Krystyna Gozdalska

On affine planes non-extensible to Laguerre planes and some related problems

Mathematica Bohemica, Vol. 116 (1991), No. 1, 2–11

Persistent URL: http://dml.cz/dmlcz/126197

ON AFFINE PLANES NON-EXTENSIBLE TO LAGUERRE PLANES AND SOME RELATED PROBLEMS

KRYSTYNA GOZDALSKA, Bydgoszcz

(Received March 13, 1987)

Summary. Some examples of affine planes non-extensible to a Laguerre plane are studied and conditions for the uniqueness of a Laguerre extension are given.

Keywords: Laguerrian extensions, the Hughes projective plane over nearfield.

AMS Classification: 51B15.

INTRODUCTION

It is well known that every Laguerre plane induces affine planes (cf. [2], p. 260) but the converse of this statement is not true in general. In the article [3] I have announced a necessary and sufficient condition for an affine plane to be extensible to a Laguerre plane. This condition requires the existence of non-empty family of point sets of an affine plane with some simple properties. The sets of points (called L-ovals in [3]) may be introduced analogously as some classes of parabolas in the real plane.

As the family of L-ovals cannot be generally described in analytical terms the synthetic method has to be applied. Thank to this method it was possible to describe irregular cases to which simple algebraization is not applicable.

A certain non-typical Laguerre plane associated with the affine Moulton plane and a family of L-ovals analogous to "modified parabolas"was constructed on the basis of axioms of the family of L-ovals (see [3]). The example presented above gives an answer to a crucial question:

Are not the conditions characterizing the family of L-ovals so strict that they are fulfilled only by those affine planes on which L-ovals are simply parabolas?

The problem of uniqueness is evidently connected with extension. Actually, we have two problems as the construction of a Laguerre extension is done in two significantly different steps.

The first step is the construction of an affine plane where the family of L-ovals („structure of parabolas") is selected. The second step of a Laguerre extension consists in completing the affine plane with the „structure of parabolas" by improper points to a Laguerre plane.

Two crucial problems concerning additional conditions for L-ovals (which should be added in order to eliminate ambiguities) and the type of extension (i.e. relations between L-ovals and lines on the affine plane and the Laguerre chains) are related to these two steps of a Laguerre plane extension.

The former problem was solved on the basis of Artzy's condition (Π^*) formulated in terms of 4-pascalian ovals (see [1]). This condition associates the oval on the projective plane with the quadratic equation in a certain Hall's coordinate system.

The latter problem of the uniqueness of the extension was solved on the basis of two conditions concerning additional objects completing the affine plane to the Laguerre one.

Another crucial question consists in the presentation of examples of affine planes non-extensible to a Laguerre plane. This question will be discussed in the second part of my paper.

The first part is devoted to the above mentioned extensibility problems and to the solution of some of them. The detailed proofs are omitted and can be found in my papers [3] and [4]. The results concerning the construction of non-extensible affine planes are presented in a more detailed way.

It seems that it is rather difficult to find examples of finite affine planes which are not extensible to Laguerre planes. In the case of Möbius or Minkowski planes the theorem about non-existence of non-Miquelian planes of even order is well known. This each non-Desarguesian affine plane of oven order is extensible to neither Möbius nor Minkowski plane. The analogous theorem concerning Laguerre planes has not been known so far.

1. THE SYNTHETICAL CHARAKTERIZATION OF EXTENSIBILITY AND TWO UNIQUENESS PROBLEMS

The Laguerre plane will be denoted by $\mathbb{P} = \langle \mathscr{P}, \mathscr{C}, - \rangle$ where $\mathscr{P}$ is the set of points, $\mathscr{C} \subset 2^{\mathscr{P}}$ is the set of elements called chains and the relation $- \subset [(\mathscr{P} \cup \mathscr{C}) \times (\mathscr{P} \cup \mathscr{C})]$ is the touch relation (cf. [2], p. 258). The equivalence class of the touch relation (on points) containing the point P will be denoted by $[P]_-$.

