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A NOTE CONCERNING A PAPER BY L. E. SNYDER

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In a recent paper [2] L. E. Snyder has given a sufficient condition for every boundary function defined by the approximate Stolz angle method to be in the first Baire class. In this paper the result by Snyder is improved in such a way as to give the necessary and sufficient condition.

R^2 detones the Euclidean plane. Let $W = \{(x, y) : (x, y) \in R^2, y > 0\}$. By the Stolz angle S_x we mean an angle in W with a vertex in $(x, 0)$ which is symmetric about the half-line $\{(x, y) : (x, y) \in W, y \geqslant 0\}$ and its size is not greater than π. Let $\Theta(x)$ be a size of the S_x, i. e. $0 < \Theta(x) < \pi$. For $r > 0$ we define $S_x^2 = \{(u, v) : (u, v) \in S_x, v < r\}$. The point $(x, 0)$ is said to be a point of density of E relative to S_x, if

$$\liminf_{r \to 0+} \frac{|E \cap S_x^r|}{|S_x^r|} = 1$$

holds. $|A|$ means the 2-dimensional Lebesgue measure of the set A.

L. E. Snyder has proved ([2], p. 420, Corollary 2):

Let $\Phi : W \to (-\infty, \infty)$ and let there exist for each $x \in (-\infty, \infty)$ a set $E_x \subset W$ such that

(i) $(x, 0)$ *is a point of density of E_x relative to S_x*

(ii) $\lim\limits_{\substack{(u,v)\to(x,0)\\(u,v)\in E_x}} \Phi(u, v) = f(x)$ *exists for each $x \in (-\infty, \infty)$.*

If the function Θ: $(-\infty, \infty) \to (0, \pi)$ associated with the family of Stolz angles is upper semicontinuous, then the boundary function f is in the first Baire class.

The proof of this corollary is indirect. It is supposed that there is a nonempty perfect set P for which the partial function f/P has no point of continuity. The upper semicontinuity of the function Θ is used only to guarantee the existence of an open interval J for which $\inf\{\Theta(x) : x \in J \cap P\} > 0$. The existence of such an interval J is sufficient to conclude a contradiction. Hence from the proof of Corollary 2 it is clear that the following Theorem is true:

Theorem 1. *Let $\Phi : W \to (-\infty, \infty)$ be a function, let there exist for each $x \in (-\infty, \infty)$ a set $E_x \subset W$ such that*

(i) $(x, 0)$ *is a point of density of* E_x *relative to* S_x

(ii) $\lim\limits_{\substack{(u,v)\to(x,0)\\(u,v)\in E_x}} \Phi(u, v) = f(x)$ *exists for each* $x \in (-\infty, \infty)$.

If the function $\Theta : (-\infty, \infty) \to (0, \pi)$ *associated with the family of Stolz angles has the following property*:

(iii) *for each perfect set* P *there exists an open interval* J *such that* $J \cap P \neq \emptyset$ *and* $\inf\{\Theta(x) : x \in J \cap P\} > 0$, *then the function* f *is in the first Baire class.*

Every function $\Theta : (-\infty, \infty) \to (0, \pi)$ which is in the first Baire class has the property (iii). In fact: If Θ is in the first Baire class and P is a perfect set, then there exists a point $x_0 \in P$ such that the partial function f/P is continuous in x_0. Since $\Theta(x_0) > 0$, there exists an open interval J which contains the point x_0 and $\Theta(x) > \frac{\Theta(x_0)}{2} > 0$, for each $x \in J \cap P$. Therefore we have: $\inf\{\Theta(x) : x \in J \cap P\} \geqslant \frac{\Theta(x_0)}{2} > 0$. We remark further that every upper semicontinuous function is in the first Baire class ([1], p. 249).

Theorem 1 is the best possible result in this respect, since the following holds:

Theorem 2. *Let a family of Stolz angles* S_x *for* $x \in (-\infty, \infty)$ *be given. Let* Θ *be a function*: $(-\infty, \infty) \to (0, \pi)$ *associated with the family of Stolz angles which does not possess the property* (iii), *i. e., there exists a nonempty perfect set* P *such that* $\inf\{\Theta(x) : x \in J \cap P\} = 0$ *for every open interval* J *for which* $J \cap P \neq \emptyset$.

Then there exists a function $\Phi : W \to (-\infty, \infty)$ *and a set* E_x *for each* $x \in (-\infty, \infty)$ *such that*

(i) $(x, 0)$ *is a point of density of a set* E_x *relative to* S_x

(ii) $\lim\limits_{\substack{(u,v)\to(x,0)\\(u,v)\in E_x}} \Phi(u, v) = f(x)$ *exists for each real number* x *and* f *is not in the first Baire class.*

Proof. Let P be a nonempty perfect set with the property that $\inf\{\Theta(x) : x \in J \cap P\} = 0$ for each open interval J for which $J \cap P \neq \emptyset$. From the existence of the countable base for $(-\infty, \infty)$ it follows that there exists a sequence $\{J_n\}_{n=1}^{\infty}$ of open intervals such that $J_n \cap P \neq \emptyset$ for $n = 1, 2, 3, \ldots$ and for every open interval J and for each point $x \in J \cap P$ there exists an open interval J_p for which $x \in J_p \cap P \subset J \cap P$. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence of points of P such that $\Theta(r_n) < \frac{1}{2}$, $\operatorname{tg} \frac{\Theta(r_{n+1})}{2} < \frac{1}{2} \operatorname{tg} \frac{\Theta(r_n)}{2}$ for $n = 1, 2, 3, \ldots$

and every open interval J_n $(n = 1, 2, 3, \ldots)$ contains an infinity of terms of this sequence. We take $E_{r_1} = S_{r_1}^{h_1}$, where $h_1 < 1$. From $(r_{n+1}, 0) \notin$

