the cardinality of a basis). Clearly $\ell(\mathscr{V}) = \ell(\mathscr{V}_p)$ in the regular case. If $\mathscr{U}$ is a submodule of a regular module $\mathscr{V}$ and either $\ell(\mathscr{U}_p)$ or $\ell((\mathscr{V}/\mathscr{U})_p)$ are finite constants independent of $p \in \mathbf{J}$, then the inclusion $\mathscr{U} \subset \mathscr{V}$ may be assumed regular (after appropriate modification of the underlying space, cf. Section 1).

(vi) The *differential* $\mathrm{d}f$ ($f \in \mathscr{F}$) is a familiar concept; it may be identified with the family of all classes $\mathrm{d}_p f$ of the functions $f - f(p) \in \mathscr{F}$ in the factormodules $\mathscr{F}/\mathfrak{m}_p^2\mathscr{F}$. Then the $\mathscr{F}$-module $\Phi = \Phi(\mathbf{J})$ of *differential 1-forms*, and in general, the $\mathscr{F}$-module $\Psi_s = \Psi_s(\mathbf{J})$ of *exterior differential s-forms* arise by pure algebra (we use the abbreviations $\mathscr{F} = \Psi_0$, $\Phi = \Psi_1$). The $\mathscr{F}$-module $\mathscr{T} = \mathscr{T}(\mathbf{J}) = \Phi^\wedge$ of the *vector fields* on $\mathbf{J}$ arises by the duality.

(v) The *exterior differential* $\mathrm{d}: \Psi_s \to \Psi_{s+1}$, the *interior derivative* $Z \urcorner: \Psi_{s+1} \to \Psi_s$ ($Z \in \mathscr{T}$; we apologize for the unusual notation caused by typographical troubles), the *Lie derivative* $\mathscr{L}_Z = \mathrm{d}Z\urcorner + Z \urcorner \mathrm{d}: \Psi_s \to \Psi_s$, and the *commutant* (*Lie bracket*) of vector fields $[\cdot,\cdot]$ can be easily introduced for the underlying spaces and satisfy the common rules. In particular,

$$(1) \qquad \mathscr{L}_X(Y \urcorner \xi) = [X, Y] \urcorner \xi + Y \urcorner \mathscr{L}_X \xi \,.$$

We also recall the notation $Z \urcorner \zeta = \zeta(Z)$ ($\zeta \in \Phi$), $Zf = \mathscr{L}_Z f = Z \urcorner \mathrm{d}f$ ($f \in \mathscr{F}$).

(viii) In terms of the coordinates $f^1, f^2, \ldots$ on $\mathbf{V} \subset \mathbf{J}$ (we omit the subscript α), the differential forms ζ are uniquely expressible by finite sums $\zeta = \sum z_i \,\mathrm{d}f^i$, and the vector fields $Z \in \mathscr{T}$ by weakly convergent infinite series $Z = \sum^{\infty} z^i \partial/\partial f^i$. Here $\partial/\partial f^1$, $\partial/\partial f^2, \ldots \in \mathscr{T}$ is the dual weak basis to the basis $\mathrm{d}f^1, \mathrm{d}f^2, \ldots$ of Φ.

(ix) Let $\Omega \subset \Phi$ be a regular inclusion. The submodule Ω is called *formally* *integrable* (or: *flat*) if any of the following equivalent requirements

$$(2)_{1-4} \qquad \mathrm{d}\Omega = 0 \ (\text{modulo } \Omega), \quad \mathscr{H} \rightharpoondown \mathrm{d}\Omega \subset \Omega, \quad \mathscr{L}_{\mathscr{H}}\Omega \subset \Omega, \quad [\mathscr{H}, \mathscr{H}] \subset \mathscr{H} \quad (\mathscr{H} = \Omega^{\perp} \subset \mathscr{T})$$

is valid. We shall use the term an (*locally*) *integrable* submodule Ω if there exists a basis of Ω consisting of total differentials of functions, at least locally. If $\ell(\mathscr{F}/\Omega) < \infty$, then these concepts are equivalent (*Frobenius theorem*).

(x) Let $\Omega \subset \Phi$ be a regular inclusion. Let Adj $\Omega \subset \Phi$ be the submodule of all forms of the kind $\mathscr{L}_Y\omega = Y \rightharpoondown \mathrm{d}\omega$ ($\omega \in \Omega$, $Y \in \mathscr{H}$). Assuming the regularity of the inclusion Adj $\Omega \subset \Phi$, we have $Z \in (\text{Adj }\Omega)^{\perp}$ if and only if $\mathscr{L}_{fZ}\Omega \subset \Omega$ for all $f \in \mathscr{F}$ (cf. [4]). This implies the formal integrability of Adj Ω and, if $\ell(\Omega) < \infty$, the integrability of Adj Ω. In this case, let $\mathrm{d}g^1, \mathrm{d}g^2, \ldots$ be a basis of Adj Ω. Then the functions $g^1, g^2, \ldots$ (more rigorously, any function $g \in \mathscr{F}$ satisfying $\mathrm{d}g \in \text{Adj }\Omega$) are called *adjoint variables* of Ω. There exists a basis of Ω expressible in terms of these adjoint variables.

(xi) If $\mathscr{K}$ is a ring, then the concepts of a basis and a weak basis for $\mathscr{K}$-modules are obvious. We shall be interested only in the case $\mathbb{R} \subset \mathscr{K} \subset \mathscr{F}$ and in various $\mathscr{K}$-submodules $\mathscr{W} \subset \mathscr{V}$ of an $\mathscr{F}$-module $\mathscr{V}$; every $\mathscr{F}$-module may be automatically regarded as a $\mathscr{K}$-submodule by the restriction of the coefficient ring $\mathscr{F}$ to $\mathscr{K}$. Then the least $\mathscr{K}$-submodule of the $\mathscr{F}$-module $\mathscr{V}$ containing a subset $A \subset \mathscr{V}$ will be expressively designed by $\mathscr{K} \otimes A$. In particular, $\mathscr{F} \otimes \mathscr{W}$ is the least $\mathscr{F}$-submodule of the module $\mathscr{V}$ that contains $\mathscr{W}$ and, say, $\Phi = \mathscr{F} \otimes \mathrm{d}\mathscr{F}$ where $\mathrm{d}\mathscr{F}$ denotes the set of all differentials $\mathrm{d}f$ ($f \in \mathscr{F}$).

(xii) Continuing the above notation, a $\mathscr{K}$-module $\mathscr{W}$ is called *regular* if the $\mathscr{F}$-module $\mathscr{F} \otimes \mathscr{W}$ is regular (in the sense of (ii)) and if there exists a sequence $\zeta^1, \zeta^2, \ldots \in \mathscr{W}$ which may simultaneously serve for a $\mathscr{K}$-basis of $\mathscr{W}$ and an $\mathscr{F}$-basis of $\mathscr{F} \otimes \mathscr{W}$. In this case the sequence $Z_1, Z_2, \ldots \in \mathscr{W}^\wedge$ ($\mathscr{W}^\wedge$ is the $\mathscr{K}$-module of all $\mathscr{K}$-linear mappings $Z: \mathscr{W} \to \mathscr{K}$) defined by the duality $\zeta^i(Z_j) \equiv \delta^i_j$ provides a common weak basis of $\mathscr{W}^\wedge$ and $(\mathscr{F} \otimes \mathscr{W})^\wedge$. It follows that $(\mathscr{F} \otimes \mathscr{W})^\wedge$ is the space of all sums $Z = \sum^{\infty} z^i Z_i$ ($z^i \in \mathscr{F}$) and $Z \in \mathscr{W}^\wedge$ if and only if $z^i \in \mathscr{K}$ for all i. In particular $\mathscr{W}^\wedge \subset (\mathscr{F} \otimes \mathscr{W})^\wedge$.

(xiii) In order not to produce confusions, we abstain from introducing the concepts of localization, (formal) integrability, Adj, and the orthogonal complement in the area of $\mathscr{K}$-modules. These concepts will be avoided by the use of other means.

3. Generalized diffieties [3, 4, 17]. A formally integrable $\mathscr{F}$-submodule $\Omega \subset \Phi$ will be called a *diffiety* if Ω is finitely generated by the operators $\mathscr{L}_Z$ ($Z \in \mathscr{H}$). In greater detail, we suppose (2) together with the existence of *good filtrations* of Ω, i.e., there are filtrations Ω^*: $\Omega^0 \subset \Omega^1 \subset \ldots \subset \Omega = \bigcup \Omega^l$ by regular submodules Ω^l with $\ell(\Omega^l) < \infty$ and such that

$$(3)_{1,2} \qquad \Omega^{l+1} \subset \Omega^l + \mathscr{L}_{\mathscr{H}}\Omega^l \text{ (all } l), \quad \Omega^{l+1} = \Omega^l + \mathscr{L}_{\mathscr{H}}\Omega^l \ (l \geqq l_0(\Omega^*))$$

where $l_0 = l_0(\Omega^*) < \infty$. Occasionally we put $\Omega^l = 0$ $(l < 0)$ for technical reasons. (In comparison with [3, 4], the axiom $\mathscr{D}im$ is omitted so that $\ell(\mathscr{H}) = \infty$ may well happen.)

Let another underlying space $\mathbf{J}'$, a diffiety $\Omega' \subset \Phi' = \Phi(\mathbf{J}')$, and a mapping $\mathbf{g}: \mathbf{J}' \to \mathbf{J}$ be given. If $\mathbf{g}$ is a fibering and every space $\mathscr{H}'_p$ $(= (\Omega'^{\perp})_p, p \in \mathbf{J}')$ is surjectively projected onto $\mathscr{H}_q$ $(q = \mathbf{g}(p))$ by the tangent mapping $\mathbf{g}_*$ induced from $\mathbf{g}$, then Ω is called a *factordiffiety* of Ω'. Analogously, if $\mathbf{g}$ is an inclusion and $\mathbf{g}_*\mathscr{H}'_p \equiv \mathscr{H}_q$ $(q = \mathbf{g}(p))$, then Ω' is called a *subdiffiety* of Ω. (In less formal terms, we identify $\mathbf{J}' = \mathbf{g}(\mathbf{J})$ with a subset of $\mathbf{J}$ for the inclusion $\mathbf{g}$ (cf. Section 1). Then the tangent mapping $\mathbf{g}_*$ is injective and we have $\mathscr{H}' = \mathscr{H}$ along the subspace $\mathbf{J}' \subset \mathbf{J}$.) The concepts of *automorphism* and *isomorphism* of diffieties are obvious. We abstain from introducing the morphisms of diffieties in full generality, already the isomorphisms may be unforeseenly complicated. (For the particular case when $\ell(\mathscr{H}) = \ell(\mathscr{H}') < \infty$, the properties [4, formula (4)] can be verified so that no confusion arises.)

3. Algebraic tools. Let $\mathscr{A} = \odot\mathscr{H} = \oplus\mathscr{A}^l$ $(\mathscr{A}^0 = \mathscr{F}, \mathscr{A}^1 = \mathscr{H}, \mathscr{A}^2 = \mathscr{H} \odot \odot \mathscr{H}, \ldots)$ be the free graded commutative algebra over the $\mathscr{F}$-module $\mathscr{H}$. Then the graded $\mathscr{F}$-module $\mathscr{M} = \operatorname{Grad} \Omega^* = \oplus\mathscr{M}^l$ $(\mathscr{M}^l = \Omega^l/\Omega^{l-1})$ turns into a graded $\mathscr{A}$-module with the multiplication

$$(4) \qquad Z \, . \, [\omega] = [\mathscr{L}_Z\omega] = [Z \urcorner \, d\omega] \quad (Z \in \mathscr{H}, \ \omega \in \Omega)$$

where the square brackets denote the classes in the module $\operatorname{Grad} \Omega^* = \mathscr{M}$. Owing to (iii), Section 2, the localization at $p \in \mathbf{J}$ yields the module $\mathscr{M}_p = M = \oplus M^l$ $(M^l \equiv \mathscr{M}^l_p)$ over the polynomial ring $A = \mathscr{A}_p = \oplus A^l$ $(A^0 = \mathbb{R}, A^1 = H, A^2 = H \odot H, \ldots$; here $H = \mathscr{H}_p)$. Alas, if $\ell(\mathscr{H}) = \ell(H) = \infty$, then the methods of commutative algebra cannot be directly applied.

For any graded $\mathscr{A}$-module $\mathscr{M}$ (in particular, for $\mathscr{M} = \operatorname{Grad} \Omega^*$), the *Koszul homologies* $\mathrm{H}(\mathscr{M})^l_s$ arise from the complex

$$(5) \qquad \ldots \to \mathscr{M}^{l-1} \otimes (\wedge^{s+1}\mathscr{H}) \to \mathscr{M}^l \otimes (\wedge^s\mathscr{H}) \to \mathscr{M}^{l+1} \otimes (\wedge^{s-1}\mathscr{H}) \to \ldots$$

with the differentials

$$(6) \qquad \partial(\mu \otimes Z_0 \wedge \ldots \wedge Z_s) = \sum (-1)^i Z_i\mu \otimes Z_0 \wedge \ldots \wedge Z_{i-1} \wedge \wedge Z_{i+1} \wedge \ldots \wedge Z_s \, .$$

The homologies $\mathrm{H}(M)^l_s$ for a graded A-module M may be introduced quite analogously or by the localization: $\mathrm{H}(\mathscr{M}_p)^l_s \equiv (\mathrm{H}(\mathscr{M})^l_s)_p$. It is known that the zeroth homologies specify the generators, e.g., $\mathrm{H}(M)^l_0 = M^l/\mathrm{H}M^{l-1}$ are the generators lying in M^l. Owing to $(3)_2$ we have $\mathrm{H}(\operatorname{Grad} \Omega^*)^l_0 = 0$ $(l \geqq l_0)$, but the important property $\mathrm{H}(\operatorname{Grad} \Omega^*)^l_s = 0$ (l large enough) is not self-evident.

5. The reduction argument. Assume $\ell(\mathscr{H}) = \infty$ for a moment; in the other case the results of this Section are trivial. Assume moreover that an appropriate series of differentials $dx^2, dx^2, \ldots \in d\mathscr{F}$ may serve for a basis of the module $\Phi/\Omega = \mathscr{H}^\wedge$ (we use the regularity of the inclusion $\Omega \subset \Phi$ here). Then the duality $dx^i(\partial_j) = \partial_j x^i \equiv \delta^i_j$ (together with $\omega(\partial_j) = 0$, $\omega \in \Omega$) provides a certain weak basis of $\mathscr{H}^\wedge$; note that clearly $[\partial_i, \partial_j] \equiv 0$. (Such *standard bases* are rather useful and will be often refered to.)

Let $\omega^1, \ldots, \omega^m$ be a basis of the module Ω^{l_0}. If $l \geqq l_0$, then the forms

$$\omega^j_{i_1 \ldots i_s} = \mathscr{L}_{\partial_{i_1}} \ldots \mathscr{L}_{\varrho_{i_s}} \omega^j \quad (j = 1, \ldots, m;\ s = 0, \ldots, l - l_0) \tag{7}$$

generate the module Ω^l. Note that $\omega_I = \omega_{I'}$ if the multiindices $I = i_1 \ldots i_s$, $I' = i'_1 \ldots i'_s$ differ only by the order of the terms.