Lemma 1.1. *Let Q be any point of a Laguerre plane $\mathbb{P} = \langle \mathscr{P}, \mathscr{C}, - \rangle$ and let $\mathscr{A} = \mathscr{P} \setminus \{P \in \mathscr{P}: P - Q\}$, $\mathbb{L} = \{\alpha \setminus \{Q\}: Q - \alpha \in \mathscr{C}\} \cup \{[P]_-: Q + P \in \mathscr{P}\}$. The incidence structure $\mathbb{P}_Q = \langle \mathscr{A}, \mathbb{L}, \in \rangle$ is the affine plane, where $\mathscr{A}$ is the set of points, $\mathbb{L}$-the set of lines (induced by a point Q of a Laguerre plane). The set $\mathscr{L}_I = \{[P]_-: Q + P \in \mathscr{P}\}$ constitutes the equivalence class of parallel lines.*

Lemma 1.2. *If $Q \in \mathscr{P}$, $Q \notin (\alpha \in \mathscr{C})$ in a Laguerre plane $\mathbb{P} = \langle \mathscr{P}, \mathscr{C}, - \rangle$ then $(\alpha \cap \mathscr{A}) \cup \{\mathscr{L}_I\}$ is an oval in the projective supplement $\mathbb{P}_Q^-$ of $\mathbb{P}_Q$.*

Definition 1.3. An affine plane $\mathbb{A} = \langle \mathscr{A}, \mathbb{L}, \in \rangle$ will be said to be *extensible to a Laguerre plane* if there exists a Laguerre plane $\mathbb{P} = \langle \mathscr{P}, \mathscr{C}, - \rangle$ with a point Q such that $\mathbb{A} = \mathbb{P}_Q$.

Definition 1.4. Let (Y) be any fixed direction, i.e. any fixed class of parallel lines on an affine plane $\mathbb{A} = \langle \mathscr{A}, \mathbb{L}, \in \rangle$.

a) Any subset α of $\mathscr{A}$ is called an *L^Y-oval* (short: *L-oval*) if $\alpha \cup \{(Y)\}$ is an oval in the projective supplement $\mathbb{A}^-$ of $\mathbb{A}$.

b) Two L-ovals α, β are called *congruent* if and only if they satisfy one of the following conditions:

1°. $\alpha = \beta$.

2°. $\alpha \cap \beta = \{P\}$, $P \in \mathscr{A}$ and the tangents of α and β at P are distinct.

3°. $\alpha \cap \beta = \emptyset$ and there exists an L-oval ϑ such that $|\alpha \cap \vartheta| \cap |\vartheta \cap \beta| = 1$; α, ϑ and ϑ, β are pairs of congruent L-ovals in the sense defined by case 2°.

c) Two points $P, Q \in \mathscr{A}$ are called *touching* if there exists $l \in (Y)$ such that $P, Q \in l$.

We use the notation $\alpha \sim \beta$ if L-ovals α and β are congruent. For the touch relation on points of $\mathscr{A}$ and its negation we use the same symbols as in a Laguerre plane ($-$ or $+$, respectively).

A necessary and sufficient condition for an affine plane to induce a Laguerre plane is presented in

Theorem 1.5. *An affine plane $\mathbb{A} = \langle \mathscr{A}, \mathbb{L}, \in \rangle$ is extensible to a Laguerre plane if and only if there is a non-empty set $\mathscr{X}$ of L-ovals on $\mathbb{A}$ satisfying axioms* X1 – X6:

X1. *For any three non-collinear points P, Q, R of $\mathscr{A}$ such that $(P, Q, R)_+$ there exists a unique L-oval $\alpha \in \mathscr{X}$ containing P, Q, R.*

X2. *For every line $l \in \mathbb{L} \setminus (Y)$ and points $P, Q \in \mathscr{A}$ such that $P \in l$, $Q \notin l$, $P + Q$ there is exactly one L-oval $\alpha \in \mathscr{X}$ such that $Q \in \alpha$, $\alpha \cap l = \{P\}$.*

X3. *For every L-oval $\alpha \in \mathscr{X}$ and two non-touching points $P, Q \in \mathscr{A}$ there exists exactly one L-oval $\beta \in \mathscr{X}$ such that $\{P, Q\} \subset \beta$ and $\alpha \sim \beta$.*