$\overline{\cup \{E_{r_i} : i = 1, 2, 3, \ldots, n\}}$ it follows that there exists an $h_{n+1} < \frac{1}{2^{n+1}}$ such that $S_{r_{n+1}}^{h_{n+1}} \cap (\cup \{E_{r_i} : i = 1, 2, 3, \ldots, n\}) = \emptyset$. We put $E_{r_{n+1}} = S_{r_{n+1}}^{h_{n+1}}$. In such a way we obtain by induction a sequence $\{E_{r_n}\}_{n=1}^{\infty}$ of disjoint sets for which $E_{r_n} = S_{r_n}^{h_n}$, where $h_n < \frac{1}{2^n}$, for $n = 1, 2, 3, \ldots$. It is obvious that $(r_n, 0)$ is a point of density of E_{r_n} relative to S_{r_n} for $n = 1, 2, 3, \ldots$.

Let $x \notin \{r_n : n = 1, 2, 3, \ldots\}$. We put $E_x = S_x^1 - \cup \{E_{r_n} : n = 1, 2, 3, \ldots\}$. We shall show that $(x, 0)$ is a point of density of E_x relative to S_x. Let $\varepsilon > 0$ and $\frac{1}{2^K} < \varepsilon$. Since $\Theta(x) > 0$, we can choose an N such that $\operatorname{tg} \frac{\Theta(r_N)}{2} < \frac{1}{2^{K+1}} \operatorname{tg} \frac{\Theta(x)}{2}$. Then the following holds:

$$\operatorname{tg} \frac{\Theta(r_{N+i})}{2} < \frac{1}{2^i} \operatorname{tg} \frac{\Theta(r_N)}{2} < \frac{1}{2^{K+i+1}} \operatorname{tg} \frac{\Theta(x)}{2}$$

for $i = 1, 2, 3, \ldots$. Since $(x, 0) \notin \overline{\cup \{E_{r_n} : n = 1, 2, \ldots, N-1\}}$ there exists an h, $0 < h \leqq 1$ such that $S_x^h \cap (\cup \{E_{r_n} : n = 1, 2, 3, \ldots, N-1\}) = \emptyset$. Let $0 < h' < h$. Then $E_x \cap S_x^{h'} = S_x^{h'} - (\cup \{E_{r_n} : n = N, N+1, \ldots\}) = S_x^{h'} - (\cup\{E_{r_n} \cap S_{r_n}^{h'} : n = N, N+1, \ldots\})$. Hence it follows that

$$\frac{|E_x \cap S_x^{h'}|}{|S_x^{h'}|} = \frac{h'^2 \operatorname{tg} \frac{\Theta(x)}{2} - |\cup\{E_{r_n} \cap S_{r_n}^{h'} : n = N, N+1, \ldots\}|}{h'^2 \operatorname{tg} \frac{\Theta(x)}{2}} \leqq$$

$$\leqq \frac{h'^2 \operatorname{tg} \frac{\Theta(x)}{2} - \sum_{n=N}^{\infty} h'^2 \operatorname{tg} \frac{\Theta(r_n)}{2}}{h'^2 \operatorname{tg} \frac{\Theta(x)}{2}} > \frac{\operatorname{tg} \frac{\Theta(x)}{2} - \sum_{n=N}^{\infty} \frac{\operatorname{tg} \frac{\Theta(x)}{2}}{2^{K+(n-N+1)}}}{\operatorname{tg} \frac{\Theta(x)}{2}} =$$

$$= 1 - \sum_{i=0}^{\infty} \frac{1}{2^{K+i+1}} = 1 - \frac{1}{2^K} > 1 - \varepsilon \text{ for } 0 < h' < h.$$

Hence $(x, 0)$ is a point of density of E_x relative to S_x.

We define now $\Phi : W \to (-\infty, \infty)$ as follows:

$$\Phi(u, v) = \Theta(r_n) \text{ if } (u, v) \in E_{r_n} \text{ for } n = 1, 2, 3, \ldots$$
$$\Phi(u, v) = 1 \text{ if } (u, v) \in W - (\cup\{E_{r_n} : n = 1, 2, 3, \ldots\}).$$

It is obvious that there exist

$$f(r_n) = \lim_{\substack{(u,v)\to(r_n,0)\\(u,v)\in E_{r_n}}} \Phi(u, v) = \Theta(r_n) \text{ for } n = 1, 2, 3, \ldots$$

and

$$f(x) = \lim_{\substack{(u,v)\to(x,0)\\(u,v)\in E_x}} \Phi(u, v) = 1 \text{ for } x \notin \{r_n : n = 1, 2, 3, \ldots\}.$$

The function f is not in the first Baire class because f/P has no point of continuity on P ([1], p. 254). If J is an open interval which has a nonempty intersection with P, then $\inf\{f(x) : x \in J \cap P\} = 0 < 1 = \sup\{f(x) : x \in J \cap P\}$, because $\{r_n : n = 1, 2, 3, \ldots\} \cap (J \cap P)$ is a countable set and $J \cap P$ is an uncountable set. The function f/P has then the oscilation equal to 1 in each point of P and therefore it is nowhere continuous on P.

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