One can then observe that $\omega^j_i = \partial_i \urcorner d\omega^j = 0$ if i is large enough. (This follows from the trivial fact that every form can be expressed only by a finite number of variables.) Consequently $\omega^j_I \neq 0$ may occur only for multiindices $I = i_1 \ldots i_s$ with all terms i_k small enough. In particular, the submodule $\mathscr{C}(\mathscr{M}) = \mathscr{H} \cap \operatorname{Ann} \mathscr{M} \subset \mathscr{H}$ (consisting of all $Z \in \mathscr{H}$ which satisfy $Z . \mathscr{M} = 0$) is of a finite codimension and $\mathscr{M} = \operatorname{Grad} \Omega^*$ may be regarded as a module over the ring of polynomials from $\mathscr{A}' = \odot\mathscr{H}/\mathscr{C}(\mathscr{M})$ with a finite number of indeterminates. Quite analogous conclusions can be drawn if one deals with a finite number of filtrations. It follows that *the methods of the theory of Noetherian modules may be applied and all algebraic results of* [3] *can be accepted for our generalized case.*

After the reduction to the Noetherian case of polynomials like $\mathscr{A}'$, the homological algebra does not cause any troubles. However, one can also deal with the homology of the original $\mathscr{A}$-modules. Slightly modifying the problem, let us consider a graded $\odot\mathscr{H}$-module $\mathscr{M}$ under the assumption that there is a submodule $\mathscr{H}' \subset \mathscr{H}$ which operates trivially on $\mathscr{M}$ (i.e., $\mathscr{H}' . \mathscr{M} = 0$) so that $\mathscr{M}$ may be regarded as a module over $\odot \mathscr{H}/\mathscr{H}'$. Besides the original homologies $H(\mathscr{M})^l_s$ defined by (5), we have other homologies denoted $H'(\mathscr{M})^l_s$ arising from the complex (5) with $\mathscr{H}$ replaced by $\mathscr{H}/\mathscr{H}'$. The result is that $H(\mathscr{M})^l_s$ *is the direct sum of certain number of copies of* $H'(\mathscr{M})^l_r$ *with* $0 \leqq r \leqq s$. (A sketch of proof: By abuse of notation, let $\mathscr{H}/\mathscr{H}' \subset \mathscr{H}$ be a submodule complementary to the submodule $\mathscr{H}' \subset \mathscr{H}$. Then the terms of (5) can be decomposed into the direct sums

$$\mathscr{M}^l \otimes (\Lambda^s \mathscr{H}) = \oplus \mathscr{M}^l \otimes (\Lambda^r \mathscr{H}/\mathscr{H}') \otimes (\Lambda^{s-r} \mathscr{H}') \quad (0 \leqq r \leqq s),$$

where the factors $\Lambda^{s-r}\mathscr{H}'$ do not affect the differential (6). It follows that (5) may be replaced by the direct sum of the complexes $\ldots \to \mathscr{M}^l \otimes (\Lambda^r \mathscr{H}/\mathscr{H}') \to \ldots$, their number being $\ell(\Lambda^{s-r}\mathscr{H}')$, with homologies $H'(\mathscr{M})^l_r$ as claimed.)

Applying this result to the particular case $\mathscr{M} = \operatorname{Grad} \Omega^*$, $\mathscr{H}' = \mathscr{C}(\mathscr{M})$, one may employ the familiar property $H'(\mathscr{M})^l_r = 0$ for all l large enough (cf. [3, Section 34]) to conclude that $H(\mathscr{M})^l_s = 0$ for such l. In more explicit terms, we have $H(\operatorname{Grad} \Omega^*)^l_s =$

$= 0$ if $l \geqq l_s = l_s(\Omega^*)$ with certain uniformly bounded constants $l_s \leqq l(\Omega^*) < \infty$. This provides a useful generalization of the property $(3)_2$.

6. The Cauchy characteristics. We shall continue the above reasoning in a direction nontrivial even if $\ell(\mathscr{H}) < \infty$ (thus correcting [3, Section 24]). For a given filtration Ω^*, let $\mathscr{C}(\mathscr{M}^l) = \mathscr{H} \cap (\mathrm{Adj}\, \Omega^l)$ be the module of all $Z \in \mathscr{H}$ for which $Z \,.\, \mathscr{M}^l = 0$ (cf. [4, (v) in Section 9]). Clearly $\mathscr{C}(\mathscr{M}) = \bigcap \mathscr{C}(\mathscr{M}^l)$. One can then see that $\mathrm{H}(\mathscr{M})_0^{l+1} = 0$ implies $\mathscr{C}(\mathscr{M}^{l+1}) \supset \mathscr{C}(\mathscr{M}^l)$ so that $\mathscr{C}(\mathscr{M}^l) \supset \mathscr{C}(\mathscr{M})$ for all $l \geqq l_0$; in particular $\mathscr{C}(\mathscr{M}^0) = \mathscr{C}(\mathscr{M})$ if we suppose $l_0 = l_0(\Omega^*) = 0$. (On the other hand, assuming $\ell(\mathscr{H}) = n < \infty$, one can verify that $\mathrm{H}(\mathscr{M})_n^{l+1} = 0$ implies $\mathscr{C}(\mathscr{M}^{l+1}) \subset \mathscr{C}(\mathscr{M}^l)$; cf. [3, (v) in Section 26] for the interpretation of the n-order homologies. In the case $\ell(\mathscr{H}) = \infty$, the reduction argument can be applied with the same result.) The module $\mathscr{C}(\mathscr{M})$ depends on the choice of the filtration Ω^*, of course.

Let us temporarily introduce the family $C(\Omega^l)$ of all submodules $\mathscr{Z} \subset \mathscr{H}$ satisfying $\mathscr{L}_{\mathscr{Z}}^m \Omega^l \subset \Omega^{l+c}$ for all $m \geqq 0$ and an appropriate $c = c(\mathscr{Z})$. Assuming $k \geqq l$, the inclusion $C(\Omega^k) \subset C(\Omega^l)$ follows by a trivial argument (with c replaced by $c + k - 1$). On the other hand, assume $\mathscr{Z} \in C(\Omega^l)$ and $l \geqq l_0$. Then, according to $(3)_2$, we have $\mathscr{L}_{\mathscr{Z}}^m \Omega^{l+1} = \mathscr{L}_{\mathscr{Z}}^m(\Omega^l + \mathscr{L}_{\mathscr{Z}}\Omega^l)$ where

$$(8) \qquad \begin{aligned} &\mathscr{L}_{\mathscr{Z}}^m \mathscr{L}_{\mathscr{H}} \Omega^l \subset \mathscr{L}_{\mathscr{Z}}^{m-1}(\mathscr{L}_{\mathscr{H}}\mathscr{L}_{\mathscr{Z}} + \mathscr{L}_{[\mathscr{H},\mathscr{Z}]})\,\Omega^l \subset \\ &\subset (\mathscr{L}_{\mathscr{Z}}^{m-1}\mathscr{L}_{\mathscr{H}}\mathscr{L}_{\mathscr{Z}} + \mathscr{L}_{\mathscr{Z}}^{m-1}\mathscr{L}_{\mathscr{H}})\,\Omega^l \subset \ldots \\ &\ldots \subset (\mathscr{L}_{\mathscr{H}}\mathscr{L}_{\mathscr{Z}}^m + \mathscr{L}_{\mathscr{H}}\mathscr{L}_{\mathscr{Z}}^{m-1} + \ldots + \mathscr{L}_{\mathscr{H}})\,\Omega^l \subset \Omega^{l+c+1}, \end{aligned}$$

so that $C(\Omega^l) \subset C(\Omega^{l+1})$. Altogether, $C(\Omega^l) = C(\Omega^{l_0})$ is stable for $l \geqq l_0 = l_0(\Omega^*)$. (At this stage, one should prove that the family $C(\Omega^l)$, $l \geqq l_0$, does not in reality depend on the choice of the filtration, but we are passing to a better result.)

As the interrelations between $\mathscr{C}(\mathscr{M}^l)$ and $C(\Omega^l)$ are concerned, we mention the inclusion $\mathscr{Z} + \mathscr{C}(\mathscr{M}^{l_0}) \in C(\Omega^{l_0})$ valid for any $\mathscr{Z} \in C(\Omega^{l_0})$. (Choosing $Z \in \mathscr{Z}$, $Y \in \mathscr{C}(\mathscr{M}^{l_0})$, the desired inclusion easily follows by direct calculation of $\mathscr{L}_{Z+Y}^m \Omega^{l_0}$ with the use of $\mathscr{L}_{Z+Y}^m = (\mathscr{L}_Z + \mathscr{L}_Y)^m$, $\mathscr{L}_Y \Omega^{l_0} \subset \Omega^{l_0}$.) Conversely, any $\mathscr{Z} \in C(\Omega^{l_0})$ is a submodule of a certain module $\mathscr{C}(\mathrm{Grad}\, \bar{\Omega}^*)$ for a proper choice of the filtration $\bar{\Omega}^*$: $\bar{\Omega}^0 \subset \bar{\Omega}^1 \subset \ldots \subset \Omega = \bigcup \bar{\Omega}^l$ of the diffiety Ω. (For the proof, let us take $\bar{\Omega}^l = \sum \mathscr{L}_{\mathscr{Z}}^m \Omega^l$. Then $\bar{\Omega}^l \subset \bar{\Omega}^{l+1}$ and $\Omega^l \subset \bar{\Omega}^l \subset \Omega^{l+c}$, hence $\Omega = \bigcup \bar{\Omega}^l$. Moreover,

$$\mathscr{L}_{\mathscr{H}} \bar{\Omega}^l = \sum \mathscr{L}_{\mathscr{H}} \mathscr{L}_{\mathscr{Z}}^m \Omega^l \subset \ldots \subset \sum \mathscr{L}_{\mathscr{Z}}^m \mathscr{L}_{\mathscr{H}} \Omega^l \subset \bar{\Omega}^{l+1}$$

($\ldots$ are arrangements as in (8)) with equality for l large; cf. Lemma 9 below. So the filtration $\bar{\Omega}^*$ is good and clearly $\mathscr{L}_{\mathscr{Z}} \bar{\Omega}^l \subset \Omega^l$, hence $\mathscr{Z} \subset \mathscr{C}(\mathrm{Grad}\, \bar{\Omega}^*)$ as required.)

Altogether, we may conclude that there is a unique maximal element in the family $C(\Omega^{l_0})$. At the same time, this element is identical with the greatest module of the kind $\mathscr{C}(\mathrm{Grad}\, \bar{\Omega}^*)$ for various filtrations $\bar{\Omega}^*$ of the diffiety Ω. Consequently, since it does not depend on the choice of the filtration, it may be denoted by $\mathscr{C}(\Omega)$ and called the *Cauchy characteristic module of the diffiety* Ω. Now the family $C(\Omega^{l_0})$ may be forgotten: it consists of merely of submodules of $\mathscr{C}(\Omega)$.

7. Theorem. *The set $\mathscr{C}(\Omega)$ of all vector fields $Z \in \mathscr{H}$ satisfying $\mathscr{L}_Z^m \Omega^l \subset \Omega^{l+c}$ with an appropriate $c = c(Z)$ is a submodule of $\mathscr{H}$ independent of the choice of the filtration Ω^* and the level l provided $l \geqq l_0(\Omega^*)$. Moreover, $\mathscr{C}(\Omega) \subset \mathscr{C}(\text{Grad } \Omega^*)$ with equality for a suitable choice of the filtration.*

8. Aside. In the recent paper [5], a subset $\boldsymbol{G} \subset \mathscr{T}(\mathbf{J})$ is introduced such that $Z \in \boldsymbol{G}$ if and only if, for every fixed $f \in \mathscr{F}$, the totality of all functions $f, Zf, Z^2f, \dots$ can be expressed by a finite number of variables. One can easily see that $Z \in \boldsymbol{G}$ if and only if the series of forms $\varphi, \mathscr{L}_Z\varphi, \mathscr{L}_Z^2\varphi, \dots$ can be expressed by a finite number of variables for every given $\varphi \in \Phi$. Using this criterion, one can prove $\mathscr{C}(\Omega) = \boldsymbol{G} \cap \mathscr{H}$. Besides, note that $\boldsymbol{G}$ consists of the set of all infinitesimal transformations of one-parameter groups acting on $\mathbf{J}$, at least locally and in a little peculiar sense (which cannot be explained here).

9. Lemma. *Let $\bar{\Omega}^*$ be a filtration of a diffiety Ω (not necessarily a good one) satisfying $(3)_1$ (that is, $\bar{\Omega}^{l+1} \supset \mathscr{L}_{\mathscr{H}}\bar{\Omega}^l$ for all l). Let Ω^* be a good filtration of Ω such that $\Omega^l \subset \bar{\Omega}^l \subset \Omega^{l+c}$ for all l and a fixed c. Then $\bar{\Omega}^*$ is good.*

Proof. Assume $c = 1$ for a moment and consider the $\odot\mathscr{H}$-submodule $\mathscr{G} = \oplus \mathscr{G}^l \subset \text{Grad } \Omega^* (\mathscr{G}^l \equiv \bar{\Omega}^l/\Omega^l)$. Using the reduction argument and the Hilbert base theorem, one can see that $\mathscr{G}^{l+1} = \mathscr{H} \,.\, \mathscr{G}^l$, hence $\mathscr{L}_{\mathscr{H}}\bar{\Omega}^l + \Omega^{l+1} = \bar{\Omega}^{l+1}$ for l large enough. But clearly $\mathscr{L}_{\mathscr{H}}\bar{\Omega}^l + \bar{\Omega}^l \supset \mathscr{L}_{\mathscr{H}}\Omega^l + \Omega^l \supset \Omega^{l+1}$ so that, altogether, $\mathscr{L}_{\mathscr{H}}\bar{\Omega}^l + \bar{\Omega}^l \supset \mathscr{L}_{\mathscr{H}}\bar{\Omega}^l + \Omega^{l+1} \supset \bar{\Omega}^{l+1}$. Since the opposite inclusion is postulated, we are done.

For the general case of c, one can successively apply the preceding argument on the modules $\bar{\Omega}^l + \Omega^{l+c-1}, \bar{\Omega}^l + \Omega^{l+c-2}, \dots$ instead of $\bar{\Omega}^l$.

AXIOMS FOR PSEUDOGROUPS

10. Groupieties. Let $\mathbf{X}, \tilde{\mathbf{X}}$ be isomorphic underlying spaces, $\mathbf{x}: \tilde{\mathbf{X}} \to \mathbf{X}$ a fixed invertible mapping (an isomorphism). We shall abreviate $\tilde{\varphi} = \mathbf{x}^*\varphi \in \Phi(\tilde{\mathbf{X}})$ for any $\varphi \in \Phi(\mathbf{X})$. Let $\mathbf{k}: \mathbf{X} \to \mathbf{K}$ be a fixed fibering. The functions $k = \mathbf{k}^*f$ $(f \in \mathscr{F}(\mathbf{K}))$ constitute the subring $\mathscr{K} = \mathbf{k}^*\mathscr{F}(\mathbf{K}) \subset \mathscr{F}(\mathbf{X})$ and are called *invariants*. We introduce the direct product $\mathbf{X} \times \tilde{\mathbf{X}}$ with the natural projections $\mathbf{p}: \mathbf{X} \times \tilde{\mathbf{X}} \to \mathbf{X}$, $\tilde{\mathbf{p}}: \mathbf{X} \times \tilde{\mathbf{X}} \to \tilde{\mathbf{X}}$ on the factors, and the subspace $\mathbf{J} \subset \mathbf{X} \times \tilde{\mathbf{X}}$ consisting of all points $(p, q) \in \mathbf{X} \times \tilde{\mathbf{X}}$ such that $k(p) = \tilde{k}(q)$ for all invariants $k \in \mathscr{K}$. Let $\mathbf{i}: \mathbf{J} \to \mathbf{X} \times \mathbf{X}$ be the natural inclusion. We shall identify $\varphi = \mathbf{i}^*\mathbf{p}^*\varphi$, $\tilde{\varphi} = \mathbf{i}^*\tilde{\mathbf{p}}^*\tilde{\varphi}$ so that $\Psi_s(\mathbf{X}) \subset \Psi_s(\mathbf{J})$ and $\Psi_s(\tilde{\mathbf{X}}) \subset \Psi_s(\mathbf{J})$ are regarded as submodules.

A diffiety $\Omega \subset \Phi(\mathbf{J})$ is called a *groupiety* if Ω is generated by all forms $\xi - \tilde{\xi}$ $(\xi \in \Xi)$ where $\Xi \subset \Phi(\mathbf{X})$ is a regular $\mathscr{K}$-submodule with $\mathscr{F}(\mathbf{X}) \otimes \Xi = \Phi(\mathbf{X})$. The elements $\xi \in \Xi$ are called *Maurer-Cartan forms.*

Finite-dimensional $\mathbf{X}$ are admitted, one then deals with (local) Lie groups. Zero-dimensional $\mathbf{K}$ are admitted, this is the *transitive subcase* with $\mathscr{K} = \mathbb{R}$. (We shall soon see that the theory may be reduced to finite-dimensional $\mathbf{K}$ without any essential loss of generality.)

