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CONGRUENCE RELATIONS ON THE LATTICE OF PARTITIONS IN A SET

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O. Ore [9] has shown that the symmetric partition lattice $\Pi(M)$ (the lattice of all equivalence relations on a set M) has only trivial congruence relations. The present paper deals with congruence relations on the lattice $P(M)$ of all symmetric and transitive relations in a set M, or equivalently, partitions in M (the empty partition included), contrary to partitions on M, treated by Ore. The construction of all congruence relations on $P(M)$ is described. Two congruence relations Φ, Ψ on $P(M)$ are of especial importance (see Lemma 2.3). It is shown (Remark 2.3) that the lattice $\Theta(P(M))$ of all congruence relations on $P(M)$ is a set-theoretic union of the intervals $[\Lambda, \Psi]$ and $[\Phi, \mathrm{N}]$ where Λ, N are the least and the greatest congruence relation on $P(M)$. There is an interesting duality among the quotient lattices $P(M)/\Phi$ and $P(M)/\Psi$, formulated in Theorem 4.1. Ideals which are congruence classes (normal ideals) are described. It is shown that any normal ideal in $P(M)$ belongs to at most two congruence relations, one of them is in $[\Lambda, \Psi]$ the other in $[\Phi, \mathrm{N}]$. Normal ideals belonging to exactly one congruence relation are characterized (Theorem 3.2). But there is a one-one correspondence between the elements of $\Theta(P(M))$ and the couples (J, J′), where J is a normal ideal of the sublattice $\mathrm{J}(\Psi)$ and J′ a normal dual ideal of the sublattice $\mathrm{J}'(\Phi)$, $\mathrm{J}(\Psi)$ and $\mathrm{J}'(\Phi)$ being the ideal and the dual ideal of the congruence relations Ψ and Φ respectively (Theorem 4.3). Moreover $J(\Psi) \cong 2^M$. The lattice $\Theta(P(M))$ is shown to be isomorphic to the cardinal product $2 \times \Theta(2^M)$ (Theorem 4.2). Hence it is a Boolean algebra if and only if M is finite (Theorem 4.4). No non-trivial decomposition of $P(M)$ into a cardinal product exists (Corollary 4.1). On the other hand, every interval $[O, A]$ in $P(M)$ is isomorphic to the direct product of lattices $P(A\gamma)$, where $A\gamma$ are blocks of the partition A (Theorem 4.6). In $\Pi(M)$ an analogous result holds for any interval $[A, B]$ (Corollary 4.2). Distributivity and modularity of quotient lattices of $P(M)$ are investigated (Theorem 4.7).

1. Notations and some propositions

We shall use the logical symbols „$\Rightarrow$“, „$\Leftrightarrow$“, „$\wedge$“, „$\vee$“ to denote implication, equivalence, conjunction, disjunction and the symbols $\cup$, $\cap$, $\vee$, $\wedge$ for the lattice operations.

Throughout the paper M denotes a non-empty set.

A partition in a set M is a set R of disjoint nonempty subsets R_α of M [2]. The sets R_α are called blocks of the partition R . R can also be empty. We shall call this partition an empty partition and denote it by O. A domain of a partition R is the set $D(R) = \bigcup_\alpha R_\alpha$. If $D(R) = M$, then we shall call R a partition on the set M. Throughout this paper we mean by a relation a binary relation. If α is a relation, we shall write $x\alpha y$, or $x \equiv y(\alpha)$ to denote that x and y are in the relation α. Similarly, if R is a partition, xRy or $x \equiv y(R)$ will denote that x and y are in the same block of R. There is a one-one correspondence between partitions in a set M and relations in M which are transitive and symmetric. There is a one-one correspondence between equivalence relations in M and partitions on M. We shall say that a partition R^1 is less or equal to R^2 and denote $R^1 \leqq R^2$ if $xR^1y \Rightarrow xR^2y$. Partitions in a set M form a complete lattice. For it is evident that the relation $\leqq$ is a partial ordering on M with O as the least element. It suffices to check that there exists the least upper bound to an arbitrary system of partitions in a set M (see [2], § 13).

If R^1 and R^2 are partitions, xR^1R^2y will mean that there exists such an element z that xR^1z and zR^2y. The partitions R^1, R^2 will be called permutable if xR^1R^2y implies xR^2R^1y. The following assertion is obvious. Two partitions R^1, R^2 on a set M are permutable if and only if any block R^1_1 of R^1 intersects all blocks of R^2 which are in the same block of $R^1 \cup R^2$ with R^1_1 [3, § 5]. We shall denote by R° a discrete partition on a set M, i. e. the partition in which any block consists of a single element, and by R^m the greatest partition on M, i. e. the partition with only one block M. Any congruence relation on an algebra A gives a congruence relation $\alpha \mid B$ on an subalgebra $B : x \equiv y(\alpha \mid B)$ $(x, y \in B)$ if and only if $x \equiv y(\alpha)$. A non-empty set J of a lattice S is an ideal if and only if for arbitrary elements $a, b \in S : a \in J \wedge b \in J \Leftrightarrow a \cup b \in J$. A dual ideal is defined dually. A normal ideal of a lattice S is an ideal which is a class of some congruence relation on S. We denote by $J(\alpha)$ the normal ideal belonging to the congruence relation α. The lattice of all congruence relations of a lattice S will be denoted by $\Theta(S)$. The lattice of all partitions in a fixed set M, or on M, will be denoted by $P(M)$ and $\Pi(M)$, respectively. The least (greatest) element of $\Theta(P(M))$ will be denoted by Λ (N).

Theorem 1.1 [9, p. 626]. *There are only trivial congruence relations* Λ $(x \Lambda y \Leftrightarrow x = y)$ *and* N $(x \mathrm{N} y$ *for any* $x, y \in M)$ *on the lattice* $\Pi(M)$.

