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A DECOMPOSITION OF A FUNCTIONAL AS A DIFFERENCE OF TWO POSITIVE FUNCTIONALS

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The present paper deals with a generalization of the theorem concerning the decomposition of a generalized measure as a difference of two measures and of the theorem concerning the decomposition of Daniell integrals. Functions on lattices of a certain type are examined. A special selection of lattices gives the theorem about the decomposition of the measure and of the integral. A similar method was used in papers [2] and [4].

Let us introduce some notation first. $x \vee y$, $x \wedge y$ — will denote lattice operations. $x_n \nearrow x$ ($x_n \searrow x$) will be written iff $x_n \leqq x_{n+1}$ ($x_{n+1} \leqq x_n$) for every n and $\bigvee_{n=1}^{\infty} = x$ ($\bigwedge_{n=1}^{\infty} = x$).

Let S be a distributive lattice with the operations $+$, $-$. We shall use the following conditions:

(a_1) There is an element $0 \in S$ such that $x - x = 0$ for every $x \in S$.

(a_2) If $x, y, v \in S$ and $0 \leqq x \leqq y \leqq v$, then $0 \leqq y - x \leqq v - x \leqq v$.
If $x, y, v \in S$ and $v \leqq x \leqq y \leqq 0$, then $v \leqq v - y \leqq x - y \leqq 0$.

(a_3) If $a, x, x_n \in S$ and $x_n \nearrow x$ ($x_n \searrow x$), then $x_n \wedge a \nearrow x \wedge a$ and $a - x_n \searrow a - x$ ($x_n \vee a \searrow x \vee a$ and $a - x_n \nearrow a - x$).

(a_4) $b = a + (b - a)$ if $0 \leqq a \leqq b$ or if $b \leqq a \leqq 0$.

(a_5) If $u \leqq v$ and $a \leqq b$, then $a + u \leqq b + v$.

Let I be such a function on S that, for every $a \in S$, the set

$$\{I(x)/\, a \wedge 0 \leqq x \leqq a \vee 0\}$$

is either upper or lower bounded. We shall use the following conditions:

(b_1)
$$I(a) + I(b) = I(a \vee b) + I(a \wedge b)$$

for every $a, b \in S$.

(b_2) $I(0) = 0$.

(b_3) If $0 \leqq x \leqq a \leqq b, 0 \leqq y \leqq b - a$ or if $b \leqq a \leqq x \leqq 0, b - a \leqq y \leqq 0$, then

$$I(x + y) = I(x) + I(y)\,.$$

(b_4) If $a_n \nearrow a$ or $a_n \searrow a$, and $|I(a_n)| < \infty$, for every n, then

$$\lim I(a_n) = I(a) .$$

Definition. *For $a \in S$ we define*
$I_1(a) = \sup \{I(x) \,/\, 0 \leqq x \leqq a\}$, $I_2(a) = \inf \{I(x) \,/\, 0 \leqq x \leqq a\}$ if $a \geqq 0$, and
$I_3(a) = \sup \{I(x) \,/\, a \leqq x \leqq 0\}$, $I_4(a) = \inf \{I(x)\, a \leqq x \leqq 0\}$ if $a \leqq 0$.

Proposition 1. *Let S satisfy* (a_1) *and I satisfy* (b_2). *Then the following assertions hold:*

(i) $I_1(0) = I_2(0) = I_3(0) = I_4(0) = 0$,
(ii) *I_1 and I_3 are non-negative, I_2 and I_4 are non-positive,*
(iii) *If $0 \leqq a \leqq b$, then $I_1(a) \leqq I_1(b)$ and $I_2(a) \geqq I_2(b)$,*
If $a \leqq b \leqq 0$, then $I_3(a) \geqq I_3(b)$ and $I_4(a) \leqq I_4(b)$.