11. Immediate consequences. The above requirements imposed on Ξ can be a little (formally) weakened. Assume only that $\Xi \subset \Phi(\mathbf{X})$ is a $\mathscr{K}$-submodule such that there is a sequence $\xi^1, \xi^2, \ldots \in \Xi$ which may serve for an $\mathscr{F}(\mathbf{X})$-basis of $\Phi(\mathbf{X})$. Then clearly $\mathscr{F}(\mathbf{X}) \otimes \Xi = \Phi(\mathbf{X})$ which is a regular module. Let $\xi = \sum f_i \xi^i \in \Xi$ for certain $f_i \in \mathscr{F}(\mathbf{X})$. Then

$$\sum (f_i - \tilde{f}_i)\, \xi^i = (\xi - \tilde{\xi}) + \sum \tilde{f}_i(\xi^i - \tilde{\xi}^i) \in \Omega\,,$$

hence $f_i \equiv \tilde{f}_i$ and consequently $f_i \in \mathscr{K}$ for all i. It follows that $\xi^1, \xi^2, \ldots$ is also a $\mathscr{K}$-basis of Ξ, hence Ξ is a regular module. According to (vii), Section 2, the duality $\xi^i(X_j) \equiv \delta^i_j$ defines a common weak basis $X_1, X_2, \ldots$ of the $\mathscr{K}$-module $\Xi^\wedge$ and of the $\mathscr{F}(\mathbf{X})$-module $\Phi(\mathbf{X})^\wedge = \mathscr{T}(\mathbf{X})$. Turning to more important results, these bases will be of constant use.

(i) Let $\mathrm{d}\xi^i = \sum f^i_{jk} \xi^j \wedge \xi^k$ and denote $\omega^i = \xi^i - \tilde{\xi}^i \in \Omega$. Then

$$\mathrm{d}\omega^i = \sum f^i_{jk} \xi^j \wedge \omega^k + \sum \tilde{f}^i_{jk} \omega^j \wedge \tilde{\xi}^k \tag{9}$$

$+ \sum (f^i_{jk} - \tilde{f}^i_{jk})\, \tilde{\xi}^j \wedge \xi^k$. In view of $(2)_1$ we have $f^i_{jk} \equiv \tilde{f}^i_{jk}$, hence $f^i_{jk} \in \mathscr{K}$. It follows that $\mathrm{d}\xi = \sum f_{jk} \xi^j \wedge \xi^k$ with coefficients $f_{jk} \in \mathscr{K}$ for arbitrary $\xi \in \Xi$.

(ii) Let $\mathscr{X} \subset \mathscr{T}(\mathbf{X})$ be the $\mathscr{K}$-submodule of all vector fields $X \in \mathscr{T}(\mathbf{X})$ such that $X \rceil \Xi \subset \mathscr{K}$. Clearly $\mathscr{X} = \Xi^\wedge$. Choosing various i, j in the inclusion $X_j \rceil \mathrm{d}(k\xi^i) = X_j k \,.\, \xi^i + \delta^i_j \mathrm{d}k$ (where $k \in \mathscr{K}$), one concludes $X_j k \in \mathscr{K}$ (hence $\mathscr{X} \,.\, \mathscr{K} \subset \mathscr{K}$) and $\mathrm{d}k \in \Xi$ (hence $\mathrm{d}\mathscr{K} \subset \Xi$). Moreover, owing to (1) with $X, Y \in \mathscr{X}$ and $\xi \in \Xi$, one obtains $[\mathscr{X}, \mathscr{X}] \subset \mathscr{X}$.

(iii) Let $Y_1, Y_2, \ldots, Z_1, Z_2, \ldots \in \mathscr{T}(\mathbf{X} \times \tilde{\mathbf{X}})$ be vector fields defined by $\xi^i(Y_j) = \tilde{\xi}^i(Z_j) \equiv \delta^i_j$, $\tilde{\xi}^i(Y_j) = \xi^i(Z_j) = 0$. The sequence $Y_1, Z_1, Y_2, Z_2, \ldots$ is a weak basis of $\mathscr{T}(\mathbf{X} \times \tilde{\mathbf{X}})$. A vector field $Z = \sum (f^i Y_i + g^i Z_i) \in \mathscr{T}(\mathbf{X} \times \tilde{\mathbf{X}})$ is tangent to the subspace $\mathbf{J} \subset \mathbf{X} \times \tilde{\mathbf{X}}$ if and only if $Zk = Z\tilde{k}$ for every $k \in \mathscr{K}$ at all points of $\mathbf{J}$. If this is true, then the relevant vector field induced on $\mathbf{J}$ (and lying in $\mathscr{T}(\mathbf{J})$) will be denoted by the same letter Z. With this notation, $Z \in \mathscr{H}$ if and only if $\omega^i(Z) \equiv 0$, that is, $f^i = \xi^i(Z) \equiv \tilde{\xi}^i(Z) = g^i$, hence $Z = \sum f^i(Y^i + Z^i)$ along the subspace $\mathbf{J} \subset \mathbf{X} \times \tilde{\mathbf{X}}$.

(iv) Continuing the previous point, let $\omega^1, \ldots, \omega^m$ be forms that provide a family of generators of the $\mathscr{F}(\mathbf{J})$-module Ω by repeated use of the operators $\mathscr{L}_Z$ ($Z \in \mathscr{H}$); cf. (3). We may even suppose ω^j to be of the special kind $\omega^j = \xi^j - \tilde{\xi}^j$. Also the vector fields of the type $Z = Y_i + Z_i$ are quite sufficient. Calculations with the operators $\mathscr{L}_Z$ can be carried out of $\mathbf{J}$ into the surrounding space $\mathbf{X} \times \tilde{\mathbf{X}}$. Then the generators of Ω can be written as

$$(10) \qquad \mathbf{i}^* \mathscr{L}_{\mathbf{Y}_{i_1}+\mathbf{Z}_{i_1}} \ldots \mathscr{L}_{\mathbf{Y}_{i_s}+\mathbf{Z}_{i_s}} (\xi^i - \tilde{\xi}^i)^{\sim} = $$
$$= \mathscr{L}_{\mathbf{X}_{i_1}} \ldots \mathscr{L}_{\mathbf{X}_{i_s}} \xi^i - (\mathscr{L}_{\mathbf{X}_{i_1}} \ldots \mathscr{L}_{\mathbf{X}_{i_s}} \xi^i)^{\sim} = \xi^i_I - \tilde{\xi}^i_I$$

(an abbreviation with $I = i_1 \ldots i_s$). Since all forms $\xi \in \Xi$ with the property $\mathbf{i}^*(\xi - \tilde{\xi}) = 0$ are linear combinations of differentials of invariants (briefly $\xi \in \mathscr{K} \otimes \mathrm{d}\mathscr{K}$), it follows that the forms ξ^i_I together with $\mathrm{d}\mathscr{K}$ generate the $\mathscr{K}$-module Ξ.

(v) The last sentence can be reformulated as follows: there exist filtrations $\Xi^*\colon \Xi^0 \subset \Xi^1 \subset \ldots \subset \Xi = \bigcup \Xi^l$ satisfying

$$(11)_{1,2} \qquad \Xi^{l+1} \supset \Xi^l + \mathscr{L}_{\mathfrak{X}} \Xi^l \text{ (all } l\text{)}, \quad \Xi^{l+1} = \Xi^l + \mathscr{L}_{\mathfrak{X}} \Xi^l \; (l \geqq l_0(\Xi^*)),$$

where $l_0(\Xi^*) < \infty$ and Ξ^l are $\mathscr{K}$-modules *finitely generated modulo* $\mathrm{d}\mathscr{K}$ (i.e., every Ξ^l is generated by a finite number of forms and some, in general infinite, subset of $\mathrm{d}\mathscr{K}$). We will always suppose $\mathrm{d}\mathscr{K} \subset \Xi^0$, then the invariants will not cause much technical troubles.

(vi) The filtration Ξ^* can be made still better. One can ensure $l_0(\Xi^*) = 0$ (by inserting $\xi^1, \ldots, \xi^m$ already into Ξ^0) and formal integrability (hence integrability) of all modules $\mathscr{F}(\mathbf{X}) \otimes \Xi^l$ (by taking the new filtration $\bar{\Xi}^*$ defined by $\bar{\Xi}^l \equiv \Xi^{l+1}$; then

$$\mathscr{F}(\mathbf{X}) \otimes \bar{\Xi}^l = \mathscr{F}(\mathbf{X}) \otimes \Xi^{l+1} = \mathscr{F}(\mathbf{X}) \otimes (\Xi^l + \mathscr{L}_{\mathfrak{X}} \Xi^l) =$$
$$= \operatorname{Adj} \mathscr{F}(\mathbf{X}) \otimes \Xi^l$$

proves the desired integrability). For such filtrations, the inclusions

$$(12)^1 \qquad \mathrm{d}\Xi^l \subset \sum_{i+j \leqq l+1} \Xi^i \wedge \Xi^j$$

can be easily proved by induction on l using the rule $\mathrm{d}^2 = 0$. We will employ only filtrations Ξ^* of the just mentioned special kind unless otherwise mentioned.

(vii) Very advantageous are the so called *adapted filtrations* Ω^* of the groupiety Ω for which every term Ω^l is generated by all forms $\xi - \tilde{\xi}$ ($\xi \in \Xi^l$). (The invariants do not cause difficulties as the number of generators is concerned since $\mathrm{d}k - \mathrm{d}\tilde{k} = 0$ ($k \in \mathscr{K}$) along the subspace $\mathbf{J} \subset \mathbf{X} \times \tilde{\mathbf{X}}$.) We will use only adapted fitrations of groupieties unless otherwise stated.

12. Axioms for the Maurer-Cartan forms. Let $\mathbf{X}, \mathbf{K}, \mathbf{k}, \mathscr{K}$, be the same objects as at the beginning of Section 10. Let $\Xi \subset \Phi(\mathbf{X})$ be a $\mathscr{K}$-submodule and let $\mathfrak{X}$ be introduced as in (ii), Section 11. Assume moreover

A: Ξ is regular and $\mathscr{F}(\mathbf{X}) \otimes \Xi = \Phi(\mathbf{X})$,

B: $\mathfrak{X} \urcorner \mathrm{d}\Xi \subset \Xi$

(hence $\mathfrak{X} . \mathscr{K} \subset \mathscr{K}, \mathrm{d}\mathscr{K} \subset \Xi, \mathscr{L}_{\mathfrak{X}} \Xi \subset \Xi, [\mathfrak{X}, \mathfrak{X}] \subset \mathfrak{X}$; see (ii), Section 11),

C: there exist filtrations Ξ^* as in (v), Section 11 .

With these data given, the underlying spaces $\mathbf{X} \times \tilde{\mathbf{X}}$, $\mathbf{J} \subset \mathbf{X} \times \tilde{\mathbf{X}}$ and the groupiety Ω can be uniquely reconstructed. It follows that the properties **A**, **B**, **C** characterize the Maurer-Cartan forms.

Instead of **C**, one may postulate the existence of filtrations of better kind:

D: there exist filtrations Ξ^* as in (vi), Section 11 .

The conclusions are just the same as before.

13. Dual axioms characterizing the modules $\mathscr{X}$ are also available but we restrict ourselves to the case $\ell(\mathscr{F}(\mathbf{X}) \otimes \mathrm{d}\mathscr{K}) < \infty$, for brevity of exposition. Then **A** is dualized by **A**^: there exists a common weak basis of $\mathscr{X}$ and $\mathscr{T}(\mathbf{X})$. The dual axiom **B**^ may be expressed by the requirement that $\mathscr{X}$ is a Lie subalgebra of $\mathscr{T}(\mathbf{X})$. Finally, **C**^ undertakes for the existence of a non-decreasing filtration

$$\mathscr{X}^*: \mathscr{X} = \mathscr{X}^{-1} \supset \mathscr{X}^0 \supset \mathscr{X}^1 \supset \ldots, \quad \bigcap \mathscr{X}^l = 0,$$

by $\mathscr{K}$-submodules $\mathscr{X}^l \subset \mathscr{X}$ of finite codimension. We take $\mathscr{X}^l = \mathscr{X} \cap \Xi^{l\perp}$, of course, and the properties (11) are translated into the dual language as

$$(13)_1 \qquad [\mathscr{X}, \mathscr{X}^{l+1}] \subset \mathscr{X}^l,$$

$$(13)_2 \qquad Y \in \mathscr{X}^l \text{ and } [\mathscr{X}, Y] \subset \mathscr{X}^l \text{ imply } Y \in \mathscr{X}^{l+1} \ (l \text{ large enough});$$

use (1) for the proof. Conversely, if $\mathbf{X}, \mathbf{K}, \mathbf{k}, \mathscr{K}$ and a $\mathscr{K}$-submodule $\mathscr{X} \subset \mathscr{T}(\mathscr{X})$ are given satisfying all the above requirements, then the relevant module Ξ of Maurer-Cartan forms satisfying **A**, **B**, **C** can be found by using the property $\mathscr{X} \rceil \Xi \subset \mathscr{K}$.

One can also introduce the strenghtened axiom **D**^. Then (13) can be improved to

$$(14)_{1,2} \qquad [\mathscr{X}^i, \mathscr{X}^j] \subset \mathscr{X}^{i+j}, \quad Y \in \mathscr{X}^l \text{ and } [\mathscr{X}, Y] \subset \mathscr{X}^l \text{ imply } Y \in \mathscr{X}^{l+1}$$

valid for all i, j, l. These are the common axioms appearing in the recent theory of filtered Lie algebras, cf. [7, 10, 14]. (Nevertheless, this is only a formal coincidence. Apart from the fact that we include also the intransitive case usually omitted, the main difference is that the primary objects of our theory are not the infinitesimal transformations $T \in \mathscr{T}(\mathbf{X})$ of the pseudogroup (they are defined by the *Lie equations* $\mathscr{L}_T \Xi = 0$, $\mathscr{L}_T \mathscr{K} = 0$) but the objects $\Xi, \mathscr{X}$ popular under the name *moving frames*, at least for the finite-dimensional $\mathbf{X}$. It is to be noted that the existence of such T cannot be guaranteed without strong additional assumptions (cf. [8]) but the moving frames are quite explicit objects.)

14. Interrelations between Ξ and $\mathscr{X}$ were already discussed; they are produced by the duality. As the connections to the groupieties are concerned, we have the correspondence $\xi^i \leftrightarrow \omega^i = \xi^i - \tilde{\xi}^i$ bijective modulo $\mathrm{d}\mathscr{K}$ between the basis of Ξ and the generators of Ω. (More precisely, a basis of the $\mathscr{K}$-module $\Xi / \mathscr{K} \otimes \mathrm{d}\mathscr{K}$ turns into a basis of Ω.) Using the notation of (iii), Section 11, we have even the bijectivity $X_i \leftrightarrow Y_i + Z_i$ between the weak bases of $\mathscr{X}$ and $\mathscr{H}$. Passing to the adapted filtrations, we obtain the bijectivity $[\xi^i] \leftrightarrow [\omega^i]$ modulo $[\mathrm{d}\mathscr{K}] = \mathrm{d}\mathscr{K}$ (we use $\mathrm{d}\mathscr{K} \subset \Xi^0$) between the basis of the graded $\mathscr{K}$-module $\mathscr{N} = \oplus \mathscr{N}^l = \mathrm{Grad}\, \Xi^*$ ($\mathscr{N}^l = \Xi^l/\Xi^{l-1}$) and generators of the common $\mathscr{F}(\mathbf{J})$-module $\mathscr{M} = \mathrm{Grad}\, \Omega^*$. All these bijections are rather simple and will be frequently used without much hesitation.

Together with the familiar $\odot\mathscr{H}$-module structure of $\mathscr{M}$, there is the obvious $\odot\mathscr{X}$-module structure on $\mathscr{N}$ defined quite analogously as (4). From the point of view of commutative algebra, these modules $\mathscr{M}, \mathscr{N}$ are almost the same. Indeed, the fact that the correspondence between bases of $\mathscr{M}$ and $\mathscr{N}$ is bijective only modulo $\mathrm{d}\mathscr{K}$ does not matter much since

$$X \,.\, [\mathrm{d}\mathscr{K}] = [X \urcorner \mathrm{d}^2\mathscr{K}] = 0 \quad (X \in \mathscr{X})\,.$$

It follows that all concepts and results concerning the commutative algebra of $\mathscr{M}$ can be carried over to the module $\mathscr{N}$ and conversely. The same is true as concerns the familiar homologies $\mathrm{H}(\mathscr{M})^l_s$ and the homologies $\mathrm{H}(\mathscr{N})^l_s$ arising from the complex $\ldots \to \mathscr{N}^l \otimes (\bigwedge^s \mathscr{X}) \to \ldots$ analogous to (5) with differentials like (6).