Theorem 1.2 [4, II, Corollary 3.12]. *Let $h : A \to B$ be a homomorphism of an algebra A onto an algebra B and let α be the corresponding congruence relation on A ($x \equiv y(\alpha) \Leftrightarrow h(x) = h(y)$). There exists a one-one correspondence between congruence relations on B and those congruence relations α_1 on A which are $\geqq \alpha$. If $\alpha_1 \geqq \alpha$ is a congruence relation on A and $\bar{\alpha}_1$ is the corresponding congruence relation on B, then $x \equiv y(\alpha_1) \Leftrightarrow h(x) \equiv h(y)$ $(\bar{\alpha}_1)$.*

Theorem 1.3 [8, § 32]. *Let B be a Boolean algebra, J an ideal in B. Set $x \equiv y(\beta)$ if and only if there is an element $a \in J$ such that $a \cup x = a \cup y$. Then β is a congruence relation on B. Any congruence relation ω on B is determined by the ideal $J = \{x \in B \mid x \equiv 0(\omega)\}$ in the above described way.*

2. Congruence relations on the lattice P(M)

Lemma 2.1. *Let Φ, Ψ be relations on $P(M)$ defined as follows: $R^1 \equiv R^2(\Phi) \Leftrightarrow \Leftrightarrow D(R^1) = D(R^2)$. $R^1 \equiv R^2(\Psi) \Leftrightarrow$ (for any $x, y \in M$, $x \neq y$, $x \equiv y(R^1) \Leftrightarrow \Leftrightarrow x \equiv y(R^2)$) (that is the partitions R^1, R^2 have all blocks, having more than one element, identical). Then Φ, Ψ are congruence relations on $P(M)$.*

Remark 2.1. In the following sections Φ, Ψ mean the congruence relations of Lemma 2.1.

Proof. Obviously Ψ is an equivalence relation. It is sufficient to show for arbitrary $T \in P(M)$ that: $R^1 \equiv R^2(\Psi) \Rightarrow R^1 \cup T \equiv R^2 \cup T(\Psi)$ and $R^1 \cap T \equiv \equiv R^2 \cap T(\Psi)$. But if $R^1 \equiv R^2(\Psi)$, then R^1, R^2 have all blocks with more than one element identical, and the same holds for $R^1 \cap T$, $R^2 \cap T$. It follows that $R^1 \cap T \equiv R^2 \cap T(\Psi)$. Let $R^1 \equiv R^2(\Psi)$, $T \in P(M)$. If for $x, y \in M$, $x \neq y$ $x \equiv y(R^1 \cup T)$, then there is a sequence $x_0, x_1, \ldots, x_n \in M$, $x_0 = x$, $x_n = y$, $x_{i-1} \equiv x_i(A^i)$, where A^i is either R^1 or T. We can suppose $x_j \neq x_k$ for $j \neq k$. If $x_{i-1} \equiv x_i(R^1)$, $x_{i-1} \neq x_i$, then $x_{i-1} \equiv x_i(R^2)$, thus $x \equiv y(R^2 \cup T)$. Similarly, $x \neq y$, $x \equiv y(R^2 \cup T) \Rightarrow x \equiv y(R^1 \cup T)$. Thus we get $R^1 \cup T \equiv R^2 \cup T(\Psi)$. Φ is also an equivalence relation. Let $R^1 \equiv R^2(\Phi)$, $Z \in P(M)$. Then $D(R^1) = = D(R^2)$. Because $D(R^1 \cap Z) = D(R^1) \cap D(Z) = D(R^2) \cap D(Z) = D(R^2 \cap Z)$, we get $R^1 \cap Z \equiv R^2 \cap Z(\Phi)$. Likewise $D(R^1 \cup Z) = D(R^1) \cup D(Z) = D(R^2) \cup \cup D(Z) = D(R^2 \cup Z)$ we get $R^1 \cup Z \equiv R^2 \cup Z(\Phi)$.

Remark 2.2. The mapping $D : P(M) \to 2^M$ (as we have just seen) is a homomorphism of the lattice $P(M)$ onto the Boolean algebra 2^M. Hence $P(M) / \Phi \cong 2^M$.

Lemma 2.2. *The congruence relations Φ, Ψ on $P(M)$ are complemented, i. e. $\Phi \cap \Psi = \Lambda$, $\Phi \cup \Psi = \mathrm{N}$.*

Proof. $R^1 \equiv R^2(\Phi \cap \Psi) \Rightarrow R^1 \equiv R^2(\Phi) \wedge R^1 \equiv R^2(\Psi) \Rightarrow D(R^1) = D(R^2) \wedge \wedge$ (R^1, R^2 have all blocks with more than one element identical) $\Rightarrow R^1$, R^2

have all blocks identical $\Rightarrow R^1 = R^2 \Rightarrow R^1 \equiv R^2(\Lambda)$. Thus $\Phi \cap \Psi = \Lambda$. Let R^1, R^2 be arbitrary partitions from $P(M)$. Let us take first $R^1 \leqq R^2$ and let T^1 be a partition which has all blocks with more than one element identical with R^1 and each element a of the set $D(R^2) - D(R^1)$ form a block $\{a\}$ of T^1. Thus $D(R^2) = D(T^1)$ and $R^1 \equiv T^1(\Psi)$, $T^1 \equiv R^2(\Phi)$, which implies $R^1 \equiv R^2$ $(\Psi \cup \Phi)$. Now let R^1, R^2 be arbitrary. Then $R^1 \leqq R^1 \cup R^2$, $R^2 \leqq R^1 \cup R^2$ and $R^1 \equiv R^2 \cup R^1(\Psi \cup \Phi)$, $R^1 \cup R^2 \equiv R^2(\Phi \cup \Psi)$. If follows $R^1 \equiv R^2(\Psi \cup \Phi)$. Hence $\Psi \cup \Phi = \mathrm{N}$.

Lemma 2.3. *If Φ_i is a congruence relation on $P(M)$ letting all elements of $\Pi(M)$ in the same class and Ψ_j is a congruence relation on $P(M)$ separating each two elements of $\Pi(M)$, then $\Psi_j \leqq \Psi$, $\Phi \leqq \Phi_i$.*

Proof. Let $R^1 \equiv R^2(\Psi_j)$. Then $R^1 \cup R^\circ \equiv R^2 \cup R^\circ(\Psi_j)$ and $R^1 \cup R^\circ$, $R^2 \cup R^\circ \in \Pi(M)$. Thus $R^1 \cup R^\circ = R^2 \cup R^\circ$, hence the blocks of R^1 and R^2 with more than one element are identical, and $R^1 \equiv R^2(\Psi)$. If $R^1 \equiv R^2(\Phi)$, $R^1, R^2 \in P(M)$, then $D(R^1) = D(R^2)$. If $D(R^1) = M$, then obviously $R^1 \equiv R^2$ (Φ_i). Let $D(R^1) \neq M$ and let X be a partition consisting of exactly one block $M - D(R^1) = M - D(R^2)$. Then $R^1 \cup X$, $R^2 \cup X \in \Pi(M)$, hence $R^1 \cup X \equiv R^2 \cup X(\Phi_i)$. Now let Y be a partition consisting of exactly one block $D(R^1) = D(R^2)$. It is obvious that $R^1 \leqq Y$, $R^2 \leqq Y$. We have $Y \cap$ $\cap (R^1 \cup X) \equiv Y \cap (R^2 \cup X)\ (\Phi_i)$. Using Theorem 2.4 [5] we get $Y \cap$ $\cap (R^1 \cup X) = R^1 \cup (X \cap Y) = R^1 \cup O = R^1$. Analogously $Y \cap (R^2 \cup X) =$ $= R^2$. Hence $R^1 \equiv R^2(\Phi_i)$.