Proposition 2. *Let S satisfy* (a_1), (a_2), (a_4) *and I satisfy* (b_2) *and* (b_3).
If $0 \leqq v \leqq x \leqq u$, $\varepsilon > 0$, $I_1(u) < \infty$ ($I_2(v) > -\infty$) and $I_1(u) \leqq I(x) + \varepsilon$ ($I_2(u) \geqq I(x) - \varepsilon$), then $-\varepsilon \leqq I(v)$ ($I(v) \leqq \varepsilon$). If $u \leqq x \leqq v \leqq 0$, $\varepsilon > 0$, $I_3(u) < \infty$ ($I_4(u) > -\infty$) and $I_3(u) \leqq I(x) + \varepsilon$ ($I_4(u) \geqq I(x) - \varepsilon$), then $-\varepsilon \leqq$ $\leqq I(v)$ ($I(v) \leqq \varepsilon$).

Proof. We shall prove the assertion only for I_1. The proofs for I_2, I_3 and I_4 are analogous.

Let $0 \leqq v \leqq x \leqq u$, $\varepsilon > 0$, $I_1(u) < \infty$ and $I_1(u) \leqq I(x) + \varepsilon$ and let $I(v) < -\varepsilon$. Since $0 \leqq v \leqq x$, it follows from (b_3) and (a_2) that

$$I(x) = I(v) + I(x - v), \quad 0 \leqq x - v \leqq u \tag{1}$$

and

$$I(v) + I(x - v) \leqq I_1(u) + I(v) < I_1(u) - \varepsilon .$$

From this and from (1) it follows

$$I(x) < I_1(u) - \varepsilon ,$$

which contradicts the assumption. Hence $I(v) \geqq -\varepsilon$.

Proposition 3. *Let S satisfy* (a_1), (a_2) *and* (a_4), *and let I satisfy* (b_1), (b_2) *and* (b_3). *Then*

$$I_j(a) + I_j(b) = I_j(a \vee b) + I_j(a \wedge b) \tag{2}$$

for $j = 1, 2, 3, 4$ where $a, b \geqq 0$ in case $j = 1, 2$, and $a, b \leqq 0$ in case $j = 3, 4$.

Proof. The proposition will be proved only for $j = 1$. The proof for $j = 2, 3, 4$ is analogous.

Let $a, b \geqq 0$, and let $I_1(a \vee b) = \infty$. Let t be such that

$$0 \leqq t \leqq a \vee b ,$$

then (since S is distributive)

$$t = (a \wedge t) \vee (b \wedge t) .$$

From the definition of I_1 and from (b_1), (b_3), (a_4) and (a_2) we obtain

$$I(t) = I(t \wedge a) + I(t \wedge b) - I(t \wedge a \wedge b) = \\ = I(t \wedge a) + I((t \wedge b) - (t \wedge a \wedge b))$$

and

$$0 \leqq (b \wedge t) - (a \wedge b \wedge t) \leqq b .$$

Since $0 \leqq t \wedge a \leqq a$, we have

$$I(t) \leqq I_1(a) + I_1(b) .$$

Further,

$$\infty = I_1(a \vee b) = \sup \{I(t) \,/\, 0 \leqq t \leqq a \vee b\} \leqq I_1(a) + I_1(b) ,$$

hence (2) holds.

Now if $I_1(a \vee b) < \infty$, then also $I_1(a), I_1(b), I_1(a \wedge b) < \infty$. Let $a, b \in S$, $a, b \geqq 0$ and let ε be any positive number. Choose $x, y \in S$ with

$$I_1(a) \leqq I(x) + \varepsilon/2 \qquad 0 \leqq x \leqq a ,$$

$$I_1(b) \leqq I(y) + \varepsilon/2 \qquad 0 \leqq y \leqq b ,$$

then using (b_1) we obtain

$$I_1(a) + I_1(b) \leqq I(x) + I(y) + \varepsilon = I(x \vee y) + I(x \wedge y) + \varepsilon ,$$

where

$$0 \leqq x \vee y \leqq a \vee b, \qquad 0 \leqq x \wedge y \leqq a \wedge b .$$