The main difference with respect to the theory of diffieties arises from the Lie algebra structure of $\mathscr{X}$ and the induced graded Lie algebra on $\mathrm{Grad}\, \mathscr{X}$. Denoting $\mathrm{Grad}\, \mathscr{X} = \mathscr{G} = \oplus \mathscr{G}^l$ ($\mathscr{G}^l = \mathscr{X}^l/\mathscr{X}^{l-1}$), the induced brackets satisfy

$$[\mathscr{G}^i, \mathscr{G}^j] \subset \mathscr{G}^{i+j}\,, \quad y \in \mathscr{G}^l \ \text{ and } \ [\mathscr{G}, y] \subset \mathscr{G}^l \ \text{ imply } \ y = 0$$

(as follows from (14)). Unfortunately, these induced Lie algebras highly depend on the choice of the filtration. (We shall see later on that there may exist filtrations Ξ^*, $\mathscr{X}^*$ of the intrinsical kind. Then the relevant graded Lie algebras are meaningful.) Some important information can be nevertheless found. For this aim, let us look at the adjoint representation $\mathrm{Ad}\, x = [x, \cdot]\colon \mathscr{G} \to \mathscr{G}$ and the relevant dualization $(\mathrm{Ad}\, x)^\wedge\colon \mathscr{N} \to \mathscr{N}$ (we use $\mathscr{N}^\wedge = \mathscr{G}$), where $x \in \mathscr{G}$. In particular, let $x = [X] \in \mathscr{G}^i$ be the class of certain $X \in \mathscr{X}^i$. Then $\mathrm{Ad}\, x\colon \mathscr{G}^j \to \mathscr{G}^{i+j}$, hence $(\mathrm{Ad}\, x)^\wedge\colon \mathscr{N}^{i+j+1} \to \mathscr{N}^{j+1}$ (we use $\mathscr{N}^{l\wedge} \equiv \mathscr{G}^{l-1}$) for the dualized mappings. All these mappings are in general of little interest except $i = -1$ or $i = l$, $j = -1$. In the former case (where $x = [X]$, $X \in \mathscr{X}^{-1} = \mathscr{X}$) we have $(\mathrm{Ad}\, x)^\wedge\colon \mathscr{N}^j \to \mathscr{N}^{j+1}$ which is identical with the multiplication by X leading to the above $\odot\mathscr{X}$-module structure of $\mathscr{N}$. In the latter case (hence $x = [X]$, $X \in \mathscr{X}^l$) we obtain the mapping $(\mathrm{Ad}\, x)^\wedge\colon \mathscr{N}^l \to \mathscr{N}^0$ which will play an important role in Sections 16–18 and in next chapter. (We cannot mention the homologies of the Lie algebra $\mathscr{X}$; filtrations $\mathscr{X}^*$ give spectral sequences with the initial term closely related to the above homologies $\mathrm{H}(\mathscr{N})^l_s$).

15. The uniqueness of Maurer-Cartan foms. If Ξ is known, then the relevant groupiety Ω can be easily found. The converse fact seems to be not quite trivial: we shall be interested in the reconstruction of Ξ assuming the knowledge of the spaces $\mathbf{X}$, $\mathbf{J}$, the inclusion $\mathbf{i}\colon \mathbf{J} \subset \mathbf{X} \times \tilde{\mathbf{X}}$, and the groupiety $\Omega \subset \Phi(\mathbf{J})$.

Above all, the module $\mathscr{K}$ will be known if one is able to determine the space $\mathbf{K}$

and the fiber map $\mathbf{k}: \mathbf{X} \to \mathbf{K}$, that is, the family of all leaves $\mathbf{k}^{-1}(z)$, $z \in \mathbf{K}$. But the leaf meeting a point $p \in \mathbf{X}$ clearly is $\mathbf{L}(q) = \mathbf{p} \circ (\tilde{\mathbf{p}} \circ \mathbf{i})^{-1} (q)$ where $q = \mathbf{w}^{-1}(p) \in \tilde{\mathbf{X}}$ is the (known) duplicate of p and $\mathbf{p}, \tilde{\mathbf{p}}$ are projections of the direct product $\mathbf{X} \times \tilde{\mathbf{X}}$ onto the components.

With $\mathscr{K}$, $\mathrm{d}\mathscr{K}$ and hence $\mathscr{K} \otimes \mathrm{d}\mathscr{K}$ already known, it is sufficient to determine the restriction of the sought module Ξ on the above mentioned leaves. We shall begin with the separate leaf $\mathbf{L}(q)$. Let $\tilde{\xi}(q) \in \Phi_q(\tilde{\mathbf{X}})$ be a fixed differential form of the cotangent space at the point $q \in \tilde{\mathbf{X}}$. Imitating the method already used we introduce the form

$$\tilde{\eta}(p, q) = (\tilde{\mathbf{p}} \circ \mathbf{i})^* \, \tilde{\xi}(q) \in \Phi_{(p,q)}(\mathbf{J})$$

at the point $(p, q) \in \mathbf{J}$ and consider forms $\eta(p, q) \in \Phi_{(p,q)}(\mathbf{J})$ such that $\eta(p, q) - \tilde{\eta}(p, q) \in \Omega$ at the point (p, q). With $p \in \mathbf{L}(q)$ variable, there exist forms $\xi(p) \in \Phi_p(\mathbf{X})$ satisfying $(\mathbf{p} \circ \mathbf{i})^* \, \xi(p) = \eta(p, q)$ and they determine the sought restriction of a form $\xi \in \Xi$ to the leaf $\mathbf{L}(q)$.

If the construction is carried out with every leaf $\mathbf{L}(q) \subset \mathbf{X}$ that is, if the construction is made with the point q varying in a cross-section of the family of all leaves in $\mathbf{X}$ (of the family of duplicates of the leaves $\mathbf{L}(q)$), then the resulting form ξ is determined on the whole space $\mathbf{X}$ and we obtain the whole module Ξ. (One can observe that the above construction is merely a slight adaptation of the familiar method of calculation of the left invariant forms on a Lie group; the usual role of the left translations is a little suppressed.)

16. Essential invariants $k \in \mathscr{K}$ (for a filtration Ω^* of a groupiety Ω) are defined by the property $\mathrm{d}k \in \mathscr{C}(\mathscr{M})^\perp$ where $\mathscr{M} = \mathrm{Grad}\, \Omega^*$. Recall that we deal only with adapted filtrations satisfying $l_0 = 0$ and (12). Then $l_0 = 0$ implies $\mathscr{C}(\mathscr{M}) = \mathscr{C}(\mathscr{M}^0) = \mathscr{H} \cap (\mathrm{Adj}\, \Omega^0)$; cf. Section 6. So we have $\mathscr{C}(\mathscr{M})^\perp = \Omega + \mathrm{Adj}\, \Omega^0$ and the essential invariants are also characterized by the inclusion $\mathrm{d}k \in (\Omega + \mathrm{Adj}\, \Omega^0) \cap \Phi(\mathbf{X}) = \mathrm{Adj}\, \Omega^0 \cap \Phi(\mathbf{X})$.

Let $\Gamma \subset \Xi^0$ be the submodule of all forms $Y \urcorner \mathrm{d}\xi$ where $Y \in \mathscr{X}^0$, $\xi \in \Xi^0$. Then $(12)^0$ can be made more precise: $\mathrm{d}\Xi^0 \subset \Xi^0 \wedge \Xi^0 + \Gamma \wedge \Xi^1$, and since we deal with adapted filtrations, the inclusion $\mathrm{d}\Omega^0 \subset \Phi(\mathbf{J}) \wedge \Omega^0 + \Gamma \wedge \Omega^1$ can be easily obtained. It follows that $\mathrm{Adj}\, \Omega^0$ is generated by the forms from the modules Ω^0 and Γ (briefly $\mathrm{Adj}\, \Omega^0 = \{\Omega^0, \Gamma\}$) so that the essential invariants k are characterized by the inclusion $\mathrm{d}k \in \Gamma$.

Besides, note that the inclusions

$$(15)_{1,2} \qquad \mathrm{d}\Xi^l \subset \Xi^l \wedge \Xi^l + \Gamma \wedge \Xi^{l+1}, \quad \mathrm{d}\Omega^l \subset \Phi(\mathbf{J}) \wedge \Omega^l + \Gamma \wedge \Omega^{l+1}$$

generalizing the above improvement of $(12)^0$ can be obtained by induction on l. Recall moreover that $\mathrm{Adj}\, \Omega^0$ is a formally integrable (hence integrable) module so that $\mathscr{F}(\mathbf{X}) \otimes \Gamma = \mathrm{Adj}\, \Omega^0 \cap \Phi(\mathbf{X})$ is integrable by a simple geometric argument.

The previous concept depends on the choice of the filtration. However, we may introduce the *essential invariants* $k \in \mathscr{K}$ of the groupiety by the intrinsical property $\mathrm{d}k \in \mathscr{C}(\Omega)^{\perp}$. By virtue of the inclusion $\mathscr{C}(\Omega) \supset \mathscr{C}(\mathscr{M})$, we obtain *no more* essential invariants than before. According to Theorem 7, there exist filtrations $\bar{\Omega}^*$ of Ω giving just the essential invariants of Ω. We also have the criterion: $k \in \mathscr{K}$ is an essential invariant of Ω if and only if every family of inclusions $\mathscr{L}_X^m \Xi^0 \subset \Xi^c$ $(m = 1, 2, \dots;$ $X \in \mathscr{X}$; appropriate $c = c(X))$ implies $Xk = 0$.

Our next aim is to verify that there is a basis of Ω independent of all invariants which are not the essential ones. We apologize for the lengthy discussion of this well-known result in the next Section but the only available expositions [1, pp. 616–620] and [9] seem to be absolutely insufficient.

17. Adapted coordinates (i) We recall the standard bases (cf. Sections 5, 6) which can be made very explicit in our particular case. Indeed, every basis $\mathrm{d}x^1, \mathrm{d}x^2, \dots$ of $\Phi(\mathbf{X})$ may be regarded (after the identification $\mathrm{d}x^i \equiv \mathbf{p}^* \mathrm{d}x^i$) as a basis of $\mathscr{H}^\wedge = \Phi(\mathbf{J})/\Omega$. Moreover, we may suppose that $\mathrm{d}x^1, \dots, \mathrm{d}x^a$ is a basis of the module $\mathscr{F}(\mathbf{X}) \otimes \Gamma$. Then the part $\partial_{a+1}, \partial_{a+2}, \dots$ of the dual weak basis $\partial_1, \partial_2, \dots$ of $\mathscr{H}$ may serve for a weak basis of $\mathscr{C}(\mathscr{M})$.

(ii) Let $\mathrm{d}y^1, \dots, \mathrm{d}y^m$ be a basis of the (integrable) module $\mathrm{Adj}\, \Omega^0 = \mathscr{F}(\mathbf{J}) \otimes \Gamma + \Omega^0$. Then Ω^0 itself has a basis of the kind

$$\omega^j = \mathrm{d}y^j - \sum y^i_j \,\mathrm{d}x^i \quad (j = 1, \dots, m;\ \text{sum over } i = 1, \dots, a) \tag{16}$$

and, in general, Ω^l is generated by the forms (7) where $l_0 = 0$, $\omega^j_I = \mathrm{d}y^j_I - \sum^{i} y^j_{Ii} \,\mathrm{d}x^i$ with the recurrence $y^j_{Ii} = \partial_i y^j_I$.

(iii) We shall improve the notation. By virtue of $\mathscr{F}(\mathbf{X}) \otimes \Gamma = \{\mathrm{d}x^1, \dots, \mathrm{d}x^a\}$, we may suppose $(\mathscr{F}(\mathbf{X}) \otimes \Gamma) \cap (\mathscr{K} \otimes \mathrm{d}\mathscr{K}) = \{\mathrm{d}x^{b+1}, \dots, \mathrm{d}x^a\}$ for a certain $b \leqq a$ so that (roughly speak) $x^{b+1} = k^1, \dots, x^a = k^{a-b}$ are just the essential invariants (In more rigorous terms, just the composed functions $F(k^1, \dots, k^{a-b})$ are such, of course.) On the other hand, $\Gamma \subset \Xi^0$ is a submodule, hence $\Gamma = \{\xi^1, \dots, \xi^b,$ $\mathrm{d}k^1, \dots, \mathrm{d}k^{a-b}\}$ for appropriate forms $\xi^1, \dots, \xi^b \in \Xi^0$ which may be taken linearly independent and of the special kind $\xi^i \in \{\mathrm{d}x^1, \dots, \mathrm{d}x^b\}$.

(iv) If $\tilde{\xi}^i \in \{\mathrm{d}\tilde{x}^1, \dots, \mathrm{d}\tilde{x}^b\} \subset \Phi(\tilde{\mathbf{X}})$ are duplicates of ξ^i, then one can easily find certain forms $\mathrm{d}\tilde{x}^i - \sum g^j_i \,\mathrm{d}x^i \in \{\xi^1 - \tilde{\xi}^1, \dots, \xi^b - \tilde{\xi}^b\} \subset \Omega^0$ $(j = 1, \dots, b;$ sum over $i = 1, \dots, b)$. So we may suppose $y^j \equiv \tilde{x}^j$ $(j = 1, \dots, b)$ in the formula (16).

(v) Let $\xi^1, \dots, \xi^b, \xi^{b+1}, \dots, \xi^m$ be a basis modulo $\mathrm{d}\mathscr{K}$ of the module Ξ^0, where $\xi^1, \dots, \xi^b$ are the same forms as in (iii). Let us look at the integrable module $\Phi(\mathbf{X}) \otimes \Xi^0$. Since $\Gamma \subset \Xi^0$, we may suppose

$$\mathscr{F}(\mathbf{X}) \otimes \Xi^0 = \{\mathrm{d}\mathscr{K}, \mathrm{d}x^1, \dots, \mathrm{d}x^b, \mathrm{d}z^1, \dots, \mathrm{d}z^c\}$$

where $z^1, \dots, z^c$ are appropriate functions. Assuming $\mathrm{d}z^1, \dots, \mathrm{d}z^c$ linearly independent, we clearly have $b + c = m = \ell(\Omega^0)$. (All troubles in the following discussion are caused by the forms $\mathrm{d}z^k$. The particular case $c = 0$ is nevertheless of fundamental importance for geometric applications and the structure theory; cf. Sections 23–26.)

(vi) We shall abbreviate some groups of variables by a single symbol. For instance $(k) = (k^1, k^2, \ldots)$ are all invariants, $(k^{\text{ess}}) = (k^1, \ldots, k^b)$ are the essential invariants $(x) = (x^1, \ldots, x^b)$, $(z) = (z^1, \ldots, z^c)$, $(\tilde{x}) = (\tilde{x}^1, \ldots, \tilde{x}^b)$ are the duplicates lying in $\mathscr{F}(\tilde{\mathbf{X}})$, and so on. Then the functions $y^j = y^j(k, x, \tilde{x}, z, \tilde{z})$ appearing in (16) depend on the above mentioned arguments. (They are defined if the values x, z are close enough to $\tilde{x}, \tilde{z}$.) Our next aim is to eliminate all inessential invariants k^j $(j > a - b)$ by an appropriate change of the variables $z, \tilde{z}$. This was already done with the functions y^j $(j = 1, \ldots, b)$, cf. (iv).