Remark 2.3. Let α be a congruence relation on $P(M)$. Then the congruence relation $\alpha \mid \Pi(M)$ on $\Pi(M)$, induced by α, is trivial [Theorem 1.1]. From this and Lemma 2.3, it follows: A lattice $\Theta(P(M))$ is the set-theoretic union of (disjoint) intervals $[\Lambda, \Psi]$ and $[\Phi, \mathrm{N}]$. By Lemma 2.2, Φ, Ψ are complemented and $\Theta(P(M))$ is distributive [7] and it follows that the mappings $\Psi_i \to \Phi \cup \Psi_i$, $\Phi_j \to \Psi \cap \Phi_j$ are mutually inverse isomorphisms between these intervals [8, § 13].

Example 1. A lattice of partitions in a three-element set $\{\alpha, \beta, \gamma\}$ has a diagram in Figure 1. The lattice has the following congruence relations: $K_0 =$ $= \Lambda$, $K_{11} = \mathrm{N}$, K_1: $\{O, a\}, \{b, d\}, \{j, f\}, \{i, m\}, \{c, e\}, \{g\}, \{h\}, \{l\}, \{n\}, \{k\}$. $K_2 : \{O, b\}, \{c, f\}, \{e, j\}, \{a, d\}, \{l, h\}, \{g\}, \{i\}, \{k\}, \{m\}, \{n\}$. $K_3 : \{O, c\}, \{b, f\}$, $\{a, e\}, \{d, j\}, \{g, k\}, \{h\}, \{i\}, \{l\}, \{m\}, \{n\}$. $K_4 : \{O, a, b, d\}, \{c, e, f, j\}, \{h, l\}$, $\{m, i\}, \{g\}, \{k\}, \{n\}$. $K_5 : \{O, b, c, f\}, \{a, d, e, j\}, \{h, l\}, \{g, k\}, \{i\}, \{m\}, \{n\}$. $K_6 : \{O, a, c, e\}, \{b, d, f, j\}, \{g, k\}, \{m, i\}, \{h\}, \{n\}, \{l\}$. $K_7 = \Psi : \{O, a, b, c, d, e,$ $f, j\}, \{g, k\}, \{m, i\}, \{h, l\}, \{n\}$. $K_8 : \{O, a, b, d, g\}, \{f, j, k, l, m, n, i, e, c, h\}$. $K_9 : \{O, b, c, f, i\}, \{a, e, d, j, m, n, l, k, h, g\}$. $K_{10} : \{O, a, c, e, h\}, \{j, l, n, k, m, d,$ $g, i, b, f\}$. $K_{12} : \{O, c\}, \{b, f, i\}, \{a, e, h\}, \{d, g, j, k, l, m, n\}$. $K_{13} : \{O, a\}, \{b, d, g\}$,

$\{e, c, h\}$, $\{k, l, m, n, i, j, f\}$. $K_{14} : \{O, b\}$, $\{c, f, i\}$, $\{a, d, g\}$, $\{k, j, l, m, n, e, h\}$. $K_{15} = \Phi : \{O\}$, $\{a\}$, $\{b\}$, $\{c\}$, $\{e, h\}$, $\{g, d\}$, $\{f, i\}$, $\{k, l, m, n, j\}$. A lattice of congruence relations of this lattice is 2^4.

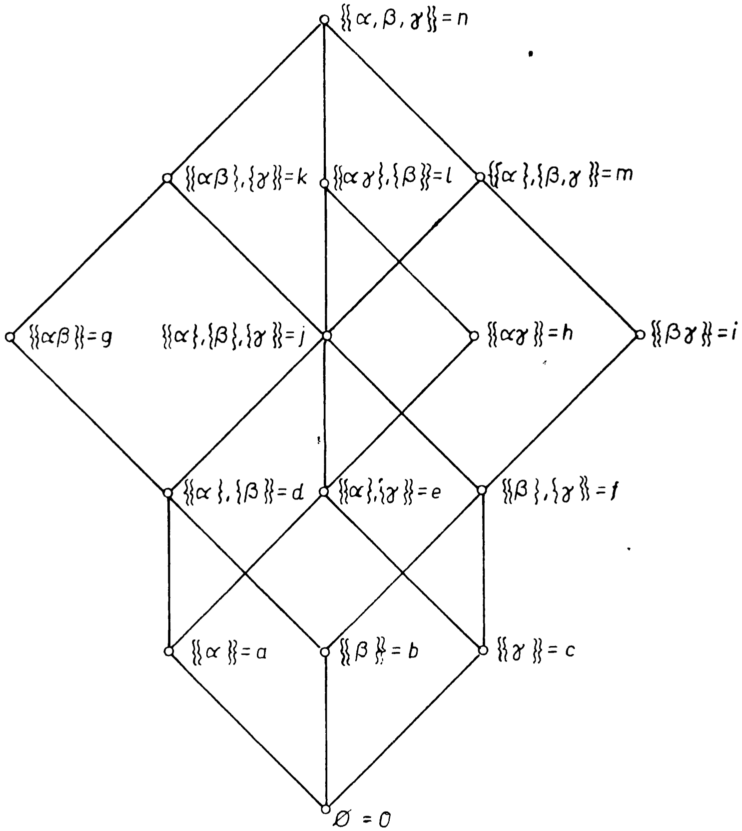

Fig. 1.

3. Normal ideals in P(M)

Normal ideals in $P(M)$ are exatly the zero-classes of congruence relations on $P(M)$. With respect to Remark 2.3, it is sufficient to consider the zero-classes of congruence relations belonging to the intervals $[\Lambda, \Psi]$, $[\Phi, \mathrm{N}]$.