Therefore

$$I_1(a) + I_1(b) \leqq I_1(a \vee b) + I_1(a \wedge b) + \varepsilon .$$

The inequality holds for every $\varepsilon > 0$, hence

(4) $$I_1(a) + I_1(b) \leqq I_1(a \vee b) + I_1(a \wedge b) .$$

Now we prove the opposite inequality. Let $\varepsilon > 0$. Choose $x \in S$ with

(5) $$I_1(a \vee b) \leqq I(x) + \varepsilon, \quad 0 \leqq x \leqq a \vee b ,$$

then

(6) $$I_1(a \vee b) + I_1(a \wedge b) \leqq I(x) + I_1(a \wedge b) + \varepsilon .$$

Clearly $x = (x \wedge a) \vee (x \wedge b)$ (since S is distributive). This gives

(7) $$I(x) + I_1(a \wedge b) + \varepsilon = I((x \wedge a) \vee (x \wedge b)) + I_1(a \wedge b) + \varepsilon .$$

From (6), (7) and from (b_1) we have

$$I_1(a \vee b) + I_1(a \wedge b) \leqq I(x \wedge a) + I(x \wedge b) - I(x \wedge a \wedge b) + \\ + I_1(a \wedge b) + \varepsilon ,$$

where $0 \leqq x \wedge a \leqq a$, $0 \leqq x \wedge b \leqq b$, therefore

(8) $\quad I_1(a \vee b) + I_1(a \wedge b) \leqq I_1(a) + I_1(a \wedge b) - I(x \wedge a \wedge b) + I_1(b) + \varepsilon .$

Choose $y \in S$ with

$$I_1(a \wedge b) \leqq I(y) + \varepsilon, \quad 0 \leqq y \leqq a \wedge b,$$

then

$$I_1(a \wedge b) - I(x \wedge a \wedge b) \leqq I(y) - I(x \wedge a \wedge b) + \varepsilon .$$

Owing to (b_1) we have

$$I_1(a \wedge b) - I(x \wedge a \wedge b) \leqq I(x \vee y) + I(x \wedge y) - I(x) - I(x \wedge a \wedge b) + \varepsilon ,$$

where $0 \leqq x \vee y \leqq a \vee b$, therefore

$$I_1(a \wedge b) - I(x \wedge a \wedge b) \leqq I_1(a \vee b) - I(x) - (I(x \wedge a \wedge b) - I(x \wedge y)) + \varepsilon .$$

From (5) and from the condition (b_3) it follows that

(9) $$I_1(a \wedge b) - I(x \wedge a \wedge b) \leqq 2\varepsilon - I(x \wedge a \wedge b - x \wedge y) ,$$

where $0 \leqq x \wedge y \leqq x \wedge a \wedge b$. Then from ($a_2$) it follows

$$0 \leqq x \wedge a \wedge b - x \wedge y \leqq x \leqq a \vee b .$$

If in Proposition 2 we put

$$u = a \vee b \quad \text{and} \quad v = x \wedge a \wedge b - x \wedge y ,$$

we can see that

(10) $$-I(x \wedge a \wedge b - x \wedge y) \leqq \varepsilon .$$

From (8), (9) and (10) we have

$$I_1(a \vee b) + I_1(a \wedge b) \leqq I_1(a) + I_1(b) + 4\varepsilon ,$$

and so

(11) $$I_1(a \vee b) + I_1(a \wedge b) \leqq I_1(a) + I_1(b) .$$

From (4) and (11) it follows

$$I_1(a \vee b) + I_1(a \wedge b) = I_1(a) + I_1(b) .$$

Proposition 4. *Let S satisfy* (a_1), (a_2), (a_4), (a_5) *and I satisfy* (b_1), (b_2), (b_3). *Then*

$$I_j(b) = I_j(a) + I_j(b - a)$$

for $j = 1, 2$ if $0 \leqq a \leqq b$, and for $j = 3, 4$ if $b \leqq a \leqq 0$.