(vii) Coming to the crucial point of the Section, we shall use the classical terminology. We begin with the observation that if the variables (x), (k^{ess}) are kept constant, then Pfaff's system $\Omega^0 = 0$ implies $\mathrm{d}y^j \equiv 0$, hence $\mathrm{d}x^i \equiv 0$ (cf. (16), (iv), (vi)). Omitting the uninteresting functions and equations y^j, $\xi^j - \tilde{\xi}^j = 0$ $(j = 1, \ldots, \mathrm{b}$; cf. (iv), (vi)), the result may be expressed as

$$\mathbf{A}: \qquad y^r(k, x, x', z, z') = c'^r \quad \text{are first integrals of} \quad \xi^s = \xi'^s .$$

Here $r, s = b + 1, \ldots, m$, the duplicates of the variables are denoted by the stroke $'$ instead of the usual tilde $\tilde{}$, and the variables (k^{ess}), (x), (x') are kept fixed. Quite analogously, under the same conditions,

$$\mathbf{B}: \qquad y^r(k, x, x'', z, z'') = c''^r \quad \text{are first integrals of} \quad \xi^s = \xi''^s ;$$

this is a mere change of notation. In virtue of **A** and **B**, the system $\xi'^s = \xi''^s$ $(s = b + 1, \ldots, m)$ with (k^{ess}), (x'), (x'') kept constant has certain first integrals of the kind

$$\mathbf{C}: \qquad y^r(k, x', x'', z', z'') = c^r(k^{\text{ess}}, x, x', x'', c', c'') \quad (r = b + 1, \ldots, m)$$

where the constants c^r on the right hand side are represented as certain functions of the constants in **A**, **B** and of the variables which were kept fixed. (If **A**, **B** are written in the resolved form $z'^r = h^r(k, x, x', z, c')$, $z''^r = h^r(k, x, x'', z, c'')$, then the explicit formula

$$\begin{aligned} &c^r(k^{\text{ess}}, x, x', x'', c', c'') \equiv \\ &\equiv y^r(k, x', x'', h(k, x, x', z, c'),\ h(k, x, x'', z, c'')) \end{aligned}$$

for the constants c^r arises by a substitution. The point is that all inessential invariants disappear.)

(viii) Let $(\bar{x}) = (\bar{x}^1(k^{\text{ess}}), \ldots, \bar{x}^b(k^{\text{ess}}))$, $(\bar{z}) = (\bar{z}^1(k), \ldots, \bar{z}^c(k))$ be fixed functions (e.g. constants). We retain the variables (k), (x) but introduce new variables $t^r \equiv y^r(k, \bar{x}, x, \bar{z}, z)$ instead of (z); note that the inversion is $z^r \equiv h^r(k, \bar{x}, x, \bar{z}, t)$ (cf. (vii)). The first integral **C** turns into an identity if (c'), (c'') are replaced by the variables using **A**, **B**. But (x), (z) do not appear on the left in **C** so that (c'), (c'') may be replaced by (t'), (t'') as well; moreover, (x) may be replaced by $(\bar{x})$. In other words, if $(\bar{x}), (t'), (t'')$ are inserted instead of $(x), (c'), (c'')$, respectively, then **C** turns into the identity

$$\mathbf{D}: \qquad y^r(k, x', x'', z', z'') \equiv c^r(k^{\text{ess}}, \bar{x}, x', x'', t', t'') .$$

So the functions $c^r(k^{\text{ess}}, \bar{x}, x, \tilde{x}, t, \tilde{t})$ play the role of the original functions $y^r(k, x, \tilde{x}, z, \tilde{z})$; after the change of variables, all inessential invariants k^j $(j > a - b)$ are eliminated.

(ix) Returning to the original notation of variables, we may assume $y^j \equiv \equiv y^j(k^{\text{ess}}, x, \tilde{x}, z, \tilde{z})$ from the very beginning. Our next task is to ensure an analogous result for the functions y^j_i, then the forms (16) will be made independent of the inessential invariants. This can be achieved by further change of variables, even with $(k), (x)$ and (z) retained. The method is just the same as before so we restrict ourselves to brief indications. In (ii), one introduces the module Adj $\Omega^1 = \mathscr{F}(\mathbf{J}) \otimes \otimes \Gamma + \Omega^1$. This module contains (in addition to (16)) some new forms $\mathrm{d}u^s - - \sum u^s_i \,\mathrm{d}x^i$; pne may take a maximal linearly independent subset of the family of forms ω^j_i so that the variables u^s are identical with some of the functions y^j_i while the others are functionally dependent. In (v), one must deal with the module

$$\mathscr{F}(\mathbf{X}) \otimes \Xi^1 = \{\mathrm{d}\mathscr{K}, \mathrm{d}x^1, \ldots, \mathrm{d}x^b, \mathrm{d}z^1, \ldots, \mathrm{d}z^c, \mathrm{d}v^1, \ldots, \mathrm{d}v^d\} ,$$

where $v^1, \ldots, v^d$ are appropriate new functions. Passing to (vii), certain additional first integrals $u^s(k, x, x', z, z', v, v') \equiv c'^s$ of the system $\xi = \xi'$ $(\xi \in \Xi^1)$ appear if the variables $(k^{\text{ess}}), (x)$ are kept constant. Finally, in (viii) the new variables $t^s \equiv \equiv u^s(k, \bar{x}, x, \bar{z}, z, \bar{v}, v)$ replacing (v) are introduced. Then the relevant relations **D** permit to conclude that all inessential invariants are eliminated: $u^s \equiv u^s(k^{\text{ess}}, x, \tilde{x}, z, \tilde{z}, v, \tilde{v})$. Since y^j_i may be expressed in terms of the variables u^s, we have the desired result.

(x) The construction may be continued with the higher order functions y^j_I in the same manner so that all these functions (hence all forms ω^j_I) are made independent of the inessential invariants.

(xi) We shall express the result in a more explicit form. Unfortunately, the functions y^j_I cannot be all included among the coordinates since they may be functionally dependent. The sought coordinates appear in the course of the above construction; look at the bases of $\mathscr{F}(\mathbf{X}) \otimes \Xi^l$ and recall that $\Phi(\mathbf{X}) = \mathscr{F}(\mathbf{X}) \otimes \Xi = \bigcup \mathscr{F}(\mathbf{X}) \otimes \Xi^l$ We may take the functions

$$k^1, \ldots, k^{a-b};\ k^{a-b+1}, \ldots;\ x^1, \ldots, x^b$$

together with

$$x^{b+1} = z^1, \ldots, x^{b+c} = z^c;\ x^{b+c+1} = v^1, \ldots, x^{b+c+d} = v^d;\ \ldots$$

(new notation) for the coordinates on **X**. Then the family $k^j, x^j, \tilde{x}^j$ $(j = 1, 2, \ldots)$ provides a coordinate system of **J**. Using these coordinates, the final result can be expressed by

$(18)_{1,2}$ $\qquad \partial y_I^j/\partial k^i = 0\,, \quad \mathscr{L}_{\partial/\partial k^i}\omega_I^j = 0 \quad (i > a - b)\,,$

which explicitly declares the independence of the inessential invariants.

(xii) At the very end, one can employ the method of Section 15 to obtain certain basis $\mathrm{d}k^1, \mathrm{d}k^2, \ldots, \xi^1, \xi^2, \ldots$ of the module Ξ satisfying

$$(19) \qquad \mathscr{L}_{\partial/\partial k^i}\xi^j = 0 \quad (i > a - b)\,.$$

The details are obvious and may be omitted.

18. Summarizing remarks. We shall denote by $\mathbf{K}^{\text{ess}}(\mathbf{K}^{\text{in}})$ the space of essential (inessential) invariants. There are obvious decompositions $\mathbf{K} = \mathbf{K}^{\text{ess}} \times \mathbf{K}^{\text{in}}$, $\mathbf{X} = \mathbf{K}^{\text{in}} \times \mathbf{X}'$, $\mathbf{J} = \mathbf{K}^{\text{in}} \times \mathbf{J}'$ into direct products. (For instance, coordinates in $\mathbf{X}'$ are $k^1, \ldots, k^{a-b}$; $x^1, x^2, \ldots$.) There are the natural projection $\mathbf{k}' : \mathbf{X}' \to \mathbf{K}' = \mathbf{K}^{\text{ess}}$ and the submodule $\mathscr{K}' = \mathbf{k}'^*(\mathscr{F}(\mathbf{K}')) \subset \mathscr{F}(\mathbf{X}')$ which may be also regarded as a submodule of $\mathscr{K}$. Omitting useless formalism, we may consider the groupiety $\Omega' \subset \Phi(\mathbf{J}')$ on the underlying space $\mathbf{J}'$ generated by the forms ω_I^j; owing to $(18)_2$ these forms are independent of the inessential invariants so that they may be regarded as forms on $\mathbf{J}'$. The relevant $\mathscr{K}'$-module Ξ' of Maurer-Cartan forms (a submodule of $\Phi(\mathbf{X}')$) is generated by $\mathrm{d}k^1, \ldots, \mathrm{d}k^{a-b}$ and all forms ξ^j (recall (19)); the inessential invariants are eliminated.

Evaluating the final result, the diffieties Ω and Ω' may be identified by means of the pull-back of the natural projection $\mathbf{J} \to \mathbf{J}'$; they differ only in the trivial component lying in $\mathbf{K}^{\text{in}}$. Also the modules Ξ and Ξ' are rather close to each other.

From the technical point of view, the removal of the inessential invariants is easy and consists in introducing an arbitrary cross-section $k^j = k^j(k^1, \ldots, k^{a-b})$, $j > a - b$, of the underlying spaces $\mathbf{J}$ or $\mathbf{X}$. (One may take k^j equal to constants. This is the familiar act in the method of moving frames, cf Section 34.) The converse construction (the adjoining of the inessential invariants) is even easier and need not be discussed.

THE INTERIOR STRUCTURE

19. A subgroupiety Ω' of a groupiety Ω arises if certain additional functions $k' \in \Phi(\mathbf{X})$ are taken for new invariants so that the original underlying space $\mathbf{J}$ is reduced to a subspace $\mathbf{J}' \subset \mathbf{J}$ and the coefficient ring is enlarged, $\mathscr{K}' \supset \mathscr{K}$. The point is however than we must have $\mathscr{H}' = \mathscr{H}$ along the subspace $\mathbf{J}' \subset \mathbf{J}$ (look at the concept of a subdiffiety in Section 3) which leads to important conditions for the set of the new invariants.

Let $k' \in \mathscr{K}'$ be such a new invariant. Then $k' - \tilde{k}' = 0$ along the subspace $\mathbf{J}' \subset \mathbf{J}$, hence

$$0 = \mathrm{d}k' - \mathrm{d}\tilde{k}' = \sum X_i k' \,.\, \xi^i - \sum (X_i k' \,.\, \xi^i)^\sim =$$

$$= \sum X_i k' . (\xi^i - \tilde{\xi}^i) + \sum (X_i k' - (X_i k')^\sim) \tilde{\xi}^i$$

and we conclude $\sum X_i k' . (\xi^i - \tilde{\xi}^i) = 0$, $X_i k' \equiv (X_i k')^\sim$. The former condition is trivial but the latter gives $X_i k' \in \mathscr{K}'$. So we have the requirement: if $k' \in \mathscr{K}'$, then $X_i k' \in \mathscr{K}'$ (briefly $\mathscr{X}\mathscr{K}' \subset \mathscr{K}'$). The latter requirement is also sufficient for a subspace $\mathbf{J}' \subset \mathbf{J}$ consisting of all points $(p, q) \in \mathbf{J} \subset \mathbf{X} \times \tilde{\mathbf{X}}$ with $k'(p) \equiv \tilde{k}'(q)$ to be an underlying space of a subgroupiety Ω' (the requirement clearly ensures $\mathscr{H}' = \mathscr{H}$ along $\mathbf{J}'$).

The naive approach to subgroupieties consists in declaring some new functions to be additional invariants but, together with any invariant k', also all functions Xk' ($X \in \mathscr{X}$) must be added to the invariants.

An alternative method uses the integrable module $\Theta = \mathscr{F}(\mathbf{X}) \otimes \mathrm{d}\mathscr{K}'$; recall that $\mathrm{d}\mathscr{K} \subset \Theta \subset \Phi(\mathbf{X})$. It is of the following special kind: There exists a basis

$$\vartheta^j = \xi^j + \sum_{i \neq j} f_i^j \xi^i \quad \text{(appropriate } j = j_1, j_2, \ldots) \tag{20}$$

of the module Θ with coefficients $f_i^j \in \mathscr{K}'$. (For the proof, consider the forms $\mathrm{d}k'/X_j k' = \xi^j + \sum X_i k'/X_j k' . \xi^i$ ($k' \in \mathscr{K}'$) for $X_j k' \neq 0$.) Such a basis clearly satisfies

$$\mathscr{X} \urcorner \vartheta^j \subset \mathscr{K}', \quad \mathscr{X} \urcorner \mathrm{d}\vartheta^j \subset \mathscr{K}' \otimes \mathrm{d}\mathscr{K}' \subset \Theta, \tag{21$_{1,2}$}$$

which provides very strong conditions for the coefficients f_i^j in the expressions (20) of the basis.

Let us overlook the final state of the matter. We have a certain fibering $\mathbf{k}': \mathbf{X} \to \mathbf{K}'$ which may be continued to the original one $\mathbf{k}: \mathbf{X} \to \mathbf{K}' \to \mathbf{K}$ by the use of an auxiliary fibering $\mathbf{K} \to \mathbf{K}'$; this ensures the inclusion $\mathscr{K}' = \mathbf{k}'^*\mathscr{F}(\mathbf{K}') \subset \mathscr{K} = \mathbf{k}^*\mathscr{F}(\mathbf{K})$ between the coefficient rings. Then the underlying space $\mathbf{J}' \subset \mathbf{J} \subset \mathbf{X} \times \tilde{\mathbf{X}}$ is determined by the above equations $k'(p) \equiv k'(q)$ and Ω' is the natural restriction of Ω on the subspace $\mathbf{J}' \subset \mathbf{J}$. As the Maurer-Cartan forms are concerned, the underlying spaces $\mathbf{X}' = \mathbf{X}$ are the same. One can then see that $\Xi' = \mathscr{K}' \otimes \Xi$ by the direct construction of Section 15 applied to Ω'. (Note that the inclusion $\mathrm{d}\mathscr{K}' \subset \mathscr{K}' \otimes \Xi$ needed for the verification follows from the existence of the basis (20).) Alternatively, one can observe that $\Xi' \supset \Xi$, hence $\Xi' \supset \mathscr{K} \otimes \Xi$. But the $\mathscr{K}'$-module $\mathscr{K}' \otimes \Xi$ satisfies all axioms of Section 12 so that necessarily $\Xi' = \mathscr{K}' \otimes \Xi$ in virtue of the uniqueness of Ξ' (cf. Section 15 applied to Ω'). It follows that every $\mathscr{K}$-basis of Ξ may serve for a $\mathscr{K}'$-basis of Ξ' as well. Consequently, the inclusion $\mathscr{X} \subset \mathscr{X}'$ is valid for the relevant dual objects with the analogous property of the weak bases. Warning: the filtrations Ξ'^* cannot be obtained by a mere extension of the coefficient ring from Ξ^* but one may take $\Xi'^l = \mathscr{K}' \otimes \Xi^l + \mathscr{K}' \otimes \mathrm{d}\mathscr{K}'$ (to ensure $\mathrm{d}\mathscr{K}' \subset \Xi'^0$).

20. A normal subgroupiety appears as a particular case of the subgroupiety if the above module Θ admits generators lying in Ξ. In other terms, we suppose $\Theta = \mathscr{F}(\mathbf{X}) \otimes (\Theta \cap \Xi)$ or, equivalently, $f_i^j \in \mathscr{K}$ (hence $\vartheta^j \in \Xi$) for an appropriate basis (20).

In the naive (but efficient) approach to normal subgroupieties, one may start with an arbitrary finitely generated submodule $\Theta^0 \subset \Xi$ and recurrently take $\Theta^{l+1} = \Theta^l + \mathscr{L}_{\mathfrak{X}}\Theta^l$. Then $\Theta = \mathscr{F}(\mathbf{X}) \otimes \bigcup \Theta^l + \mathscr{F}(\mathbf{X}) \otimes \mathrm{d}\mathscr{K}$ is the sought module. (The integrability of Θ follows from Adj $\mathscr{F}(\mathbf{X}) \otimes \Theta^l = \mathscr{F}(\mathbf{X}) \otimes \Theta^{l+1}$; see an analogous reasoning in (vi), Section 11.) However, direct calculation of integrable modules Θ with the basis (20) lying in Ξ and satisfying $(21)_2$ ($(21)_1$ is trivial since we assume $f^j_i \in \mathscr{K}$) is also possible and leads to algebraic equations for the unknown coefficients $f^j_I \in \mathscr{K}$, cf. [1].