Lemma 3.1. *If we assign to any ideal J of the Boolean algebra 2^M a set $h(J) \subset \subset P(M)$ defined as follows: $R \in h(J) \Leftrightarrow D(R) \in J$, then h is a one-one correspondence between the set of all ideals of the lattice 2^M and the set of all zero-classes of congruence relations on $P(M)$, belonging to the interval $[\Phi, \mathrm{N}]$.*

Proof. Let J be an ideal in 2^M, β_J the congruence relation on 2^M which has J as a zero-class. If we define on $P(M)$ $R^1 \equiv R^2(\beta') \Leftrightarrow D(R^1) \equiv D(R^2)$ (β_J), then according to Theorem 1.2 and Remark 2.2, β' is a congruence relation on $P(M)$ belonging to the interval $[\Phi, \mathrm{N}]$ and its zero-class is just $h(J)$. Conversely, let $\beta \in [\Phi, \mathrm{N}]$ and let J_1 be its zero-class. With respect to Theorem 1.2 a congruence relation $\bar{\beta}$ on 2^M corresponding to β has a zero-class $J = \{D(R) \mid \mid R \in J_1\}$ and it is obvious that $h(J) = J_1$.

Lemma 3.2. *The zero-classes of congruence relations from the interval* $[\Lambda, \Psi]$ *are just the sets having the form* $J(\Psi) \cap J(\Phi_1)$, *where* $\Phi_1 \in [\Phi, \mathrm{N}]$.

Proof. By Remark 2.3, any congruence relation from $[\Lambda, \Psi]$ has the form $\Psi \cap \Phi_1$, $\Phi_1 \in [\Phi, \mathrm{N}]$. Its zero-class is obviously $J(\Psi) \cap J(\Phi_1)$.

Theorem 3.1. *A set J is a normal ideal in* $P(M)$ *if and only if it has one of the following forms*:

(a) $J = \{R \mid D(R) \in J_1$, *where* J_1 *is an ideal of the lattice* $2^M\}$.

(b) $J = J(\Psi) \cap J_2$, *where* J_2 *is an ideal of type* (a).

Proof. The Theorem follows from Lemmas 3.1 and 3.2. Ideals of the form (a) (or (b)) are zero-classes of congruence relations from $[\Phi, \mathrm{N}]$ (or $[\Lambda, \Psi]$).

We shall now investigate the following question: When two congruence relations on $P(M)$ have the same zero-class?

Lemma 3.3. Φ_1, $\Phi_2 \in [\Phi, \mathrm{N}]$, $\Phi_1 \neq \Phi_2 \Rightarrow J(\Phi_1) \neq J(\Phi_2)$. Ψ_1, $\Psi_2 \in [\Lambda, \Psi]$, $\Psi_1 \neq \Psi_2 \Rightarrow J(\Psi_1) \neq J(\Psi_2)$.

Proof. If $\Phi_1 \neq \Phi_2$, Φ_1, $\Phi_2 \in [\Phi, \mathrm{N}]$, then the congruence relations Φ_1', Φ_2' on 2^M, corresponding to Φ_1, Φ_2 are different with respect to Theorem 1.2. According to Theorem 1.3 we get $J(\Phi_1') \neq J(\Phi_2')$. Then by Lemma 3.1, $J(\Phi_1) \neq \neq J(\Phi_2)$. Let Ψ_1, $\Psi_2 \in [\Lambda, \Psi]$, $\Psi_1 \neq \Psi_2$. Let us denote $\Phi_1 = \Phi \cup \Psi_1$, $\Phi_2 = = \Phi \cup \Psi_2$. By Remark 2.3, $\Phi_1 \neq \Phi_2$ and then with respect to the above result $J(\Phi_1) \neq J(\Phi_2)$. Then there exists $R \in J(\Phi_1)$, $R \notin J(\Phi_2)$ (or symmetrically). Let us recall that $\Psi_1 = \Psi \cap \Phi_1$, $\Psi_2 = \Psi \cap \Phi_2$ (Remark 2.3), hence $J(\Psi_1) = J(\Psi) \cap J(\Phi_1)$, $J(\Psi_2) = J(\Psi) \cap J(\Phi_2)$. $J(\Psi)$ is an ideal consisting of all partitions in M which have no block with more than one element. Let R' be the discrete partition on a set $D(R)$. From Φ_1, $\Phi_2 \geqq \Phi$ and $R' \equiv R(\Phi)$ it follows $R' \equiv R(\Phi_1)$, $R' \equiv R(\Phi_2)$. Then $R \in J(\Phi_1)$ implies $R' \in J(\Phi_1)$. It is evident that $R' \in J(\Psi)$. It follows that $R' \in J(\Psi_1) = J(\Phi_1) \cap J(\Psi)$. Since $R \notin J(\Phi_2)$, $R' \notin J(\Phi_2)$. Thus $R' \notin J(\Psi_2) = J(\Phi_2) \cap J(\Psi)$. Hence $J(\Psi_1) \neq J(\Psi_2)$.

Remark 3.1. A lattice-theoretical join of normal ideals of $P(M)$ need not be a normal ideal of $P(M)$. See Example 2.

Example 2. If we take congruence relations K_7, K_8 in Example 1, then $K_7 \cup K_8 = K_{11} = \mathrm{N}$. But $J(K_8) \cup J(K_7) = \{O, a, b, d, g\} \cup \{O, a, b, d, c, e, f, j\} = \{O, a, b, d, g, c, e, f, j, k\} \neq J(K_{11})$. $J(K_8) \cup J(K_7)$ is a class of no congruence relation $K_0 - K_{15}$, hence it is not a normal ideal.

Now we shall characterize the normal ideals which are classes of two or more congruence relations.