X4. *For every L-oval $\alpha \in \mathscr{X}$, every line $l \in \mathbb{L} \setminus (Y)$ and every point $P \in l$ there is a unique L-oval $\beta \in \mathscr{X}$ such that $\beta \cap l = \{P\}$ and $\alpha \sim \beta$.*

X5. *For each L-oval $\alpha \in \mathscr{X}$ and each point $P \notin \alpha$ there exists a unique L-oval $\beta \in \mathscr{X}$ such that $P \in \beta$, $\alpha \sim \beta$ and $\alpha \cap \beta = \emptyset$.*

X6. *If $\alpha \sim \beta$, $\beta \sim \vartheta$, $\alpha \cap \beta = \emptyset$ and $|\beta \cap \vartheta| = 1$, then $|\alpha \cap \vartheta| = 1$.*

The question of finding a necessary and sufficient condition for the extensibility is connected with many additional problems. Some of them concern the analysis of non-standard examples of L-ovals. The existence of non-standard families of L-ovals underlines the fact that the synthetical description involves a larger class of planes than the well known analytical description.

Let us consider the affine Moulton plane $\langle \mathbb{A}_F, L_F, \in \rangle$ over an ordered Euclidean field $\mathscr{F} = \langle F, +, \cdot, < \rangle$ where

$$\mathbb{A}_F = F^2, \quad L_F = L_{1F} \cup L_{2F} \cup L_{3F} \quad \text{and}$$

$$L_{1F} = \{\{x, y\} \in F^2: x = a\}, a \in F\},$$

$$L_{2F} = \{\{x, y\} \in F^2: y = ax + b\}; 0 \leqq a \in F, b \in F\},$$

$$L_{3F} = \left\{\left\{(x, y) \in F^2: y = \begin{cases} ax + b \text{ for } x \geqq 0 \\ kax + b \text{ for } x < 0 \end{cases}\right\}; \begin{matrix} 0 > a \in F; 1 \neq k > 0, \\ b, \quad k \in F \end{matrix}\right\}.$$

We define the family of L-ovals $\mathscr{X} = \mathscr{X}_1 \cup \mathscr{X}_2$ putting

$$\mathscr{X}_1 = \{\{(x, y) \in F^2: y = ax^2 + bx + c\}; a, b, c \in F, a \neq 0, b \geqq 0\}.$$

$$\mathscr{X}_2 = \left\{\left\{(x, y) \in F^2: y = \begin{cases} ax^2 + bx + c & \text{for } x \geqq 0 \\ ax^2 + kbx + c & \text{for } x < 0 \end{cases}\right\}; \begin{matrix} a, b, c, k \in F, b < 0, \\ a \neq 0, k > 1 \end{matrix}\right\}.$$

We have

Theorem 1.6. *The family $\mathscr{X}$ of L-ovals satisfies the conditions $X_1, \ldots, X_6$ from Theorem* 1.5.

The existence of such a family guarantees that the affine plane $\langle \mathbb{A}_F, L_F, \in \rangle$ is extensible to a Laguerre plane. (This Laguerre plane does not satisfy the bundle theorem and consequently it is a non-Miquelian plane; for some details, see [3].) Some examples of standard and non-standard affine planes inducing different families of generalized parabolas were analysed by Hartman in [5] and [6]. (The necessary and sufficient condition for the extensibility does not guarantee that the family of L-ovals is uniquely determined. It ensures only that the affine plane induces at least one structure of parabolas.) Adding to the axioms $X_1, \ldots, X_6$ of L-ovals the condition stating that all L-ovals are 6-pascalian we can guarantee the uniqueness in a rather trivial way. However it turns out that an affine or of Artzy's condition Π^* (concerning 4-pascalian ovals – see [1]) added to axioms $X_1, \ldots, X_6$ also guarantees the uniqueness of the structure of parabolas.

Let us consider the condition

$\Pi^*_{(Y)}$. For any L_Y-oval α and its three different points A, B, C the conditions $B' \in AC$, $C' \in AB$, $BB' \in (Y)$, $CC' \in (Y)$ imply that the line $B'C'$ is parallel to the straight line AA tangent to α in A.