21. A factorgroupiety Ω'' is related to the original groupiety in the following manner: There is a fibering $\mathbf{g}\colon \mathbf{J} \to \mathbf{J}''$ of the relevant underlying spaces which may be embedded into commutative diagrams

$$(22)\qquad \begin{array}{ccccc} \mathbf{J} & \to & \mathbf{X} & \to & \mathbf{K} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{J}'' & \to & \mathbf{X}'' & \to & \mathbf{K}'' \end{array} \qquad\qquad \begin{array}{ccccc} \mathbf{J} & \to & \tilde{\mathbf{X}} & \to & \mathbf{K} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{J}'' & \to & \tilde{\mathbf{X}}'' & \to & \mathbf{K}'' \end{array}$$

(with obvious rows) consisting of certain fiberings uniquely determined by $\mathbf{g}$. The relation between Maurer-Cartan forms is especially simple. The middle mapping $\mathbf{j}\colon \mathbf{X} \to \mathbf{X}''$ identifies the ring $\mathscr{K}''$ with a subring of $\mathscr{K}$, $\mathscr{K}'' = \mathbf{j}^*\mathbf{K}'' \subset \mathscr{K}$, and the $\mathscr{K}''$-module Ξ'' with a submodule of Ξ, $\Xi'' = \mathbf{j}^*\Xi \subset \Xi$. As the groupieties are concerned, Ω'' is generated by the differences $\xi'' - \tilde{\xi}''$ $(\xi'' \in \Xi'')$ where $\mathbf{j}^*\xi'' = \xi \in \Xi$, hence $\mathbf{g}^*(\xi'' - \tilde{\xi}) = \mathbf{j}^*\xi'' - (\mathbf{j}^*\xi'')^{\sim} = \xi - \tilde{\xi} \in \Omega$. It follows that Ω'' (considered as a particular kind of a diffiety) is a factordiffiety of Ω in the sense of Section 3.

Except the trivial case $\mathbf{J} = \mathbf{J}''$, $\Omega = \Omega''$, the inclusion $\mathbf{j}^*\Xi'' \subset \Xi$ is proper. But the inclusion $\mathbf{j}^*\mathscr{K}'' \subset \mathscr{K}$ between the rings of invariants can be always turned into equality if appropriate inessential invariants are adjoined to the groupiety Ω''. (This is merely a technical convention, of course, but a rather useful one.) This is achieved if the spaces $\mathbf{J}''$, $\mathbf{X}''$, $\mathbf{K}''$ are replaced by direct products with an appropriate space $\mathbf{K}'$ (a direct factor of $\mathbf{K}$) in order to ensure the commutative diagram (an extension of (22))

$$(23)\qquad \begin{array}{ccccc} \mathbf{J} & \longrightarrow & \mathbf{X} & \longrightarrow & \mathbf{K} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{K}' \times \mathbf{J}'' & \to & \mathbf{K}' \times \mathbf{X}'' & \to & \mathbf{K}' \times \mathbf{K}'' \end{array}$$

(the vertical arrows can be completed from the right to the left) and its duplicate as in (22). Then $\mathbf{K}' \times \mathbf{X}''$ is taken for the underlying space of Maurer-Cartan forms involving the previous forms Ξ'' and moreover the differentials $\mathrm{d}k'$ $(k' \in \mathscr{F}(\mathscr{K}'))$ of the new (inessential) invariants. The relevant diffiety on the underlying space $\mathbf{K}' \times \mathbf{J}''$ is a mere pull-back of Ω'' with respect to the canonical projection $\mathbf{K}' \times \mathbf{J}'' \to \mathbf{J}''$. So we may assume $\mathscr{K}'' = \mathscr{K}$ which simplifies some reasonings.

22. The factorization. The normal subgroupieties and the factorgroupieties are complementary objects in the anticipated classical sense. Using the notation of Sections 20 and 21, the invariants $\mathscr{K}'$ (i.e., first integrals of the integrable system $\Theta = 0$) are identified with the variables $\mathscr{F}(\mathbf{X}'')$, $\mathscr{K}' = \mathbf{j}^*\mathscr{F}(\mathbf{X}'')$. In terms of differential forms we have $\mathscr{K}' \otimes \mathrm{d}\mathscr{K}' = \mathbf{j}^*\Phi(\mathbf{X}'')$ or $\Theta = \mathscr{F}(\mathbf{X}) \otimes \mathrm{d}\mathscr{K}' = \mathscr{F}(\mathbf{X}) \otimes \otimes \mathbf{j}^*\Phi(\mathbf{X}'')$.

If a normal subgroupiety Ω' is given then clearly $\Xi'' = \Xi \cap \Theta = \Xi \cap (\mathscr{K}' \otimes \mathrm{d}\mathscr{K}') = \Xi \cap \mathbf{j}^*\Phi(\mathbf{X}'')$ (recall the basis (20)) and $\mathscr{K}'' = \mathscr{F}(\mathbf{X}'') \cap \mathscr{K}$. But we may assume $\mathscr{K}'' = \mathscr{K}$ and then Ω' uniquely determines Ω''. Conversely, if Ω'' is given and $\mathscr{K}'' = \mathscr{K}$ (hence $\mathbf{j}^*\Xi'' \supset \mathrm{d}\mathscr{K}'' = \mathrm{d}\mathscr{K}$), then $\Theta = \mathscr{F}(\mathbf{X}) \otimes \mathbf{j}^*\Xi''$ (or even $\Theta = \mathscr{F}(\mathbf{X}) \otimes \otimes \mathbf{j}^*\Phi(\mathbf{X}''))$ uniquely determines the invariants $\mathscr{K}'$, hence the relevant normal subgroupiety Ω'.

We shall write $\Omega'' = \Omega/\Omega'$ for the corresponding objects.

23. Towards the composition series. Let us recall the submodules $\mathscr{C}(\mathscr{M}) \subset \mathscr{H}$, $\mathscr{C}(\Omega) \subset \mathscr{H}$ once more but translated into the language of Maurer-Cartan forms. Following the instructions of Section 14, if we deal with a filtration Ξ^* and the relevant graded module $\mathscr{N} = \mathrm{Grad}\, \Xi^*$, we have to introduce the submodule $\mathscr{C}(\mathscr{N}) = \mathscr{X} \cap \mathrm{Ann}\, \mathscr{N} \subset \mathscr{X}$ consisting of all $X \in \mathscr{X}$ with $X \,.\, \mathscr{N} = 0$ or, equivalently, with $\mathscr{L}_X \Omega^l \subset \Omega^l$ for all l. However, we suppose $l_0(\Xi^*) = 0$ so that the (equivalent) requirements $X \,.\, \mathscr{N}^0 = 0$, $\mathscr{L}_X \Xi^0 \subset \Xi^0$ are sufficient. Then the submodule $\Gamma = \mathscr{C}(\mathscr{N})^\perp \cap \Xi \subset \Xi^0$ consists of all forms $\mathscr{L}_Y \Xi^0$ $(Y \in \mathscr{X}^0)$ and was already considered in Section 16. This module will be more rigorously denoted $\Gamma(\mathscr{N})$ since it depends on the choice of the filtration. In view of Theorem 7, there exists a (unique) greatest element in the family of all possible modules $\mathscr{C}(\mathscr{N})$. It will be denoted $\mathscr{C}(\Xi)$ and it consists of all $X \in \mathscr{X}$ such that $\mathscr{L}^m_X \Xi^0 \subset \Xi^c$ for all $m = 1, 2, \ldots$ and an appropriate $c = c(X)$. In dual terms, we have the smallest element denoted $\Gamma(\Xi) = \mathscr{C}(\Xi)^\perp \cap \Xi$ in the family of all possible modules $\Gamma(\mathscr{N})$. (Clearly, $\Gamma(\Xi) = \mathscr{C}(\Omega)^\perp \cap \Xi$ also holds.)

One can observe that the validity of the formula $(15)^0_1$ implies $\Gamma \supset \Gamma(\mathscr{N})$, hence $\Gamma \supset \Gamma(\Xi)$. On the other hand, $(15)^0_1$ was proved for the module $\Gamma = \Gamma(\mathscr{N})$. It follows that $\Gamma = \Gamma(\Xi)$ is the smallest module for which a formula like $(15)^0_1$ (hence all formulae $(15)^l_1$) is true for an appropriate choice of the filtration.

We pass to the main theme. Using (a little adapted) naive method of Section 20, we may introduce the intrinsical normal subgroupiety Ω' of Ω by taking

$$\Xi''^0 = \Gamma(\Xi) + \mathscr{K} \otimes \mathrm{d}\mathscr{K}\,, \quad \Xi''^{l+1} = \Xi''^l + \mathscr{L}_{\mathfrak{X}} \Xi''^l + \mathscr{K} \otimes \mathrm{d}\mathscr{K} \tag{24}$$

and choosing $\Theta = \mathscr{F}(\mathbf{X}) \otimes \Xi''^l$ for the module which determines the invariants $\mathscr{K}'$ (cf. Section 20). In other terms, (24) is a filtration of the module $\Xi'' = \bigcup \Xi''^l$ of Maurer-Cartan forms to the relevant factorgroupiety $\Omega'' = \Omega/\Omega'$. Clearly $\Gamma(\Xi) \subset \subset \Xi''^0 \subset \Xi'' \subset \Phi(\mathbf{X}'') = \mathscr{K}' \otimes \mathrm{d}\mathscr{K}'$ (cf. Section 22).

On the other hand, if Ω' is a normal subgroupiety of Ω satisfying $\Gamma(\Xi) \subset \mathscr{K}' \otimes \otimes \mathrm{d}\mathscr{K}'$, then $\Xi''^0 = \Gamma(\Xi) + \mathscr{K} \otimes \mathrm{d}\mathscr{K} = \Phi(\mathbf{X}'')$, hence $\Xi'' = \bigcup \Xi''^l \subset \Phi(\mathbf{X}'')$ (apply (24) and use $\mathscr{L}_{\mathfrak{X}} \Phi(\mathbf{X}'') \subset \Phi(\mathbf{X}''))$ so that $\mathscr{K}' \otimes \mathrm{d}\mathscr{K}' = \mathscr{F}(\mathbf{X}) \otimes \Xi'' \subset$

$\subset \Phi(X'') = \mathscr{K}' \otimes d\mathscr{K}'$. So we have $\mathscr{K}' \subset \mathscr{K}'$ and Ω' may be regarded as a subgroupiety of Ω'. (One can also observe that $\Gamma(\Xi') \subset \Gamma(\Xi)$ (triviality), hence $\Gamma(\Xi') \subset \subset \mathscr{K}' \otimes d\mathscr{K}'$ (which is an intrinsical property of Ξ'). But unfortunately, there may exist other very large subgroupieties of Ω with the same property.) Summarizing, we have the result:

24. Lemma. *In the family of all normal subgroupieties Ω' of Ω satisfying $\Gamma(\Xi) \subset \mathscr{K}' \otimes d\mathscr{K}'$ is a (unique) greatest one Ω' and it satisfies $\Gamma(\Xi') \subset \mathscr{K} \otimes d\mathscr{K}'$. The module Ξ" of Maurer-Cartan forms to the relevant factorgroupiety Ω" $= \Omega/\Omega'$ has the (intrinsical) filtration* (24).

Only few comments are needed. If some subgroupieties Ω', Ω' of a groupiety Ω are considered, we write $\Omega' < \Omega'$ if the opposite (set) inclusion $\mathscr{K}' \supset \mathscr{K}'$ between the ring of invariants is true. Clearly, $\Omega' < \Omega'$ means that Ω' may be regarded as a subgroupiety of Ω'. We shall moreover write $\Omega' \lhd \Omega'$ if Ω' is a normal subgroupiety of Ω' and $\Omega' \prec \Omega$ if Ω' is the (unique) greatest subgroupiety of Ω of the kind specified in Lemma 24.

The requirement $\Gamma(\Xi) \subset \mathscr{K} \otimes d\mathscr{K}$ means that there exist certain filtrations Ξ^* satisfying

$$(25) \qquad d\Xi^l \subset \Xi^l \wedge \Xi^l + d\mathscr{K} \wedge \Xi^{l+1}$$

(a mere transcription of (15) with $\mathscr{K} \otimes d\mathscr{K}$ instead of Γ). Such a groupiety (hence the groupiety Ω' in Lemma 24) is of a very special kind. On the contrary, the factorgroupiety Ω" of Lemma 24 may be quite arbitrary but it is equipped with the intrinsical (with respect to its behaviour in Ω) filtration.

The factorization in Lemma 24 is trivial if either $\Omega = \Omega$' (then we put $\Omega'' = 0$ and $\Omega = \Omega'$ is of a very special kind) or $\Omega = \Omega''$ (then Ω is intrinsically filtered by (24), which is a very interesting achievement, cf. [1, p. 568_{12-8}]). Except these cases, the factorization may be repeated with the groupiety Ω" instead of Ω. In explicit transcription, Theorem 24 gives $\Omega_{(1)} \prec \Omega_{[1]}$ (here $\Omega_{(1)} = \Omega'$, $\Omega_{[1]} = \Omega$), then the next step $\Omega_{(2)} \prec \Omega_{[2]}$ (where $\Omega_{(2)} = \Omega_{[1]}/\Omega_{(1)} = \Omega''$), then $\Omega_{(3)} \prec \prec \Omega_{[3]}(=\Omega_{[2]}/\Omega_{(2)})$, and so on. The procedure terminates after a finite number of steps since $\Gamma(\Xi'') \subset \Gamma(\Xi)$, and if $\Gamma(\Xi'') = \Gamma(\Xi)$ then the next factorization is trivial.

25. Theorem. *For every groupiety Ω we have a unique finite series of groupieties $\Omega_{(k)}, \Omega_{[k]}$ $(k = 1, \ldots, K+1)$ satisfying $\Omega_{[1]} = \Omega$, $\Omega_{(k)} \prec \Omega_{[k]}$, $\Omega_{[k+1]} = \Omega_{[k]}/\Omega_{(k)}$ with either $\Omega_{[K+1]} = 0$ or $\Omega_{[K+1]} = \Omega_{[K]}$ (thus $\Omega_{(K)} = 0$). All modules $\Omega_{[k]}$ $(k \geqq 1)$ have intrinsical filtrations with the initial terms $\Xi^0_{[k]} = \Gamma(\Xi^0_{[k-1]})$.*

Our next task is to replace $\Omega_{(k)}$ by appropriate normal subgroupieties Ω_k of Ω. This can be done as follows: every $\Omega_{[k+1]}$ is a factordiffiety of the foregoing $\Omega_{[k]}$, hence of $\Omega_{[1]} = \Omega$, so that dually, every $\Omega_{(k+1)}$ may be regarded as a normal subgroupiety of Ω containing $\Omega_{(k)}$. (In rigorous terms, the following common isomorphism theorem should be used: For a normal subgroupiety $\Omega' \lhd \Omega$ there is a bijective correspondence between the groupieties Ω_a, $\Omega' < \Omega_a < \Omega$, and the subgroupieties $\Omega_{(a)} < \Omega/\Omega'$ such that $\Omega' \lhd \Omega_a$ and $\Omega_a/\Omega' \sim \Omega_{(a)}$ are isomorphic in the (obvious) natural sense. Moreover, $\Omega_a \lhd \Omega$ if and only if $\Omega_{(a)} \lhd \Omega/\Omega'$. We omit the proof which consists in explicit examination of invariants.) The final result reads as follows:

26. Theorem. *Every groupiety has a unique decomposition series* $\Omega_1 \lhd \dots \dots \lhd \Omega_K = \Omega$ *(and even* $\Omega_k \lhd \Omega$ *for all* k*) with the factors* $\Omega_{k+1}/\Omega_k = \Omega_{(k)}$ *and* $\Omega/\Omega_k = \Omega_{[k]}$ $(k = 1, \dots, K-2)$ *and either* $\Omega/\Omega_{K-1} = \Omega_{(K)}$ *or* $\Omega/\Omega_{K-1} = \Omega_{[K]}$. *Here* $\Omega_{(k)}$, $\Omega_{[k]}$ *are groupieties of the special kind specified in Theorem* 25.