Theorem 3.2. *Any ideal consisting either of the empty partition alone or of the empty partition and a partition having only one one-element block is a normal ideal of just two congruence relations on $P(M)$, one of which is in $[\Lambda, \Psi]$ and the second in $[\Phi, N]$. Any other normal ideal is a class of exatly one congruence relation on $P(M)$.*

Proof. If an ideal is a class of more congruence relations, then at most one of these congruence relations can be in $[\Lambda, \Psi]$ and at most one in $[\Phi, \mathrm{N}]$ (according to Lemma 3.3). The ideal consisting of the empty partition is a class of the congruence relations Φ and Λ. Because the empty set and one-element set $\{\alpha\}$ form an ideal in 2^M, then by Lemma 3.1, the corresponding ideal in $P(M)$ (let us denote it by $J(\Phi_1)$) consists of the empty partition and the partition having only one one-element block $\{\alpha\}$ and it is a normal ideal of some congruence relation $\Phi_1 \geqq \Phi$. If we denote $\Psi_1 = \Psi \cap \Phi_1$, then $J(\Psi_1) = J(\Psi) \cap J(\Phi_1) = J(\Phi_1)$ because $J(\Phi_1) \subset J(\Psi)$. It follows that the congruence relations Ψ_1, Φ_1 have the same zero-class. If $\Phi_2 \in [\Phi, \mathrm{N}]$ and $J(\Phi_2)$ contains more than two partitions, then by Lemma 3.3, $J(\Phi_2)$ cannot be a class of a congruence relation $\Phi_3 \in [\Phi, \mathrm{N}]$, $\Phi_3 \neq \Phi_2$. We shall show that $J(\Phi_2)$ cannot be a class of a congruence relation $\Psi_1 \in [\Lambda, \Psi]$. It is evident that $J(\Phi_2)$ contains some partition R^α having only one one-element block $\{\alpha\}$ $(\alpha \in M)$ and the partition O. Let $R \in J(\Phi_2)$, $R^\alpha \neq R \neq O$. If we denote $R^1 = R^\alpha \cup R$, then $R^1 \in J(\Phi_2)$. Since $\Phi_2 \geqq \Phi$, then $J(\Phi_2)$ must contain also a partition R^2 having only one block $D(R^1)$ with more than one element. If $\Psi_1 \leqq \Psi$, then $J(\Psi_1) \subset J(\Psi)$. Now $J(\Psi_1) \neq J(\Phi_2)$ because otherwise $R^2 \in J(\Psi_1) \subset J(\Psi)$, but this is impossible because no partition, belonging to the ideal $J(\Psi)$ has blocks with more than one element. This completes the proof.

4. Further results on congruence relations on P(M)

Definition. *A symmetric partition lattice on M is the lattice $\Pi(M)$.*

Theorem 4.1. *The quotient lattice $P(M)/\Phi$ is isomorphic to the ideal $J(\Psi)$, consisting of all partitions in a set M which have no block with more than one element, and is isomorphic to 2^M. Any class of congruence relation Φ is isomorphic to a symmetric partition lattice on the domain of partitions belonging to this class.*

The quotient lattice $P(M)/\Psi$ is isomorphic to the dual ideal which is a class of the congruence relation Φ, that is with the symmetric partition lattice on M. Any class of congruence relation Ψ is a Boolean algebra.

Proof. Obviously $J(\Psi) = [O, R^\circ]$. To prove the first part of the Theorem it is sufficient to show that to any $R \in P(M)$ there is exactly one partition $R' \in J(\Psi)$ such that $R' \equiv R(\Phi)$. This partition R' is a discrete partition on the set $D(R)$. It can be immediately seen that the interval $J(\Psi) = [O, R^\circ]$ is isomorphic to 2^M. (The isomorphism $P(M)/\Phi \cong 2^M$ follows also from the homomorphism $D : P(M) \to 2^M$, see Remark 2.2.) If $\bar{R}$ is a class of the congruence relation Φ and $R \in \bar{R}$, then evidently $\bar{R}$ is isomorphic to the symmetric partition lattice on $D(R)$. To prove the second part of the Theorem it is sufficient to show that to any $R \in P(M)$ there is exactly one element $R' \in \Pi(M)$ such that $R \equiv R'(\Psi)$. This partition is $R' = R \cup R^\circ$. Let $\tilde{R}$ be a class of the congruence relation Ψ and let $V(\tilde{R})$ be the set-theoretic union of all blocks with more than one element of the partitions belong to $\tilde{R}$. We shall show that the lattice $2^{M-V(\tilde{R})}$ (of all subsets of $M - V(\tilde{R})$) is isomorphic to the sublattice $\tilde{R}$ of $P(M)$. To this purpose we assign to any subset $A \subset M - V(\tilde{R})$ a partition $R^A \in \tilde{R}$ the one element blocks of which are exactly the sets $\{a\}$ with $a \in A$. One can easily verify that this assignment yields a lattice isomorphism between $\tilde{R}$ and $2^{M-V(\tilde{R})}$.

Now we shall investigate relationship among congruence relations on the lattice $P(M)$ and congruence relations on the sublattices $\Pi(M)$ and $J(\Psi)$. Any congruence relation χ on $P(M)$ induces the congruence relations $\chi \mid J(\Psi)$ on $J(\Psi)$ and $\chi \mid \Pi(M)$ on $\Pi(M)$. $R^1 \equiv R^2\ (\chi|J(\Psi)) \Leftrightarrow R^1 \equiv R^2(\chi)$, $R^1, R^2 \in J(\Psi)$. $R^1 \equiv R^2\ (\chi \mid \Pi(M)) \Leftrightarrow R^1 \equiv R^2(\chi)$, $R^1, R^2 \in \Pi(M)$. With respect to Theorem 4.1 $J(\Psi) \cong 2^{\mathrm{M}}$, hence we know the congruence relations on $J(\Psi)$. According to Theorem 1.1 we have only trivial congruence relations Λ, N on $\Pi(M)$.

Lemma 4.1. *If χ, χ' are congruence relations on $P(M)$ and $\chi \neq \chi'$, then either $\chi \mid J(\Psi) \neq \chi' \mid J(\Psi)$ or $\chi \mid \Pi(M) \neq \chi' \mid \Pi(M)$.*

Proof. If $\chi \neq \chi'$ and $\chi \mid \Pi(M) = \chi' \mid \Pi(M)$, then we have two possibilities: 1) χ, χ' induce the least congruence relation on $\Pi(M)$. Then according to Lemma 2.3, $\chi, \chi' \in [\Lambda, \Psi]$. Thus $J(\chi)$, $J(\chi') \subset J(\Psi)$. By Lemma 3.3, $J(\chi) \neq J(\chi')$, hence $\chi \mid J(\Psi) \neq \chi' \mid J(\Psi)$. 2) χ, χ' induce the greatest congruence relation on $\Pi(M)$. By Lemma 2.3, $\chi, \chi' \in [\Phi, \mathrm{N}]$. If $\Psi_1 = \Psi \cap \chi$, $\Psi_1' = \Psi \cap \chi'$, then $\Psi_1 \neq \Psi_1'$ (Remark 2.3) and by Lemma 3.3, $J(\Psi_1) \neq J(\Psi_1')$. Then $\Psi_1 \mid J(\Psi) = \chi \mid J(\Psi)$, $\Psi_1' \mid J(\Psi) = \chi' \mid J(\Psi)$. Because $J(\Psi_1)$, $J(\Psi_1') \subset J(\Psi)$, $J(\Psi_1) \neq J(\Psi_1')$ implies $\chi \mid J(\Psi) \neq \chi' \mid J(\Psi)$.