Proof. Let $0 \leqq a \leqq b$, $\varepsilon > 0$, $I_1(b) < \infty$. Choose $x \in S$ with

$$I_1(b) \leqq I(x) + \varepsilon \quad \text{and} \quad 0 \leqq x \leqq b .$$

It follows from the last inequality, (b_1) and (b_3) that

(12) $I_1(b) \leqq J(x \wedge a) + I(x \vee a) - I(a) + \varepsilon = I(x \wedge a) + I(x \vee a - a) + \varepsilon$.

Since $0 \leqq a \leqq x \vee a \leqq b$, from ($a_2$) it follows that $0 \leqq x \vee a - a \leqq b - a$, and so from (12) we obtain

$$I_1(b) \leqq I_1(a) + I_1(b - a) + \varepsilon .$$

Therefore

(13) $$I_1(b) \leqq I_1(a) + I_1(b - a) .$$

If $I_1(b) = \infty$, then the proof of (13) is similar but we must use the fact that

$$\{I(x) \,/\, a \wedge 0 \leqq x \leqq a \vee 0\}$$

is lower bounded.

Now we prove the opposite inequality. Choose $x, y \in S$ with

$$I_1(a) \leqq I(x) + \varepsilon/2 \qquad 0 \leqq x \leqq a$$

and

$$I_1(b - a) \leqq I(y) + \varepsilon/2 \qquad 0 \leqq y \leqq b - a .$$

Then from (a_2), (b_3), (a_5) and (a_4) it follows that

$$I_1(a) + I_1(b - a) \leqq I(x) + I(y) + \varepsilon = I(x + y) + \varepsilon$$

and

$$0 \leqq x + y \leqq a + (b - a) = b .$$

Hence

(14) $$I_1(a) + I_1(b - a) \leqq I_1(b) + \varepsilon .$$

From (13) and (14) we have

$$I_1(a) + I_1(b - a) = I_1(b) .$$

The proofs for I_2, I_3 and I_4 are analogous.

Proposition 5. *Let S satisfy* (a_1), (a_2), (a_3), (a_4), (a_5), *I satisfy* (b_1), (b_2), (b_3), (b_4). *If $a, x_n \geqq 0$ and $x_n \nearrow a$ ($x_n \searrow a$), then $I_1(x_n) \nearrow I_1(a)$ and $I_2(x_n) \searrow I_2(a)$ ($I_1(x_n) \searrow I_1(a)$ and $I_2(x_n) \nearrow I_2(a)$). If $a, x_n \leqq 0$ and $x_n \nearrow a$ ($x_n \searrow a$), then $I_3(x_n) \searrow I_3(a)$ and $I_4(x_n) \nearrow I_4(a)$ ($I_3(x_n) \nearrow I_3(a)$ and $I_4(x_n) \searrow I_4(a)$).*

Proof. Let $x_n \nearrow a$, $x_n, a \geqq 0$, then for every n

(15) $$I_1(x_n) \leqq I_1(a) .$$

Let $I_1(a) < \infty$, then for every $\varepsilon > 0$ there is $t \in S$ such that

$$(16) \qquad I_1(a) \leqq I(t) + \varepsilon \quad \text{and} \quad 0 \leqq t \leqq a$$

From (a3) we have $x_n \wedge t \nearrow a \wedge t = t$.
Since $0 \leqq x_n \wedge t \leqq x_n$, on grounds of (b4) we have

$$(17) \qquad I(t) = \lim I(x_n \wedge t) \leqq \lim I_1(x_n) .$$

From (16) and (17) we obtain

$$(18) \qquad I_1(a) \leqq \lim I_1(x_n) + \varepsilon .$$

The proof follows from (15) and (18).