We will not continue the study of the structure of pseudogroups here. It is a too extensive task and the utility of the results seems to be rather doubtful for the present. Moreover, concrete nontrivial examples and possible applications are still lacking. For these reasons, we open the more urgent problem how to realize some infinite algorithms (e.g., determination of $\mathscr{C}(\Omega)$ and $\mathscr{C}(\Xi)$, calculation of subgroupieties and factorgroupieties) in a finite number of steps. This problem will be discussed in the broader context of the theory of diffieties.

SPECIAL FILTRATIONS

27. A little algebra. We return to the topics of Section 4, namely to the homology of the module M over the symmetric algebra $A = \odot H$. An exact complex $0 \leftarrow M \leftarrow \leftarrow F_0 \leftarrow F_1 \leftarrow \dots$ with free A-modules $F_0, F_1, \dots$ such that the kernels of the differentials $F_j \leftarrow F_{j-1}$ $(j = 0, 1, \dots;\ F_{-1} = M)$ are contained in the submodules $H \,.\, F_j \subset F_j$ will be called the *minimal resolution* of M. The minimal resolutions are unique and can be found by a direct construction. We shall be interested in the graded case $M = \oplus M^l$, $F_j = \oplus F_j^l$ where the kernels of the differentials $F_j^l \leftarrow F_{j-1}^l$ are contained in the submodules $H \,.\, F_j^{l-1} \subset F_j^l$. One can then see that $F_j^l = 0$ for $l > j$.

The graded A-module $\mathbb{R} = \mathbb{R} \oplus 0 \oplus 0 \oplus \dots$ has the minimal resolution $0 \leftarrow \mathbb{R} \leftarrow \leftarrow A \leftarrow A \otimes H \leftarrow A \otimes (H \wedge H) \leftarrow \dots$ with differentials as in (6). Then the tensor product with the minimal resolution of M gives the commutative diagram

(26)
$$
\begin{tikzcd}
 & 0 & 0 & & 0 & 0 & \\
0 & (M \otimes \mathbb{R})^l \arrow[l] \arrow[u] & (F_0 \otimes \mathbb{R})^l \arrow[l] \arrow[u] & \dots \arrow[l] & (F_{l-1} \otimes \mathbb{R}_j)^l \arrow[l] \arrow[u] & (F_l \otimes \mathbb{R})^l \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & M^l \arrow[l] \arrow[u] & F_0^l \arrow[l] \arrow[u] & \cdots \arrow[l] & F_{l-1}^l \arrow[l] \arrow[u] & F_{l0}^l \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & M^{l-1} \otimes H \arrow[l] \arrow[u] & F_0^{l-1} \otimes H \arrow[l] \arrow[u] & \cdots \arrow[l] & F_{l-1}^{l-1} \otimes H \arrow[l] \arrow[u] & 0 \arrow[l] & \\
 & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & &
\end{tikzcd}
$$

which is exact except (possibly) the first row and column. (This is a triviality as the rows are concerned. For the columns the fact follows from the familiar property $\mathrm{H}(F)_s^l = 0$ $(s \neq 0)$ for free F.) All mappings in the first row are zero (the property equivalent to the minimality) so that a simple diagram chasing gives

(27) $$\mathrm{H}(M)_s^{l-s} = (F \otimes \mathbb{R})^l = \mathrm{H}(F_s)_0^l\,.$$

This provides an interpretation of higher homologies of M in terms of generators of F_s. Continuing a little in this direction, let $0 \leftarrow M_j \leftarrow F_j \leftarrow M_{j+1} \leftarrow 0$ $(j = 0, 1, \ldots;\ M_0 = M)$ be the decomposition of a (not necessarily minimal) free resolution of M into short exact sequences; every M_{j+1} may be regarded as the module of all A-relations between the elements of M_j. Considering the relevant long exact homology sequences, one obtains the recurrences $\mathrm{H}(M_{j+1})_s^l \equiv \mathrm{H}(M_j)_{s+1}^l$ whenever $l \neq 0$ or $s \neq 0$ so that

(28) $$\mathrm{H}(M)_s^l \equiv \mathrm{H}(M_s)_0^l \quad (l \neq 0 \text{ or } s \neq 0)\,.$$

This provides the interpretation of homologies of M in terms of the generators of the modules of certain A-relations.

The concept of the minimal resolution will be applied in the case $M = (\mathrm{Grad}\ \Omega^*)_p$, $p \in \mathbf{J}$, for the diffieties Ω. (The localization at p need not be necessarily applied. One can deal directly with the $\mathscr{A}$-module $\mathscr{M} = \mathrm{Grad}\ \Omega^*$; the necessary modifications are clear.) In all interesting cases we have $\ell(H) = \infty$ but $\ell(M^l) < \infty$. One can moreover ensure that $\mathrm{H}(M)_s^l = 0$ $(l > 0$, all $s)$ if the original filtration Ω^* replaced is by an other one (the so called *c-prolongation*, cf. [3, Section 18])

$$\bar{\Omega}^* = \Omega^{*+c}\colon \bar{\Omega}^0 = \Omega^c \subset \bar{\Omega}^1 = \Omega^{c+1} \subset \ldots \subset \Omega = \bigcup \bar{\Omega}^l$$

with c large enough, and the original module M by the relevant new one denoted by M^{+c} (cf. [3, Section 7] for the multiplication in M^{+c}). This is a consequence of the reduction argument of Section 5 and the obvious relations $\mathrm{H}(M)_s^{l+c} = \mathrm{H}(M^{+c})_s^l$ valid for $l > 0$ (see also [3, Section 34]). In fact, weaker assumptions are sufficient for the next lemma (a prototype for more advanced analytical results to follow) which carries the algebraic calculations into the area of finite-dimensional spaces, at least in principle.

28. Lemma. *If* $\mathrm{H}(M)_0^l = 0$ $(l > 1)$, $\mathrm{H}(M)_1^l = 0$ $(l > 0)$, *then the graded A-module M can be uniquely reconstructed from the initial terms M^0, M^1 of the gradation and from the multiplication $H\colon M^0 \to M^1$.*

Proof: First assume $\mathrm{H}(M)_0^l = 0$ $(l > 0)$. Then $M = F_0/M_1$ where F_0 is freely generated by $F_0^0 = M^0$ (hence known), and the submodule $M_1 \subset F_0$ is generated from the subspace $M_1^1 \subset F_0^1$ which is the kernel of the mapping $F_0^1 = H \otimes F_0^0 = H \otimes M^0 \to M^1$ (the given multiplication is represented as a bilinear mapping $H \otimes M^0 \to M^1$).

If only $\mathrm{H}(M)_0^l = 0$ $(l > 1)$ is assumed, new generators (corresponding to $\mathrm{H}(M)_0^1$) should be added to F_0^1 as direct summands but the construction is retained.

29. A Cartan filtration Ω^* of a diffiety Ω is defined by the assumptions $\mathrm{H}(\mathscr{M})_0^l = 0$ $(l > 1)$, $\mathrm{H}(\mathscr{M})_1^l = 0$ $(l > 0)$, and

$$(29) \qquad \Omega^1 = \Omega \cap \operatorname{Adj} \Omega^0$$

(cf. [4, Section 10] for the particular case $\ell(\mathscr{H}) = 1$). For the Cartan filtration, the standard bases of Section 5 can be made more explicit. Let $f^1, \ldots, f^b$ be the adjoint variables of Ω^0 and suppose that $\mathrm{d}f^1, \ldots, \mathrm{d}f^a$ is a maximal linearly independent modulo Ω subset of the family $\mathrm{d}f^1, \ldots, \mathrm{d}f^b$. Introducing the more convenient notation $x^i = f^i$ $(i = 1, \ldots, a)$, $y^j = f^{a+j}$ $(j = 1, \ldots, m = b - a)$, the assumption (29) ensures the existence of a certain basis consisting of special forms $\omega^j = \mathrm{d}y^j - \sum y_i^j \, \mathrm{d}x^i$ to the module Ω^1. And even more generally, in virtue of the first homology assumption, the module Ω^{l+1} $(l \geqq 0)$ has the basis (7) with $l_0 = 0$ and explicitly expressible as

$$(30) \qquad \omega_I^j = \mathrm{d}y_I^j - \sum y_{Ii}^j \, \mathrm{d}x^i$$
$$(I = i_1 \ldots i_s;\ s = 0, \ldots, l;\ \text{recurrence } y_{Ii}^j \equiv \partial_i y_I^j)\,.$$

The operators ∂_i are the same as in Section 5.

The module Ω^0 causes some difficulties. Let $\pi^1, \ldots, \pi^c$ be a basis of Ω^0 expressible by the adjoint variables $f^1, \ldots, f^b$. Then

$$\pi^k \equiv \sum h_j^k \omega^j\,, \quad \mathrm{d}\pi^k = \sum f_{ij}^k \, \mathrm{d}x^i \wedge \omega^j + \sum g_{pq}^k \omega^p \wedge \omega^q$$

with the coefficients h_j^k, g_{pq}^k (but not necessarily f_{ij}^k) expressible by the adjoint variables. Let us consider the $\mathscr{A}$-module $\mathscr{M} = \oplus \mathscr{M}^l = \operatorname{Grad} \Omega^*$. The classes $[\pi^1], \ldots, [\pi^c] \in \mathscr{M}^0$ yield a basis of $\mathscr{M}^0$. The classes $[\omega^1], \ldots, [\omega^m] \in \mathscr{M}^1$ generate $\mathscr{M}^1$, they are related by $\sum h_j^k[\omega^j] = 0$. Moreover, we have the forms $\pi_i^k = \partial_i \urcorner \mathrm{d}\pi^k = \sum f_{ij}^k \omega^j \in \Omega^1$ which give the classes $[\pi_i^k] = \partial_i \,.\, [\pi^k] \in \mathscr{M}^1$. It follows hat all relations in $\mathscr{M}^1$ between the above mentioned classes are

$$(31)^0 \qquad \sum h_j^k[\omega^j] = 0\,, \quad [\pi_i^k] - \sum f_{ij}^k[\omega^j] = 0 \quad (k = 1, \ldots, m;\ i = 1, \ldots, a)\,.$$

The classes $[\omega_I^j]$ of the forms (30) generate $\mathscr{M}^{l+1}$. Moreover, we have the recurrently defined forms $\pi_{Ii}^k = \partial_i \urcorner \mathrm{d}\pi_I^k \in \Omega^{l+1}$ determining the classes $[\pi_{Ii}^k] = \partial_i \,.\, [\pi_I^k] \in \mathscr{M}^{l+1}$. By virtue of the second homology assumption, all relations in the module $\mathscr{M}^{l+1}$ between the mentioned classes are

$$(31)^l \qquad \sum h_j^k[\omega_I^j] = 0\,, \quad [\pi_{Ii}^k] - \sum f_{ij}^k[\omega_I^j] = 0$$
$$(k = 1, \ldots, m;\ i = 1, \ldots, a;\ I = i_1 \ldots i_1)\,.$$

Indeed, $\mathrm{H}(\mathscr{M}_1)_0^l = \mathrm{H}(\mathscr{M})_1^l = 0$ $(l > 0$; cf. (28)) so that the module $\mathscr{M}_1 = \oplus \mathscr{M}_1^1$ of all $\mathscr{A}$-relations between the classes of $\mathscr{M}$ is generated by the first term $\mathscr{M}_1^0$ of the gradation, that is, by the relations $(31)^0$.

30. Theorem. *A diffiety Ω is uniquely determined by the first term Ω^0 of its (arbitrary) Cartan filtration.*

Proof. Let Ω^* be a Cartan filtration of the same diffiety as above. Let $\bar{\Omega}^*$ be a Cartan filtration of an other diffiety $\bar{\Omega}$ on an underlying space $\bar{\mathbf{J}}$ with the relevant dashed functions and forms $\bar{\pi}_I^k$, $\bar{\omega}_I^k$ satisfying the dashed relations $(31)^-$.

Assume $\Omega^0 = \bar{\Omega}^0$. Then the corresponding adjoint variables and bases may be identified: $x^i \equiv \bar{x}^i$, $y^j \equiv \bar{y}^j$, $\pi^k \equiv \bar{\pi}^k$.

Assume moreover that Ω is a subdiffiety of $\bar{\Omega}$ and $\mathbf{g}\colon \mathbf{J} \to \bar{\mathbf{J}}$ is the relevant inclusion (cf. Section 3). Then $\mathscr{H} = \bar{\mathscr{H}}$ along the subspace $\mathbf{J}$ so that $\partial_i \equiv \bar{\partial}_i$ and hence $\mathbf{g}^*\pi_I^k \equiv \bar{\pi}_I^k$, $\mathbf{g}^*\omega_I^j \equiv \bar{\omega}_I^j$. Since $\mathbf{g}^*$ is surjective, the same equations are valid for the classes in $\mathscr{M}$, $\bar{\mathscr{M}}$. At the same time, the dashed relations $(31)^-$ turn into the original (31) by applying $\mathbf{g}^*$. Since (31) are all relations between a certain family of generators of $\mathscr{M}$, we conclude that all linear relations between the classes $[\omega] = \mathbf{g}^*[\bar{\omega}] \in \mathscr{M}$ are valid for the preimages $[\bar{\omega}] \in \bar{\mathscr{M}}$. It follows that $\mathbf{J}$ cannot be a proper subspace of $\bar{\mathbf{J}}$. (In the opposite case, there would exist forms $0 \neq \bar{\omega} \in \bar{\Omega}$ with $0 = \mathbf{g}^*\bar{\omega} = \omega$. Choosing an appropriate class $[\bar{\omega}] \neq 0$, we have the contradiction $[\omega] = 0$.) So we have $\mathbf{J} = \bar{\mathbf{J}}$, $\Omega = \bar{\Omega}$.

In expressive terms, Ω is the greatest diffiety which can exist over the first term Ω^0 of its Cartan filtration. In this sense, Ω is uniquely determined.

31. A strong Cartan filtration Ω^* is defined by the more restrictive homology assumptions $\mathrm{H}(\mathscr{M})_0^l = \mathrm{H}(\mathscr{M})_1^l = 0$ $(l > 0)$ and the condition

$$\Omega^0 = \Omega \cap \Delta \tag{32}$$

where $\Delta \subset \Phi(\mathbf{J})$ is an appropriate integrable submodule. The module Δ undertakes the role of Adj Ω^0 in Section 29. One can then find a certain basis (30) for every term Ω^l of the filtration. So we have a particular case of a Cartan filtration.

32. Comments. The diffieties are lucid objects thanks to the fact that all constructive aspects are omitted. However, Theorem 30 is of an other nature. It is related to the problem of explicit reconstruction of Ω from the first term Ω^0 of the Cartan filtration. (One can see that this is a formal version of the general Pfaff's problem expressed in brief homological terms.) Making use of the result, one then has to successively calculate the greatest module Ω^{l+1} satisfying $\mathrm{d}\Omega^l = 0$ modulo Ω^{l+1}. This requirement gives an algebraic system of equations (at every $p \in \mathbf{J}$) for the coefficients y_{Ii}^j appearing in the forms ω_I^j. The sense of Theorem 30 is that the maximal solution (the whole component of a variety) must be used in order to obtain the sought Ω.

The groupieties were included as a particular case, nevertheless, reformulation in terms of Maurer-Cartan forms is desirable. Assume $\mathrm{H}(\mathscr{N})_0^l = \mathrm{H}(\mathscr{N})_1^l = 0$ $(l > 0;$ $\mathscr{N} = \mathrm{Grad}\,\Xi^*)$. Then $\mathrm{H}(\mathscr{M})_0^l = \mathrm{H}(\mathscr{M})_1^l = 0$ for the relevant adapted filtration Ω^* (cf. Section 14). If Ξ^0 is known, then Ω^0 is known (and generated by $\xi - \bar{\xi}$, $\xi \in \Xi^0$). According to (ii), Section 17, Ω^* is a strong Cartan filtration, hence it is determined by Ω^0 and Ξ can be found (Section 15). Altogether, Ξ^0 determines Ξ. (Direct use of (12) for explicit calculation of Ξ^{l+1} from Ξ^l is simpler and currently used in [1].)