Lemma 4.2. *Let χ' be an arbitrary congruence relation on $J(\Psi)$. If we define $\chi : R \equiv R'(\chi) \Leftrightarrow$ (there is $T \in J(\chi')$ such that $T \cup R = T \cup R'$) then χ is* *a congruence relation on* $P(M)$ *and* $\chi \mid J(\Psi) = \chi'$, $\chi \mid \Pi(M) = \gamma$ (γ *is the least congruence relation on* $\Pi(M)$).

Proof. First we shall show that χ is a congruence relation. Obviously χ is an equivalence relation. If $R \equiv R'(\chi)$, then there is $T \in J(\chi')$ such that $T \cup R = T \cup R'$. Let $Z \in P(M)$. Obviously $T \cup R \cup Z = T \cup R' \cup Z$. Thus $R \cup Z \equiv R' \cup Z(\chi)$. If $T \in J(\chi') \subset J(\Psi)$, then $T \leqq R^\circ$. With respect to Theorem 3.4 [5], $T \cup (R \cap Z) = (T \cup R) \cap (T \cup Z)$, $T \cup (R' \cap Z) = (T \cup R') \cap \cap (T \cup Z)$. If $T \cup R = T \cup R'$, then $T \cup (R \cap Z) = T \cup (R' \cap Z)$, consequently $R \cap Z \equiv R' \cap Z(\chi)$. Thus χ is a congruence relation. We shall now show that $\chi \mid \Pi(M) = \gamma$. If $R^1 \equiv R^2(\chi)$, $R^1, R^2 \in \Pi(M)$, then there is $T \in J(\chi') \subset J(\Psi)$ such that $T \cup R^1 = T \cup R^2$. Since $R^1, R^2 \in \Pi(M)$, evidently $R^1, R^2 \geqq R^\circ$. Since $T \in J(\Psi)$, $T \leqq R^\circ$. It follows that $R^1 = T \cup R^2 = R^2$. Thus $\chi \mid \Pi(M) = \gamma$. Now we shall show that $\chi \mid J(\Psi) = \chi'$. Let $R^1, R^2 \in J(\Psi)$, $R^1 \equiv R^2(\chi')$. As $J(\Psi)$ is a Boolean algebra this holds if and only if there exists $T \in J(\chi')$ such that $T \cup R^1 = T \cup R^2$ (see Theorem 4.1 and Theorem 1.3). This holds if and only if $R^1 \equiv R^2(\chi)$ (see Definition of χ). It follows that $\chi \mid J(\Psi) = \chi'$.

Lemma 4.3. $\Phi \mid \Pi(M)$ *is the greatest congruence relation on* $\Pi(M)$.

Proof. The assertion is evident from the Definition of the congruence relation Φ.

Remark 4.1. Obviously $R^1 \equiv R^2(\Phi) \Leftrightarrow R^\circ \cap R^1 = R^\circ \cap R^2$, because the equality on the right-hand side is equivalent to $D(R^1) = D(R^2)$.

In the next Lemma 4.4 we shall denote by $J'(\alpha)$ the dual ideal of a congruence relation α, that is the class of the elements which are congruent with the greatest partition R^m on the set M.

Lemma 4.4. *Let* χ' *be an arbitrary congruence relation on* $J(\Psi)$ *with the ideal* $J(\chi')$, χ *the congruence relation of Lemma* 4.2, *and* $\Phi \mid \Pi(M)$ *the congruence relation of Lemma* 4.3. *Then the congruence relation* $\alpha = \Phi \cup \chi$ *on* $P(M)$ *has the following properties*:

(1) $J(\alpha) \cap J(\Psi) = J(\chi') = J(\chi)$.

(2) $J'(\alpha) \cap \Pi(M) = \Pi(M)$.

(1′) $\alpha \mid J(\Psi) = \chi'$.

(2′) $\alpha \mid \Pi(M) = \Phi \mid \Pi(M)$.

Proof. The assertion (2) is trivial, because $J'(\alpha) \supset J'(\Phi) = \Pi(M)$. We shall prove (1′). Let $R \equiv R'(\alpha)$, $R, R' \in J(\Psi)$, then $R, R' \leqq R^\circ$. From the Definition of the congruence relation α, $R^1, R^2, \ldots, R^n$ exist such that $R \equiv R^1(\Phi)$ [or χ], $R^1 \equiv R^2(\chi)$ [or Φ], $\ldots$, $R^n \equiv R'(\Phi)$ [or χ]. Then $R =$