Let now $I_1(a) = \infty$, then for every N there is an element $t \in S$ such that $0 \leqq t \leqq a$ and $N \leqq I(t)$. Similarly we can easily see that

$$N \leqq I(t) \leqq \lim I_1(x_n)$$

for every N, therefore

$$\lim I_1(x_n) = \infty .$$

Let now $x_n \searrow a$. From (a3) it follows that

$$x_1 - x_n \nearrow x_1 - a \quad \text{and} \quad x_1 - x_n,\ x_1 - a \geqq 0 .$$

From the first part of the proposition it follows that

$$I_1(x_1 - x_n) \nearrow I_1(x_1 - a) .$$

According to Proposition 4 we obtain

$$I_1(x_1) = I_1(x_n) + I_1(x_1 - x_n) .$$

Hence

$$I_1(x_1) = \lim I_1(x_n) + \lim I_1(x_1 - x_n) = \lim I_1(x_n) + I_1(x_1 - a) ,$$

and so

$$I_1(x_1) - I_1(x_1 - a) = \lim I_1(x_n) ,$$

therefore from Proposition 4 it follows that

$$I_1(a) = \lim I_1(x_n) .$$

The proofs for I_2, I_3 and I_4 are analogous.

Proposition 6. *Let S satisfy* (a1), (a2), *let I satisfy* (b2), (b3). *If $a \geqq 0$, then*

$$(19) \qquad I(a) = I_1(a) + I_2(a) ,$$

if $a \leqq 0$, then

$$(20) \qquad I(a) = I_3(a) + I_4(a) .$$

Proof. Let $a \geqq 0$, then $I(a) = \infty$ $(I(a) = -\infty)$ if and only if $I_1(a) = \infty$ $(I_2(a) = -\infty)$.

That means, if $I(a)$, $I_1(a)$ or $I_2(a)$ is ∞ or $-\infty$, then (19) holds.

Let now $I(a)$, $I_1(a)$, $I_2(a)$ be finite. Let $\varepsilon > 0$. Choose $x \in S$ with

$$I_1(a) \leqq I(x) + \varepsilon \quad \text{and} \quad 0 \leqq x \leqq a,$$

then $0 \leqq a - x \leqq a$ and hence

$$I(a - x) \geqq J_2(a) .$$

Therefore it follows from (b_3), that for every $\varepsilon > 0$,

$$I_1(a) + I_2(a) \leqq I(a) - I(a - x) + \varepsilon + I_2(a) \leqq$$

$$\leqq I(a) + I(a - x) - I(a - x) + \varepsilon = I(a) + \varepsilon .$$

Hence

$$I_1(a) + I_2(a) \leqq I(a) . \tag{21}$$

Let now $\varepsilon > 0$ and x be such that

$$I(x) \leqq I_2(a) + \varepsilon \quad \text{and} \quad 0 \leqq x \leqq a .$$

Then $0 \leqq a - x \leqq a$ and from (b_3) we obtain

$$I(a) = I(x) + I(a - x) \leqq I(x) + I_1(a) \leqq I_2(a) + I_1(a) + \varepsilon .$$

It follows from the last inequality that

$$I(a) \leqq I_1(a) + I_2(a) . \tag{22}$$

The obtained inequalites (21) and (22) complete the proof. The proof for $a \leqq 0$ is analogous.

Definition. *We denote*

$$I^+(a) = I_1(a \vee 0) + I_4(a \wedge 0), \quad I^-(a) = -I_2(a \vee 0) - I_3(a \wedge 0).$$

Then the following theorem holds:

Theorem. *Let S satisfy* (a_1), (a_2), (a_3), (a_4), (a_5) *and I satisfy* (b_1), (b_2), (b_3), (b_4). *Then*

(i) $I^+(0) = I^-(0) = 0$,

(ii) *If $a \leqq b$, then $I^+(a) \leqq I^+(b)$ and $I^-(a) \leqq I^-(b)$,*

(iii) *If $x_n \nearrow a$ $(x_n \searrow a)$, then $I^+(x_n) \nearrow I^+(a)$ $(I^+(x_n) \searrow I^+(a))$ and $I^-(x_n) \nearrow I^-(a)$ $(I^-(x_n) \searrow I^-(a))$,*