33. Warning. After Sections 28–32, all essential data are contained in certain finite-dimensional modules so that a chance appears to bring to an end every algorithm after a finite number of steps. However, still one trouble remains unsolved: while the algebraic data $M^0, M^1, M^0 \to M^1$ in Lemma 28 can be quite arbitrary, this is not true for the data Ω^0 or Ξ^0 in Sections 30–32 (except the particular case $\ell(\mathscr{H}) = 1$; cf. [4, Theorem 11]). The higher homologies are rather useful in this connection but we postpone this difficult and important problem to an other place.

Three brief remarks that relate the groupieties to certain crucial concepts of analysis and geometry will conclude the paper.

CONCLUDING REMARKS

34. The general equivalence problem will be discussed as briefly as possible from the naive point of view omitting the geometric interpretations. The problem is as follows: we have a space M^0 of variables $x^1, \ldots, x^{m_0}$, a family of functions $\mathscr{F}^0 \subset \subset \mathscr{F}(M^0)$, and a family of differential forms $\Xi^0 \subset \Phi(M^0)$. Our task is to determine all (local) diffeomorphisms $\mathbf{f}^0$ of M^0 preserving all elements of $\mathscr{K}^0$ and Ξ^0 (briefly: the *symmetries* $\mathbf{f}^0$): $\mathbf{f}^{0*}k = k \; (k \in \mathscr{K}^0)$, $\mathbf{f}^{0*}\xi = \xi \; (\xi \in \Xi^0)$. The strategy is as follows: the sought symmetries $\mathbf{f}^0$ must preserve also other functions and forms which will be successively added to $\mathscr{K}^0$ and Ξ^0. As the final result, a groupiety appears resolving the problem. (Compare with [1, pp. 722–745 and 1311–1334]).

The preliminary measures consist in several elementary adaptations of $\mathscr{K}^0$ and Ξ^0 (without any change of notation). So we may ensure: (i) $\mathscr{K}^0 \subset \mathscr{F}(M^0)$ is a subring containing all composed functions $F(k^1, \ldots, k^c)$ for any F and $k^1, \ldots, k^c \in \mathscr{K}^0$, (ii) $\Xi^0 \subset \Phi(\mathbf{M}^0)$ is a $\mathscr{K}^0$-submodule, (iii) $\mathrm{d}\mathscr{K}^0 \subset \Xi^0$, (iv) for any linear relation $\sum f_i \xi^i = 0 \; (\xi^i \in \Xi^0)$ we have $f_i / f_j \in \mathscr{K}^0$ whenever $f_j \neq 0$. Taking these measures, there is a $\mathscr{K}^0$-basis $\xi^1, \ldots, \xi^{n_0}$ of the $\mathscr{K}^0$-module Ξ^0. Either $n_0 = m_0$ or $n_0 < m_0$ may occur.

Suppose $n_0 = m_0$. Then $\xi^1, \ldots, \xi^{m_0}$ is an $\mathscr{F}(\mathbf{M}^0)$-basis of $\Phi(\mathbf{M}^0)$, hence $\mathrm{d}\xi^i \equiv \equiv \sum f^i_{jk} \xi^j \wedge \xi^k$ for appropriate coefficients. All these coefficients f^i_{jk} must be added to $\mathscr{K}^0$ and the above arrangements (i)–(iv) must be applied. The calculations end. We have the Lie group of symmetries $\mathbf{f} = \mathbf{f}^0$ acting on $\mathbf{X} = \mathbf{M}^0$ and determined by the ring of invariants $\mathscr{K} = \mathscr{K}^0$ and the $\mathscr{K}$-module $\Xi = \Xi^0$ of Maurer-Cartan forms. (This relatively simple subcase covers the classical "moving frame method" of differential geometry.)

Suppose $n_0 > m_0$. Let $\xi^1, \ldots, \xi^{n^0}, \eta^1, \ldots, \eta^{m_0 - n_0}$ be an $\mathscr{F}(\mathbf{M}^0)$-basis of $\Phi(\mathbf{M}^0)$. The symmetries $\mathbf{f}^0$ need not preserve the forms η^k, of course, but produce a linear substitution of the mentioned basis. So we introduce the forms

$$\xi^j = \sum u^j_i \xi^i + \sum v^j_k \eta^k$$

$$(j = n_0 + 1, \ldots, m_0; \text{ sum over } j = 1, \ldots, n_0; \; k = 1, \ldots, m_0 - n_0)$$

on the extended space $\mathbf{M}^1$ of the original variables $x^1, \dots, x^{m_0}$ together with the parameters (new variables) u_i^j, v_k^j which may be denoted by $x^{m_0+1}, \dots, x^{m_1}$ (in a certain ordering, with a certain m_1). Moreover, we introduce the $\mathscr{K}^0$-submodule $\Xi^1 \subset \subset \Phi(\mathbf{M}^1)$ generated by the above forms $\xi^1, \dots, \xi^{m_1}$. The point of the method lies in the fact that the original symmetries $\mathbf{f}^0$ (preserving $\mathscr{K}^0$ and Ξ^0) bijectively correspond to (local) diffeomorphisms $\mathbf{f}^1$ of $\mathbf{M}^1$ preserving the elements of $\mathscr{K}^0$ and Ξ^1. It follows that we may deal with (the prolonged symmetries) $\mathbf{f}^1$ instead of the original $\mathbf{f}^0$. Clearly

$$\text{(33)} \qquad \mathrm{d}\xi^i = \sum f_{jk}^i \xi^j \wedge \xi^k \quad (i = 1, \dots, n_0;\ \text{sum over}\ \ j, k = 1, \dots, m_0)$$

and all coefficients f_{jk}^i must be assigned to $\mathscr{K}^0$ among the invariants. It is suitable to denote the arising set by $\mathscr{K}^1$ (clearly $\mathscr{K}^1 \subset \mathscr{F}(\mathbf{M}^1)$). Applying the points (i)–(iv) to $\mathscr{K}^1$, Ξ^1 (instead of $\mathscr{K}^0$, Ξ^0), we obtain a $\mathscr{K}^1$-basis $\xi^1, \dots, \xi^{n_1}$ of Ξ^1 with a certain n_1, $m_0 < n_1 \leqq m_1$. Then either $n_1 = m_1$ or $n_1 < m_1$.

If $n_1 = m_1$, then we obtain a Lie group of symmetries $\mathbf{f} = \mathbf{f}^1$ on the space $\mathbf{X} = \mathbf{M}^1$ with invariants $\mathscr{K} = \mathscr{K}^1$ and Maurer-Cartan forms $\Xi = \Xi^1$.

If $n_1 < m_1$, then the above procedure may be repeated to give further $\mathbf{M}^2$, $\mathscr{K}_2$, Ξ^2. Continuing in this way, one obtains either a Lie group or the infinite-dimensional underlying space $\mathbf{X} = \lim\operatorname{inv} \mathbf{M}^l$, the invariants $\mathscr{K} = \bigcup \mathscr{K}^l$, and the Maurer-Cartan forms $\Xi = \bigcup \Xi^l$ resolving the problem. The sought symmetries $\mathbf{f} = \lim \mathbf{f}^l$ are solutions of the system $\mathbf{f}^* k = k$ ($k \in \mathscr{K}$), $\mathbf{f}^* \xi = \xi$ ($\xi \in \Xi$). (The true existence of such $\mathbf{f}$ is however not clear.)

Two important notes should be added.

Using some higher order homology criteria (involutiveness, Spencer's second acyclicity theorem (cf. [14] or [3, Section 34]), the calculations can be stopped with certain $\mathbf{M}^l$, $\mathscr{K}^l$, Ξ^l since the continuation is uniquely determined (cf. Sections 30, 32) and would not bring any new information.

The inessential invariants $k \in \mathscr{K}$ can be eliminated already in the course of the calculations by taking appropriate cross-sections (cf. Section 18) which substantionally simplify the formulae. A little subtler analysis shows that all essential invariants can be expressed by the variables $x^1, \dots, x^{m_1}$ of $\mathbf{M}^1$. That means, if there is an invariant $k \in \mathscr{K}^l$ ($l \geqq 2$) effectively depending on some variable x^j ($j > m_1$), then we may introduce the cross-section $k = F(k^1, \dots, k^c)$ with an arbitrary F depending on $k^1, \dots, k^c \in \mathscr{K}^1$ without changing the problem.

35. Geometrical objects of various kinds belonging to a given groupiety (in classical terms, to the pseudogroup of all relevant symmetries) are all realized by the choice of a filtration. This is the ancient and well-known idea: an abstract group can be represented by permutation groups (cf. [7, 16]).

In greater detail, let $\Psi^0 \subset \Xi$ be a submodule with $\mathscr{F}(\mathbf{X}) \otimes \Xi^0$ integrable and $\ell(\Xi^0) = a < \infty$. If $x^1, \ldots, x^a$ are first integrals of the system $\Xi^0 = 0$, then for every symmetry $\mathbf{f}$, the functions $\mathbf{f}^*(x^j) = f^j(x^1, \ldots, x^a)$ are first integrals, too. It follows that $\mathbf{f}$ induces a local diffeomorphism of the space $\mathbf{M}$ $(=\mathbf{M}^0)$ of these first integrals. But $\Psi^1 = \Psi^0 + \mathscr{L}_{\mathscr{X}}\Psi^0 \subset \Xi$ has analogous properties as Ψ^0 and we obtain the space $\mathbf{M}^1$ of first integrals of the system $\Psi^1 = 0$ with the induced actions $\mathbf{f}$. The procedure can be continued again and again. The arising spaces $\mathbf{M}^l$ $(l \geqq 1)$ may be called the *l-order cotangent spaces* to the original space $\mathbf{M}^0$ of geometrical objects. The tangent spaces and in general the tensor spaces then arise by appropriate algebraic constructions applied to $\mathbf{M}^1$ (some inessential invariants must be adjoined). The spaces of differential operators (with the pseudogroup actions) arise from $\mathbf{M}^l$ by pure algebra, too. In view of the preceding Section, all invariants in these spaces (in particular all invariant differential operators) can be derived from a finite number of invariants by algebraic operations and the operators $\mathscr{L}_X$ $(X \in \mathscr{X})$. In reality, owing to an appropriate reduction argument, only a finite number of these operators is sufficient. All the above mentioned facts are implicitly contained in Cartan's work (cf., e.g., [1, p. 934]) but they are often rediscovered in very particular cases.

We know that certain groupieties possess intrinsical filtrations (Theorem 24) and such pseudogroups have canonical realizations by specific geometrical objects. (For instance, the common realization of simple pseudogroups by point transformations is of this kind.) But there are other methods that produce the geometrical objects and we mention two of them. We shall restrict ourselves to the transitive case $\mathscr{K} = \mathbb{R}$, for brevity.

Passing to the first method, let a subgroupiety $\Omega' < \Omega$ and a point $p \in \mathbf{J}$ be given. Using the notation of Section 19, let us consider the forms $\varphi^j = \xi^j + \sum f_i^j(p)\, \xi^i \in \Xi$ (the coefficients are freezed at p). They generate a formally integrable submodule $\Psi \subset \Phi(\mathbf{X})$. But if a filtration Ξ^* is given, the modules $\Psi^l = \mathscr{F}(\mathbf{X}) \otimes (\Psi \cap \Xi^l)$ are integrable and lead to certain geometrical objects (these are the *left classes* of Ω').

To outline the second method, we consider the dual filtrations Ξ^*, $\mathscr{X}^*$ (cf. Section 13). Then $\mathscr{X}^0/\mathscr{X}^1$ acts on $\mathscr{X}/\mathscr{X}^0$ by the Lie bracket (cf. Section 14). Let λ be a weight of a subalgebra $\mathscr{R} \subset \mathscr{X}^0/\mathscr{X}^1$ (e.g., of the nil radical) for this representation. Then the weight subspace of $\mathscr{X}^0/\mathscr{X}^1$ (consisting of all X satisfying $[Y, X] = \lambda(Y)\, X, Y \in \mathscr{R}$) is closed with respect to the bracket operation on $\mathscr{X}/\mathscr{X}^0$ and gives some integrable submodule $\Psi^0 \subset \Xi$, hence a geometrical object.

36. Connections. In the transitive case, the $\mathbb{R}$-module Ξ of Maurer-Cartan forms gives an absolute parallelism on the underlying space $\mathbf{X}$, hence a curvature-free connection for the tangent bundle. In the general intransitive case, we have a parallelism on every leaf $k \equiv$ const. $(k \in \mathscr{K})$, that is, the connection in the relevant foliated bundle.

These are the connections of the *second kind of equipolence* for the particular case of the Lie groups (cf. [2, pp. 673–791]); they are determined by the left-invariant forms. Connections of the *first kind of equipolence* which should be defined by the right-invariant forms do not in general exist. (Indeed, dual objects to the presumed right-invariant forms are the vector fields $T \in \mathscr{T}(\mathbf{X})$ satisfying $[T, \mathscr{X}] = 0$, that is, $T\mathscr{K} = 0$, $\mathscr{L}_T \Xi = 0$. These are the infinitesimal transformations (cf. Section 13). In general, such T are not submitted to any absolute parallelism: infinitesimal transformations at different points are unrelated.) But there are objects resembling the *equipolences of the third kind* (without torsion) which are of practical interests. They are intrinsically related to a given filtration Ξ^*, that is, to a geometrical object.

Let Ξ^* be a filtration with $\mathrm{H}(\mathscr{N})_0^l = 0$ $(l > 0)$, let $\xi^1, \ldots, \xi^a$ be a basis of Ξ^0. Clearly $\mathrm{d}\xi^k = \sum \xi^j \wedge \xi^k_j$ with certain forms ξ^k_j, generators of Ξ^1. Then $0 = \mathrm{d}^2\xi^k = = \sum \xi^p \wedge (\xi^j_p \wedge \xi^k_j - \mathrm{d}\xi^k_p)$, hence $\mathrm{d}\xi^k_p = \sum \xi^j_p \wedge \xi^k_j + \sum \xi^q \wedge \xi^k_{pq}$ with certain forms ξ^k_{pq} satisfying $\sum \xi^p \wedge \xi^q \wedge \xi^k_{pq} = 0$. Repeating the calculations, we have

$$0 = \mathrm{d}^2\xi^k_p = \sum \xi^q \wedge (\xi^j_{pq} \wedge \xi^k_j + \xi^j_p \wedge \xi^k_{jq} + \xi^j_q \wedge \xi^k_{pj}),$$

hence $\mathrm{d}\xi^k_{pq} = \sum \xi^j_{pq} \wedge \xi^k_j + \sum \xi^j_p \wedge \xi^k_{jq} + \sum \xi^j_q \wedge \xi^k_{pj} + \sum \xi^j \wedge \xi^k_{pqj}$ where ξ^k_{pqr} are certain forms satisfying $\xi^q \wedge \xi^r \wedge \xi^k_{pqr} = 0$. Containuing in this way, the final result

$$\mathrm{d}\xi^k_{i_1 \ldots i_l} = \sum \xi^i_{j_1 \ldots j_r} \wedge \xi^k_{j_{r+1} \ldots j \ldots j_s} \tag{34}$$

proves to be relatively simple: the summation is taken over all partitions $j_1 \ldots j_r$, $j_{r+1} \ldots j_s$ of the sequence $i_1 \ldots i_l$ (so that $r + s = l$) keeping the order, and with $j = 1, \ldots, a$ which is inserted at the place which should be occupied by j_1 in the sequence $j_{r+1} \ldots j_s$ (in the ordering induced from $i_1 \ldots i_l$); if $r = 0$, then j is inserted after all $j_{r+1} \ldots j_s = i_1 \ldots i_l$. For the particular case of the pseudogroup of all diffeomorphisms, the forms ξ^k_I are symmetrical in the multiindex I and formulae equivalent to (34) were stated in [1, pp. 668, 769] for $a = 1$ and $a = 2$; they contain complicated binomial coefficients.

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Souhrn

K FORMÁLNÍ TEORII DIFERENCIÁLNÍCH ROVNIC III

JAN CHRASTINA

Základy teorie Lieových-Cartanových grup jsou vyloženy v rámci nekonečných prodloužení systémů diferenciálních rovnic, to ji činí nezávislou na náhodných realizacích pomocí transformací nějakého geometrického objektu. Práce obsahuje tři axiomatické přístupy a studuje pojem podstatného invariantu, podgrupy, normální podgrupy a faktorgrupy. Pomocí nového pojetí Cauchyových charakteristik je dokázána existence kanonické komposiční řady.

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