We have

Theorem 1.7. *Let (Y) be a class of equivalence of parallel lines in an affine plane $\mathbb{A}$. There exists at most one family $\mathscr{X}$ of L_Y-ovals in $\mathbb{A}$ satisfying axioms $X_1, \ldots, X_6$ and $\Pi^*_{(Y)}$.*

Analysing the other step in the process of extension we can use some results obtained by I. Hunjiĉ, M. Polonijo and V. Volenec in [8]. Examining the Euclidean plane $\mathscr{E} = \langle A, L, \mathscr{C}, \in \rangle$ /which can be viewed as an affine plane $(A, L, \in)$ with the family of circles $\mathscr{C}$/ Hunjiĉ and others formulated the following conditions (i), (ii) which should be satisfied by the Möbius extension $(M, K, \in)$ /M – the set of points, K – the set of chains/ of the Euclidean plane:

(i) $A \subset M$,

(ii) $\forall_{\alpha \in \mathfrak{I}} \exists_{k \in K} (\alpha \subset k)$.

These conditions can be analysed with regard to the Laguerre case. It turns out that they are not sufficient to guarantee the uniqueness of the extension. (A simple counterexample can be based on the following observation: starting from the affine plane with the structure of parabolas over the field Q of rational numbers one can find the Laguerre extension forming the Laguerre "rational plane". However, the real Laguerre plane also satisfies the conditions (i) and (ii) with respect to the initial "rational affine plane with circles". Using the model – theoretical theorem of Löwenheim-Skolem one can construct similar extensions of an arbitrary infinite cardinality).

Let us define the Laguerrian extension in the following way:

Definition 1.8. Given an arbitrary affine plane $\mathbb{A} = (\mathscr{A}, \mathbb{L}, \in)$ with the family of L-ovals $\mathfrak{X}$, by its *Laguerrian extension* we mean the Laguerre plane $\mathbb{P} = \langle \mathscr{P}, \mathscr{C}, - \rangle$ satisfying the conditions

(1) $\mathscr{A} \subset \mathscr{P}$,

(2) $\forall_{\alpha \in \mathfrak{X}} \exists_{\lambda \in \mathscr{C}} (\alpha \subset \lambda)$.

Let us denote by Card a condition limiting the cardinality of new objects adjoined to L-ovals and to lines from the pencil (Y). Using the notation from Definition 1.8 we can present the condition Card as the following conjunction:

$$\text{(Card)}\ [\forall_{\alpha \in \mathfrak{X}} \forall_{\lambda \in \mathscr{C}} (\alpha \subset \lambda) \Rightarrow |\lambda \setminus \alpha| \leqq 1] \wedge [\forall_{A \in \mathscr{A}} \forall_{B \in \mathscr{P}} (B - A) \Rightarrow B \in \mathscr{A}] .$$

The essential results concerning uniqueness problemms can be formulated in the following way (cf. [3]):

Theorem 1.9. *Let $\mathbb{A}$ be an affine plane with the distinguished pencil (Y) of parallel lines. Conditions $X_1, \ldots, X_6$ and $\Pi^*_{(Y)}$ concerning the structure of parabolas (the family of L_Y-ovals) and the condition* Card *concerning the type of extension guarantee the uniqueness of the Laguerrian extension.*

2. SOME EXAMPLES OF AFFINE PLANES NON-EXTENSIBLE TO THE LAGUERRE PLANE

We will show that two affine planes associated with the Hughes projective plane over the near-field of order 9 are non-extensible to the Laguerre plane. The method of proof is rather complicated and consists in finding a 4-arc which cannot be completed to an oval. Analysing 5-arcs non-extensible to an oval it is easy to verify that the sum, of two 5-arcs of this type forms an interesting configuration (10_3), distinguished by Hilbert and Cohn-Vossen as non-realizable in the real and complex projective plane (see [3]).