$= R \cap R^\circ \equiv R^1 \cap R^\circ(\Phi)$ [or χ], $R^1 \cap R^\circ \equiv R^2 \cap R^\circ(\chi)$ [or Φ], ..., $R^n \cap R^\circ \equiv$ $\equiv R' \cap R^\circ = R'(\Phi)$ [or χ]. We have $R^i \cap R^\circ \leqq R^\circ$ for $i = 1, \ldots, n$ and thus $R^i \cap R^\circ \in J(\Psi)$. With regard to Lemma 4.2, $\chi \mid J(\Psi) = \chi'$. Then $R^i \cap R^\circ \equiv$ $\equiv R^{i+1} \cap R^\circ(\chi)$ implies $R^i \cap R^\circ \equiv R^{i+1} \cap R^\circ(\chi')$. $R^i \cap R^\circ \equiv R^{i+1} \cap R^\circ(\Phi) \Rightarrow$ $\Rightarrow R^i \cap R^\circ = R^{i+1} \cap R^\circ$ (because $R^i \cap R^\circ$, $R^{i+1} \cap R^\circ \in J(\Psi)) \Rightarrow R^i \cap R^\circ \equiv$ $\equiv R^{i+1} \cap R^\circ(\chi')$. It follows that $R \equiv R'(\chi')$. We proved $\alpha \mid J(\Psi) \leqq \chi'$. We have $\alpha \geqq \chi$, hence $\alpha \mid J(\Psi) \geqq \chi \mid J(\Psi) = \chi'$. Hence we get $\alpha \mid J(\Psi) = \chi'$. The assertion (1') implies immediately the first part of the assertion (1), that is $J(\alpha) \cap J(\Psi) = J(\chi')$. We have to show $J(\chi) = J(\chi')$. χ' is a congruence relation on $J(\Psi)$, thus $J(\chi') \subseteq J(\Psi)$. By Lemma 4.2, $\chi \mid \Pi(M) = \gamma$, thus $J(\chi) \subseteq J(\Psi)$ (Lemma 2.3), moreover $\chi \mid J(\Psi) = \chi'$ and it follows that $J(\chi') = J(\chi)$. We shall prove (2'). By Lemma 4.3, $\alpha \mid \Pi(M) \leqq \Phi \mid \Pi(M)$. Since $\Phi \leqq \alpha$, $\Phi \mid \Pi(M) \leqq \alpha \mid \Pi(M)$ and we get $\alpha \mid \Pi(M) = \Phi \mid \Pi(M)$.

Theorem 4.2. *The lattice of all congruence relations on $P(M)$ is a cardinal product of the lattice of all congruence relations on the sublattice $J(\Psi)$ and the lattice of all congruence relations on the sublattice $\Pi(M)$. Consequently, $\Theta(P(M)) \cong$ $\cong \Theta(2^M) \times 2$.*

Proof. The mapping $f: \chi \to (\chi \mid J(\Psi), \chi \mid \Pi(M))$ maps $\Theta(P(M))$ into the cardinal product $\Theta(J(\Psi)) \times \Theta(\Pi(M))$. By Lemma 4.1 f is injective and clearly isotone. To prove that f is surjective and f^{-1} isotone, let χ' and χ'' be congruence relations on $J(\Psi)$ and $\Pi(M)$, respectively. 1) If $\chi'' = \gamma$, let χ be the congruence relation $\chi \in \Theta(P(M))$ of Lemma 4.2. Then $f(\chi) = (\chi', \chi'')$. Moreover, if we repeat this process with $\chi_1' \in \Theta(J(\Psi))$, $\gamma = \chi_1'' \in \Theta(\Pi(M))$ and obtain $\chi_1 \in \Theta(P(M))$, then (according to the proof of Lemma 4.2) $(\chi', \chi'') \leqq (\chi_1', \chi_1'')$ implies $\chi \leqq \chi_1$. 2) If $\chi'' = \Phi \mid \Pi(M)$, take first χ as in the case of 1), then put $\alpha = \Phi \cup \chi$. Then according to Lemma 4.4, $f(\alpha) =$ $= (\chi', \chi'')$. Moreover, if $\alpha_1 = \Phi \cup \chi_1$ is the congruence relation obtained from the couple (χ_1', χ_1'') by the same way, we see immediately that $(\chi', \chi'') \leqq$ $\leqq (\chi_1', \chi_1'')$ imply $\chi \leqq \chi_1$ and $\alpha \leqq \alpha_1$. We have proved the surjectivity of f and, in two cases, the isotonity of f^{-1}. To finish the proof, let $(\chi', \chi'') \leqq$ $\leqq (\chi_1', \chi_1'')$ and $\chi'' = \gamma$, $\chi_1'' = \Phi \mid \Pi(M)$. Take χ_1 and $\alpha_1 = \Phi \cup \chi_1$ as in the case 2), and χ as in the case 1). Then $f(\chi) = (\chi', \chi'')$, $f(\alpha_1) = (\chi_1', \chi_1'')$ and since $\chi \leqq \chi_1$, $\chi \leqq \alpha_1$. This completes the proof.

Theorem 4.3. $\Theta(P(M)) \cong S_1 \times S_2$, *where S_1 is the lattice of normal ideals in $J(\Psi)$ and S_2 is the lattice of normal dual ideals in $\Pi(M)$.*

Proof. By Theorem 4.2 $\Theta(P(M)) \cong \Theta(J(\Psi)) \times \Theta(\Pi(M))$. $J(\Psi)$ is a Boolean algebra (Theorem 4.1), hence $\Theta(J(\Psi)) \cong S_1$ (by Theorem 1.3). It is evident that $\Theta(\Pi(M)) \cong S_2$ (see Theorem 1.1).

Remark 4.2. Using Theorems 1.3, 1.1 and 4.2, and Lemmas 4.2 and 4.3 we get the correspondences in the isomorphism of Theorem 4.3: given $\alpha \in$

$\in \Theta(P(M))$ we set $J_1 = J(\alpha) \cap J(\Psi)$, $J'_1 = J'(\alpha) \cap \Pi(M)$ to obtain the corresponding couple $(J_1, J'_1) \in S_1 \times S_2$. Conversely, given a couple (J_1, J'_1), we construct the congruence relation δ on $P(M)$ with $J(\delta) = J_1$ ($R \equiv R'(\delta)$ if and only if $T \cup R = T \cup R'$ for a $T \in J_1$) and the congruence relation η on $P(M)$ with the dual ideal J'_1 ($\eta = \Phi$ if $J'_1 = \Pi(M)$ and $\eta = \Psi$ if $J'_1 = \{R^m\}$). Then the congruence relation on $P(M)$, corresponding to (J_1, J'_1), is $\beta = \delta \cup \eta$.

Remark 4.3. If M is infinite, then any congruence relation of $\Theta(P(M))$ need not have a complement. We shall construct such congruence relation ξ. We denote ξ_{R^1, T^1} the least congruence relation with $R^1 \equiv T^1$. $2^M \cong J(\Psi)$ is also infinite and we can take an infinite sequence $O = R^1 < T^1 < \ldots < R^i < T^i < \ldots < R^\circ$ of elements of $J(\Psi)$. Then by Lemma 13 [6, p. 160] and Lemma 15 [6, p. 161] the congruence relation $\xi = \bigvee_{i=1}^{\infty} \xi_{R^i, T^i}$ has no complement in $\Theta(P(M))$.