(iv) *For every* $a, b \in S$ *we have*

$$I^+(a) + I^+(b) = I^+(a \vee b) + I^+(a \wedge b)$$

and

$$I^-(a) + I^-(b) = I^-(a \vee b) + I^-(a \wedge b).$$

(v) $I^+(b) = I^+(a) + I^+(b - a)$ *and* $I^-(b) = I^-(a) + I^-(b - a)$ *if* $0 \leqq a \leqq b$ *or if* $b \leqq a \leqq 0$.

(vi) *For every* $a \in S$ *we have*

$$I(a) = I^+(a) - I^-(a) .$$

Proof. The statements (i), (ii), (iii), (iv), (v) follow from Propositions 1, 3, 4, 5. We prove (vi). Let $a \in S$. From (b_1) and (b_2) it follows that

$$I(a) = I(a \vee 0) + I(a \wedge 0) .$$

However $a \vee 0 \geqq 0$ and $a \wedge 0 \leqq 0$, hence from proposition 6 we have

$$I(a) = I_1(a \vee 0) + I_2(a \vee 0) + I_3(a \wedge 0) + I_4(a \wedge 0) = I^+(a) - I^-(a).$$

It is a natural question whether the decomposition of I is unique. If, e. g. $I^+(a)$ is finite, then

$$I(a) = 2I^+(a) - (I^+(a) + I^-(a))$$

and this decomposition is a different one. Yet the following proposition is true:

Proposition 7. *Let* S *satisfy* (a_1), (a_2), (a_3), (a_4) *and* (a_5), I *satisfy* (b_1), (b_2), (b_3) *and* (b_4). *Let* $|I(a)| < \infty$ *and* $I = J^+ - J^-$, *where* J^+, J^- *satisfy* (i), (ii), (v) *and* (vi). *Then* $I^+(a) \leqq J^+(a)$, $I^-(a) \leqq J^-(a)$.

Proof. Let first $a \geqq 0$. Then

$$I^+(a) = I_1(a) \quad \text{and} \quad I^-(a) = -I_2(a) .$$

Since $|I(a)| < \infty$, for every $\varepsilon > 0$ there is $u \in S$ such that

$$I_1(a) \leqq I(u) + \varepsilon \quad \text{and} \quad 0 \leqq u \leqq a .$$

It follows from the last inequality and from Proposition 2 that $0 \leqq x \leqq u$ implies $I(x) \geqq -\varepsilon$, therefore $I_2(u) \geqq -\varepsilon$ and hence

$$I_1(u) \leqq I_1(a) \leqq I(u) + \varepsilon, \quad I_2(u) \geqq -\varepsilon. \tag{23}$$

Further, according to (b_3) and (23),

$$I(a - u) = I(a) - I(u) = I_1(a) + I_2(a) - I(u) \leqq I_2(a) + \varepsilon$$

$$I_1(a - u) = I(a - u) - I_2(a - u) \leqq I(a - u) - I_2(a) \leqq \varepsilon .$$

Hence

(24) $$I_2(a-u) \leqq I(a-u) \leqq I_2(a)+\varepsilon, \quad I_1(a-u) \leqq \varepsilon.$$

Since $I^+(b)-I^-(b)=J^+(b)-J^-(b)$, for every $b \in S$ we have

$$I^+(u)-I^-(u)=J^+(u)-J^-(u)\,.$$

Thus,

$$I^+(u)=J^+(u)-J^-(u)+I^-(u)\,.$$

However, $I^-(u)=-I_2(u)$, $-J^-(u) \leqq 0$, hence according to (23)