Hughes in [7] interpreted the projective plane of order 9 over a near-field as an incidence structure $\mathbb{R}(N\mathbb{F}^9) = \langle \mathcal{R}, \mathcal{L}, \in \rangle$ with the point set

$$\mathcal{R} = \{A_i, B_i, C_i, D_i, E_i, F_i, G_i\colon\ i = 0, 1, \ldots, 12\}$$

and the line set

$$\mathcal{L} = \{L_1^i, L_j^i, L_{2j}^1, L_{1+j}^i, L_{2+2j}^i, L_{1+2j}^i, L_{2+j}^i\colon\ i = 0, 1, \ldots, 12\}\ .$$

Besides,

$$\begin{aligned}
L_1^0 &\quad \{A_0, A_1, A_3, A_9, B_0, C_0, D_0, E_0, F_0, G_0\}\ ,\\
L_j^0 &= \{A_0, B_1, B_8, D_3, D_{11}, E_2, E_5, E_6, G_7, G_9\}\ ,\\
L_{2j}^0 &= \{A_0, C_1, C_8, E_7, E_9, F_3, F_{11}, G_2, G_5, G_6\}\ ,\\
L_{1+j}^0 &= \{A_0, B_7, B_9, D_1, D_8, F_2, F_5, F_6, G_3, G_{11}\}\ ,\\
L_{2+2j}^0 &= \{A_0, B_2, B_5, B_6, C_3, C_{11}, E_1, E_8, F_7, F_9\}\ ,\\
L_{1+2j}^0 &= \{A_0, C_7, C_9, D_2, D_5, D_6, E_3, E_{11}, F_1, F_8\}\ ,\\
L_{2+j}^0 &= \{A_0, B_3, B_{11}, C_2, C_5, C_6, D_7, D_9, G_1, G_8\}
\end{aligned}$$

and we obtain

$L_k^i\ /k = 1, j, 2j, 1 + j, 2 + 2j, 1 + 2j, 2 + j;\ i = 1, 2, \ldots, 12/$ by adding „i" to the indices of points lying on L_k^0 and reducing them modulo 13. The set $\Gamma = \{A_i\colon i = 0, 1, \ldots, 12\}$ forms the point set of the projective plane $\mathbb{R}(\mathbb{F}_3)$ over the field $\mathbb{F}_3 = \{0, 1, 2\}$. Thus $\mathbb{R}(\mathbb{F}_3)$ is a subplane of $\mathbb{R}(N\mathbb{F}^9)$.

The authors of [7] described a group $\mathcal{G}$ of automorphisms of $\mathbb{R}(N\mathbb{F}^9)$ containing the subgroup

$$\mathbb{E} = \{id, (BDG)(CEF), (BGD)(CFE), (BC)(DF)(EG), (BE)(CD)(FG),$$
$$(BF)(CG)(DE)\}$$

and the cyclic group $\Sigma = \{A^i\colon i = 0, 1, \ldots, 12\}$ where A^i means adding i (mod. 13) to the index of a point, for example $A^7(B_9) = B_3$. Every automorphism φ of $\mathbb{E}$ is a pair or a triple of cyclic permutations.

Definition 2.1. Let α be an arc in a projective or affine plane. α is *complete* if each point of the plane belongs to a secant of α. Points not lying on the secants of the arc α will be called adjoinable points of α.

Lemma 2.2 *Let α be a k-arc in a projective plane $\mathbb{R} = \langle R, L, \in \rangle$ of order n. Let P be an arbitrary point of α. Then α cannot be extended to an oval of $\mathbb{R}$, if one of the following conditions is fulfilled:*

(a) *There exist two different tangents p, q of α at P such that for any adjoinable point $T \in p \cup q$ the $(k + 1)$-arc $\alpha \cup \{T\}$ cannot be extended to any oval of $\mathbb{R}$.*

(b) *There exist two different tangents of α at P none of which contains any adjoinable point.*

Proof. There are $n + 1 - (k - 1) = n - k + 2$ tangents of α at P and they together contain all adjoinable points of α by Definition 2.1. If β is an arc such that $\alpha \subset \beta$ and l is the tangent of α then β contains at most one adjoinable point of α lying on l. Thus if β contains only adjoinable points of α lying on $(n - k)$ tangents of α then β consists of at most $k + (n - k) = n$ points. Hence β is not an oval.