Lemma 4.5. *Let* $\Psi_1 \in [\Lambda, \Psi]$. *Then* Ψ_1 *has a complement* Φ_1 *in* $[\Lambda, \mathrm{N}]$ *if and only if* Ψ_1 *has a complement* $\bar{\Psi}_1$ *in* $[\Lambda, \Psi]$ *and* $\Phi_1 = \bar{\Psi}_1 \cup \Phi$. *The dual theorem also holds: Let* $\Phi_1 \in [\Phi, \mathrm{N}]$. *Then* Φ_1 *has a complement* Ψ_1 *in* $[\Lambda, \mathrm{N}]$ *if and only if* Φ_1 *has a complement* $\bar{\Phi}_1$ *in* $[\Phi, \mathrm{N}]$ *and* $\Psi_1 = \Psi \cap \bar{\Phi}_1$.

Proof. Since the lattice $\Theta(P(M))$ is distributive [7], then by Remark 2.3 the Lemma follows immediately.

Definition. *We shall say that a lattice* L *is discrete if and only if any bounded chain in* L *is finite.*

Theorem 4.4. *The lattice of all congruence relations on* $P(M)$ *is a Boolean algebra if and only if* M *is finite.*

Proof. By Theorem 4.2 and Theorem 1.1 it is sufficient to find out when $\Theta(J(\Psi))$ is a Boolean algebra. $J(\Psi) \cong 2^M$ (see Theorem 4.1) ,thus $J(\Psi)$ is distributive. By the theorem of Hashimoto [7] $\Theta(J(\Psi))$ is a Boolean algebra if and only if $J(\Psi)$ is discrete. But 2^M is discrete if and only if M is finite.

Theorem 4.5. *No couple of mutually complemented congruence relations on* $P(M)$ *is permutable except the couple* (Λ, N).

Proof. Let $\Phi_1, \Psi_1 \in \Theta(P(M))$ be mutually complements and permutable. Each of the congruence relations Ψ_1, Φ_1 belongs precisely to one of the intervals $[\Lambda, \Psi]$, $[\Phi, \mathrm{N}]$ (Remark 2.3). Both Ψ_1 and Φ_1 cannot be simultaneously contained neither in $[\Lambda, \Psi]$, nor in $[\Phi, \mathrm{N}]$, since Ψ_1 and Φ_1 are mutually complements. Let e. g. $\Psi_1 \in [\Lambda, \Psi]$, $\Phi_1 \in [\Phi, \mathrm{N}]$. Suppose $\Psi_1 \neq \Lambda$. The greatest partition R^m (on M) forms a block of the congruence relation Ψ_1 because $R^m \equiv R(\Psi_1) \Rightarrow R^m \equiv R(\Psi) \Rightarrow R^m = R$. If Ψ_1, Φ_1 are permutable, then the block $\{R^m\}$ is incident with any block of the congruence relation Φ_1 (see the assertion in section 1) and then also with a block $\{R : O \equiv R(\Phi_1)\}$. It follows that $R^m \equiv O(\Phi_1)$, that is $\Phi_1 = \mathrm{N}$, which is a contradiction to the assumption.

Corollary 4.1. *The lattice $P(M)$ cannot be decomposed into a cardinal product in a nontrivial way.*

Proof. Otherwise there exists a nontrivial couple of mutually complemented and permutable congruence relations in $\Theta(P(M))$ (see [1, Th. 5, Chapter VII.]), which is a contradiction to Theorem 4.5.

Theorem 4.6. *Let A be a partition in M, and let $\{A_\gamma : \gamma \in \Gamma\}$ be the set of all its blocks. Then the interval $[O, A]$ is isomorphic to the direct product $\mathbf{X}\ \{P(A_\gamma) : : \gamma \in \Gamma\} = Q$.*

Proof. With any $B \in [O, A]$ and any $\gamma \in \Gamma$ we associate a partition B^γ in A_γ consisting of all blocks of B contained in A_γ. (If no block of B in A_γ exists we set $B^\gamma = O$.) Then $B^\gamma \in P(A_\gamma)$ and the mapping $f\colon [O, A] \to Q$ given by $(f(B))_\gamma = B^\gamma$ is surjective. It is evident that $B \leqq B'$ if and only if $B^\gamma \leqq B'^\gamma$ for every $\gamma \in \Gamma$.

Remark 4.4. It can be proved similarly that an analogous theorem holds for the lattice $\Pi(M)$ (it suffices to replace P by Π).

Lemma 4.6. *Let $A \in \Pi(M)$. Then the interval $[A, R^m]$ of the lattice $\Pi(M)$ is isomorphic to the lattice $\Pi(A)$.*

Proof (cf. [10]). With any $C \in [A, R^m]$ we associate the partition C^* on A defined as follows. Given blocks A_1, A_2 of A, $A_1 C^* A_2$ if and only if A_1 and A_2 are contained in the same block of C. It can be easily shown that the assignment $C \to C^*$ is an isomorphism of the lattices $[A, R^m]$ and $\Pi(A)$.

Corollary 4.2. *Any interval $[A, B]$ of the lattice $\Pi(M)$ is isomorphic to a direct product of symmetric partition lattices. More precisely, if $B = \{B_\gamma : \gamma \in \Gamma\}$, and if for any $\gamma \in \Gamma, M_\gamma$ denotes the set of all blocks of A contained in B_γ, then $[A, B] \cong \mathbf{X}\ \{\Pi(M_\gamma) : \gamma \in \Gamma\}$.*

Remark 4.5. On the lattice $P(\{1, 2, 3\})$ (see Figure 1) it can be seen that Lemma 4.6 and Corollary 4.2 do not hold for intervals of the lattice $P(M)$.

Theorem 4.7. *Let card $M \geqq 4$. The congruence relation $\Phi \in \Theta(P(M))$ is the least one for which the quotient lattice $P(M)/\Phi$ is modular. More precisely, for all congruence relations $\Phi_1 \geqq \Phi$ the lattice $P(M)/\Phi_1$ is a Boolean algebra and if $\Psi_1 \geqq \Phi$ does not hold, then $P(M)/\Psi_1$ is not even modular.*

Proof. It follows from Theorem 4.1 that $P(M)/\Phi$ is a Boolen algebra. If $\Psi_1 \ngeqq \Phi$, then $\Psi_1 \leqq \Psi$ (Remark 2.3). If $P(M)/\Psi_1$ were modular, then its homomorphic image $P(M)/\Psi \cong \Pi\ (M)$ would be modular too, which is a contradiction.

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