(25) $$I^+(u) \leqq J^+(u)+\varepsilon\,.$$

Similarly from the relations

$$I^-(a-u)=J^-(a-u)-J^+(a-u)+I^+(a-u)\,,$$

$$I^+(a-u)=I_1(a-u), \quad -J^+(a-u) \leqq 0$$

and from (24) we obtain

(26) $$I^-(a-u) \leqq J^-(a-u)+\varepsilon\,.$$

Using (26) and

$$I^+(a-u)=J^+(a-u)+I^-(a-u)-J^-(a-u)$$

we have

(27) $$I^+(a-u) \leqq J^+(a-u)+\varepsilon\,.$$

From (25), (27) and from the property (v) (Theorem) for I^+ and J^+ we have

$$I^+(a)=I^+(u)+I^+(a-u) \leqq J^+(u)+J^+(a-u)+2\varepsilon=J^+(a)+2\varepsilon,$$

for every $\varepsilon>0$, hence

$$I^+(a) \leqq J^+(a)\,.$$

Further

$$I^-(a)=I^+(a)-J^+(a)+J^-(a) \leqq J^-(a)\,.$$

Hence we obtained

$$I^+(a) \leqq J^+(a) \quad \text{and} \quad I^-(a) \leqq J^-(a)\,.$$

For $a \leqq 0$, the proof is similar.

Let a be an arbitrary element from S. From the validity of (i) and (iv) for I^+ and I^- we obtain

$$I^+(a)=I^+(a \vee 0)+I^+(a \wedge 0) \leqq J^+(a \vee 0)+J^+(a \wedge 0)=J^+(a)\,.$$

Similarly we obtain that $I^-(a) \leqq J^-(a)$.

The following example shows that if $|I(a)| = \infty$, then the statement of Proposition 7 need not be valid.

Example. Denote J^+, J^- by

$$J^+(0) = 0, \quad J^+(a) = -1, \quad J^-(0) = 0, \quad J^-(a) = -\infty .$$

Then $I = J^+ - J^-$ implies that $I(0) = 0$, $I(a) = \infty$, where $S = \{0, a\}$ and $0 > a$, $0 + a = a + 0 = a + a = a - 0 = a$, $0 + 0 = 0 - 0 = 0 - a = a - a = 0$. We see that $I = I^+ - I^- = J^+ - J^-$ is valid although

$$-1 = J^+(a) < J^+(a) = 0 .$$

Corollary 1. *Let S be a σ-algebra of subsets of X. Let μ be the generalized measure on S. Then there are measures μ^+ and μ^- such that*

$$\mu = \mu^+ - \mu^- .$$

Proof. If A and B are any two sets from S, then let $A \vee B$ denote their union, $A \wedge B$ denote their intersection, $A - B$ denote the relative complement of B in A, and $A + B = A \vee B$. Then (a_1), (a_2), (a_3), (a_4), (a_5), (b_1), (b_2), (b_3), (b_4) hold and the Corollary is a consequence of the Theorem.

Corollary 2. *Let S be a σ-algebra of real functions. Let μ be the Daniell integral on S. Then there are integrals μ^+ and μ^- such that $\mu^+(f) \geqq 0$ and $\mu^-(f) \geqq 0$ if $f \geqq 0$, and*

$$\mu = \mu^+ - \mu^- .$$

Proof. If f, g are any two functions, let $(f \vee g)(x) = \max\{f(x), g(x)\}$ $(f \wedge g) = \min\{f(x), g(x)\}$, $(f + g)(x) = f(x) + g(x)$, $(f - g)(x) = f(x) - g(x)$. Then (a_1), (a_2), (a_3), (a_4), (a_5), (b_1), (b_2), (b_3), (b_4) are valid and an application of the Theorem completes the proof.

REFERENCES

[1] FEDERER H.: Geometric Measure Theory. Berlin 1969.

[2] FUTÁŠ E.: Extension of continuous functionals, Mat. časop., *21*, 1971, 191—198.

[3] HALMOS P. R.: Measure Theory. New York 1950.

[4] RIEČAN B.:O непрерывном продолжении монотонных функционалов некоторого типа. Mat.-fyz. časop. 15, 1965, 116—125.

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