Using automorphisms of Hughes it is easy to verify the following lemma:

Lemma 2.3. *Let L_k^i be any line of the projective plane $\mathbb{R}(N\mathbb{F}^9) = \mathbb{R}$. If L_k^i contains four points (exactly one point) of the projective subplane $\mathbb{R}(\mathbb{F}_3)$ then the planes $\mathbb{R}_{L_k{}^i}$ and $\mathbb{R}_{L_1}(\mathbb{R}_{L_k{}^i}$ and $\mathbb{R}_{L_j{}^0})$ are isomorphic, where $\mathbb{R}_{L_k{}^i}$ is the affine plane induced by the line L_k^i.*

Lemma 2.4. *Let $\alpha \in \mathscr{X}$ be an L-oval in an affine plane $\mathbb{A} = \langle \mathscr{A}, \mathbb{L}, \in \rangle$ with the distinguished direction Y. Then the ideal line of the plane $\mathbb{A}^-$ is the tangent of the oval $\beta = \alpha \cup \{Y\}$ at Y.*

Lemma 2.5. *Let Q be a point of a Laguerre plane $\mathbb{P} = \langle \mathscr{P}, \mathscr{C}, - \rangle$ and let Y be the point of $\mathbb{P}_Q^- = \langle \mathscr{R}, \mathscr{L}, \in \rangle$ corresponding to the equivalence class of parallel lines L_I by Lemma* 1.1. *Then every* 4-*arc tangent to the ideal line at the point Y may be extended to an oval.*

Proof. Let $\alpha = \{Y, Z, U, V\}$ be a 4-arc tangent to the ideal line at the point Y in the projective plane $\mathbb{P}_Q^-$. Thus Z, U, V are points of the affine plane $\mathbb{P}_Q = \langle \mathscr{A}, \mathbb{L}, \in \rangle$ and because of Lemma 1.1 they belong to $\mathscr{P}$.

Let β be a unique chain containing Z, U, V. Lemma 1.2 implies that $\vartheta = (\beta \cap \mathscr{A}) \cup \{Y\}$ is an oval in $\mathbb{P}_Q^-$ and $\alpha \subset \vartheta$, of course.

Corollary 2.6. *If $\mathbb{A}$ is an affine plane extensible to a Laguerre plane then $\mathbb{A}^-$ contains an ideal point Y such that every* 4-arc *tangent to the ideal line at Y is extensible to an oval.*

Remark 2.7. Let α be a k-arc in a projective plane $\mathbb{R} = \langle \mathscr{R}, \mathscr{L}, \in \rangle$, let lines p, q be its tangents such that $\alpha \cap p \cap q \neq \emptyset$ and $P_1, \ldots, P_i$ $(Q_1, \ldots, Q_j)$ are the only adjoinable points of α lying on $p(q)$. Then we shall use the following notation: $/\alpha\colon p \sim P_1, \ldots, P_i;\ q \sim Q_1, \ldots, Q_j/$. Analogously $/\alpha\colon p; q/$ means that p and q contain no adjoinable point of α.

Theorem 2.8. *Let $\alpha = \{A_0, E_7, D_9, G_3\}$ where A_0, E_7, D_9, G_3 are points of a projective plane $\mathbb{R}(N\mathbb{F}^9)$. There is no oval containing the 4-arc α.*

Proof. The main idea is based on Lemma 2.2. For the sake of clarity we shall consider only tangents of the 4-arc α at the point A_0. It is easy to verify that we have the following situation:

$$/\alpha\colon\ L_1^0 \sim A_1, A_3, A_9, D_0, E_0, F_0;\ L_1^4 \sim A_5, A_7, B_4, D_4, E_4, F_4/\,.$$

Now we will show that there exists no oval on the $\mathbb{R}(\mathbb{F}^9)$ plane containing the 5-arc $\alpha \cup \{A_1\}$.

$$/\alpha \cup \{A_1, B_1\}\colon L_1^{10};\ L_1^{12} \sim A_8/\,;$$

$$/\alpha \cup \{A_1, D_3\}\colon L_1^4 \sim A_7;\ L_1^{12} \sim B_{12}/\,;$$

$$/\alpha \cup \{A_1, G_9\}\colon L_{2+2j}^0 \sim E_1;\ L_1^4 \sim A_7/\,;$$

$$/\alpha \cup \{A_1, B_6\}\colon L_{1+2j}^0 \sim D_5;\ L_j^0 \sim B_1/\,;$$

$$/\alpha \cup \{A_1, C_3\}\colon L_j^0 \sim B_1;\ L_1^{12} \sim A_8/\,;$$

$$/\alpha \cup \{A_1, E_1\}\colon L_{1+2j}^0 \sim F_8;\ L_1^{10} \sim F_{10}/\,;$$

$$/\alpha \cup \{A_1, B_1, A_8\}\colon L_{1+2j}^0\ L_1^{10}/\,;$$

$$/\alpha \cup \{A_1, D_3, A_7\}\colon L_{1+2j}^0;\ L_1^{10}/\,;$$

$$/\alpha \cup \{A_1, D_3, B_{12}\}\colon L_{2+2j}^0;\ L_1^4 \sim A_7/\,;$$

$$/\alpha \cup \{A_1, G_9, E_1\}\colon L_1^4;\ L_1^{10}/\,;$$

$$/\alpha \cup \{A_1, G_9, A_7\}\colon L_{1+2j}^0;\ L_1^{12}/\,;$$

$$/\alpha \cup \{A_1, B_6, D_5\}\colon L_1^4;\ L_1^{10}/\,;$$

$$/\alpha \cup \{A_1, B_6, B_1\}\colon L_1^{10};\ L_1^{12}/\,;$$

$$/\alpha \cup \{A_1, C_3, B_1\}\colon L_1^{10};\ L_{1+2j}^0/\,;$$

$$/\alpha \cap \{A_1, C_3, A_8\}\colon L_1^4;\ L_1^{10}/\,;$$

$$/\alpha \cup \{A_1, E_1, F_8\}\colon L_1^{10};\ L_1^4/\,;$$

$$/\alpha \cup \{A_1, E_1, F_{10}\}\colon L_j^0;\ L_{1+2j}^0/\,;$$

$$/\alpha \cup \{A_1, D_3, B_{12}, A_7\}\colon L_{2+2j}^0;\ L_{1+2j}^0/\,.$$

The procedure is similar in the case of the other 5-arcs. The full proof of this theorem can be found in [3].

Definition 2.9. Let $\mathbb{A}$ be an affine plane and Y an ideal point of $\mathbb{A}^-$, i.e. a pencil of parallel lines of an affine plane $\mathbb{A}$. A plane $\mathbb{A}$ is *locally non-extensible to a Laguerre plane with respect to the pencil* Y if there is no Laguerre plane $\mathbb{P} = \langle \mathscr{P}, \mathscr{C}, - \rangle$ with a point $Q \in \mathscr{P}$ satisfying the two following conditions:

(1) $\mathbb{P}_Q = \mathbb{A}$,

(2) $\mathscr{L}_I = \{[0]_-;\ Q + 0 \in \mathscr{P}\} = Y.$

Corollary 2.10. *An affine plane $\mathbb{A}$ is non-extensible to a Laguerre plane if and only if it is locally non-extensible with respect to every ideal point Y of $\mathbb{A}^-$.*

Corollary 2.11. *If there exists on an affine plane $\mathbb{A}$ at least one 4-arc non-extensible to an oval and tangent to the ideal line of the projective plane $\mathbb{A}^-$ at the point Y then $\mathbb{A}$ is locally non-extensible to a Laguerre plane with respect to the pencil Y.*

Theorem 2.12. *Let $\mathbb{R}(N\mathbb{F}^9)$ be the projective plane. Then the affine planes $\mathbb{R}_{L_j^0}$ and $\mathbb{R}_{L_j^0}$ are non-extensible to a Laguerre plane.*

Proof. The line $L_j^0(L_1^0)$ is the ideal line of $\mathbb{R}^-_{L_j^0}(\mathbb{R}^-_{L_1^0})$. According to Corollaries 2.10, 2.11 it is enough to show that for every point X lying on L_j^0 there exists a 4-arc α tangent to the line L_j^0 at the point X such that α cannot be extended to an oval. The same property should hold for Y lying on L_1^0. We have already shown that a 4-arc $\alpha = \{A_0, E_7, D_9, G_3\}$ occuring in Theorem 2.8 cannot be extended to an oval. α is tangent to the lines L_j^0 and L_1^0 at the point A_0. Now, Lemma 2.5 and Definition 1.3 imply that $\mathbb{R}_{L_j^0}$ as well as $\mathbb{R}_{L_1^0}$ are locally non-extensible to a Laguerre plane with respect to A_0. For the remaining points of L_j^0 and the points of L_1^0 we shall use automorphisms of $\mathscr{G}$. Each isomorphism of projectie (or affine) planes obviously preserves the incidence relation. Since the incidence relation is the only relation used in Lemma 2.2 we obtain that for every automorphism $\Psi \in \mathscr{G}$ the 4-arc $\Psi(\alpha)$ can be extended to an oval.

Let φA^i $(i = 0, 1, \ldots, 12)$ denote the superposition of automorphism φ and A^i for $\varphi \in \mathbb{E}$, $A^i \in \Sigma$ and let us denote $\varphi_1 = (BDG)(CEF)$, $\varphi_2 = (BGD)(CFE)$, $\varphi_3 = (BC)(DF)(EG)$, $\varphi_4 = (BE)(CD)(FG)$, $\varphi_5 = (BF)(CG)(DE)$. For instance, we obtain

$$\varphi_5 A^4(\alpha) = \varphi_5(\{A_4, E_{11}, D_0, G_7\}) = \{A_4, D_{11}, E_0, C_7\} = \beta$$

and β is the 4-arc tangent to L_j^0 at D_{11}. In this case we have $\varphi_5 A^4(L^9_{1+2j}) = L_j^0$ where L^9_{1+2j} is the tangent of α at E_7. So $\mathbb{R}_{L_j^0}$ is locally non-extensible with respect to the ideal point D_{11}. The procedure is similar for the other points belonging to the lines L_1^0 and L_j^0.

Theorem 2.12 and Lemma 2.3 yield the following consequence:

Corollary 2.13. *If $\mathbb{R}_l$ is an affine plane induced by any line l of the projective plane $\mathbb{R} = \mathbb{R}(\mathbb{F}^9)$ then $\mathbb{R}_l$ is non-extensible to a Laguerre plane.*

References

[1] *R. Artzy:* A symmetry theorem for Laguerre planes, Journal of Geometry, *5*(*2*) (1974), 1—8.
[2] *Yi Chen:* A characterization of some geometries of chains, Can. J. Math., No. 2, *26* (1974), 258—260.
[3] *K. Gozdalska:* Zagadnienie rozszerzalności afinicznych struktur incydencyjnych do płaszczyzn Laguerre'a, doctor thesis, Toruń 1985.
[4] *K. Gozdalska:* Affine planes which induce Laguerre planes, Geometria, Scient. Chapt. of Institute of Technology in Poznań, *15* (1986), 99—110.
[5] *E. Hartman:* Eine Klasse nicht einbettbarer reeller Laguerre-Ebenen, Journal of Geometry, *13* (*1*) (1979), 49—67.
[6] *E. Hartman:* Moulton-Laguerre-Ebenen, Arch. Math., *17* (1976), 424—435.
[7] *D. R. Hughes:* A class of non-Desarguesian projective planes, Can. J. Math., No. 3, *9* (1957), 378—388.
[8] *J. Hunjić, M. Polonijo, V. Volenec:* Uniqueness of the completion of the Euclidean plane to a Möbius plane, Glasnik Matematički, No. 1, *9* (*29*) (1974), 125—130.

Souhrn

O AFINNÍCH ROVINÁCH NEROZŠIŘITELNÝCH NA LAGUEROVY ROVINY A NĚKTERÉ PŘÍBUZNÉ PROBLÉMY

KRYSTYNA GOZDALSKA

Jsou uvedeny příklady afinních rovin nerozšiřitelných na Laguerovy roviny a podány podmínky jednoznačnosti Laguerova rozšíření.

Author's address: Institute of Mathematics and Phisics, Technical and Agricultural Academy 85-790 Bydgoszcz, Poland.